

1 An isolated nonlinear active stage for a fixed smooth profile

The result of this section concerns one isolated annulus. It replaces the discontinuous reference vorticity by one fixed smooth profile, explicitly. The smoothing length is chosen once and is independent of viscosity and of the small parameter below. In particular, the theorem does not assert the analogous nonlinear estimate for a family of sharp patch boundaries.

Theorem 1.1 (Isolated smooth activation). *There exist a real, radial, compactly supported C^∞ vorticity W_ϵ , with zero circulation, a number $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with $\sigma_\epsilon > 0$, and a nonzero complex number b_ϵ , with the following property. For every $2 < p < \infty$ and $K_* > 0$ there is $q_0 > 0$ such that the conclusions below hold for $0 < q < q_0$. Put*

$$L_q = \log(1/q), \quad \alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad \delta_q = L_q^{-4}, \quad H_q = \frac{1}{\sigma_\epsilon} \log \frac{\delta_q}{\alpha_q}.$$

There is a 2π -periodic family of smooth real seeds Φ_ϑ , supported in two fixed annular collars, such that

$$\int \Phi_\vartheta = 0, \quad B_4(\Phi_\vartheta) = 0, \quad \Phi_\vartheta(ix) = \Phi_\vartheta(x).$$

Let $\omega^{q,\vartheta}$ be the unforced Navier–Stokes solution with viscosity κ_q and initial vorticity $W_\epsilon + \alpha_q \Phi_\vartheta$. Then, uniformly in $\vartheta \in \mathbb{R}$,

$$\sup_{0 \leq t \leq H_q} \|\omega^{q,\vartheta}(t) - e^{\kappa_q t \Delta} W_\epsilon\|_{L^1 \cap L^p} \leq C \delta_q, \quad (1)$$

$$\int_0^{H_q} \|K * (\omega^{q,\vartheta}(t) - e^{\kappa_q t \Delta} W_\epsilon)\|_\infty dt \leq C \delta_q, \quad (2)$$

$$B_4(\omega^{q,\vartheta}(H_q)) = \delta_q e^{i(\vartheta + \tau_\epsilon H_q)} b_\epsilon + o(\delta_q). \quad (3)$$

Here $\|f\|_{L^1 \cap L^p} = \|f\|_1 + \|f\|_p$; all fourth absolute moments in (??) are finite. In particular, (??) is a statement about the whole moment, rather than only a compactly tested moment. The constants depend on the fixed profile, p and K_ , but not on q or ϑ .*

The positive cap δ_q will eventually be fixed by choosing one q . Its logarithmic decrease is useful for the simultaneous condition $H_q \sqrt{\delta_q} \rightarrow 0$. The theorem concerns a family of isolated initial data and does not claim nonuniqueness of any fixed-data vanishing-viscosity limit.

1.1 The fixed profile and its positive angular spectrum

Fix $\rho^2 = 3/5$ and choose a nonnegative, radial, radially decreasing $\chi \in C_c^\infty(B_1)$ with integral one. Define

$$\chi_\epsilon(x) = \epsilon^{-2} \chi(x/\epsilon), \quad W_\epsilon = \chi_\epsilon * (-\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}).$$

The convolution is smooth, radial, supported in $B_{1+\epsilon}$ and has zero integral. It is constant on $B_{\rho-\epsilon}$. Choose $\epsilon_* < (1 - \rho)/4$ and subsequently $0 < \epsilon < \epsilon_*$. Its radial derivative has the form

$$W'_\epsilon(r) = D_{\rho,\epsilon}(r) - \rho^2 D_{1,\epsilon}(r), \quad D_{a,\epsilon} \geq 0, \quad \text{supp } D_{a,\epsilon} \subset [a - \epsilon, a + \epsilon], \quad \int_0^\infty D_{a,\epsilon}(r) dr = 1. \quad (4)$$

To verify the sign, express the radial decreasing mollifier by its superlevel disks. The area of intersection of two disks is nonincreasing in their center separation: along each line parallel to their displacement, the overlap of the two interval sections is nonincreasing. Layer-cake

integration gives radial monotonicity of the disk convolution. Its value is one at the origin and zero at infinity, giving the displayed derivative mass. The support assertion follows because the convolution is exactly constant at points whose distance to the disk boundary exceeds ϵ .

Set

$$\Omega_\epsilon(r) = r^{-2} \int_0^r s W_\epsilon(s) ds.$$

On the two derivative collars, respectively,

$$\Omega_\epsilon(r) = \Omega_0(r_i) + O(\epsilon), \quad (r_1, r_2) = (\rho, 1), \quad (\Omega_0(r_1), \Omega_0(r_2)) = (-1/5, 0). \quad (5)$$

Indeed disk translation changes its L^1 norm by at most a constant times the translation length. Thus $\|W_\epsilon - W_0\|_1 \leq C\epsilon$, where $W_0 = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}$. The radial circulation formula, on radii bounded away from zero, gives $|\Omega_\epsilon - \Omega_0| \leq C\epsilon$. The piecewise smooth Ω_0 is Lipschitz there. Direct integration gives the interface values.

For a positive angular mode $m \geq 1$ use the convention $f(r)e^{im\theta}$. Put

$$h_m(r, s) = \left(\frac{\min(r, s)}{\max(r, s)} \right)^m, \quad \psi_f(r) = -\frac{1}{2m} \int_0^\infty h_m(r, s) f(s) s ds.$$

The function $\psi_f e^{im\theta}$ solves $\Delta\psi = f e^{im\theta}$, is regular at the origin and decays at infinity. This follows either by solving the radial second-order equation on each side of $r = s$, or by checking the derivative jump $1/s$ of the displayed Green kernel. Since $u_r = -im\psi_f/r$, the frozen Euler linearization at W_ϵ is

$$\mathcal{L}_m f = -im\Omega_\epsilon f + im\psi_f W'_\epsilon/r. \quad (6)$$

For $m = 0$ the linearization is zero: two radial vorticities have tangential velocities and do not advect one another. Negative sectors are the complex conjugates of the positive sectors. From this point onward all angular sectors belong to $4\mathbb{Z}$. No spectral assertion is made for other symmetry classes.

Lemma 1.2 (The fixed smooth unstable pair). *One can choose a fixed sufficiently small $\epsilon > 0$ and $0 < \eta < \sigma_\epsilon/4$ so that in the quarter-turn symmetry class: the $m = 4$ generator has one algebraically simple eigenvalue $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with positive real part greater than η ; its growing radial eigenfunction f_ϵ is smooth and supported in the two derivative collars; and all other positive-real-part spectral values, except the conjugate growing eigenvalue in $m = -4$, have real part at most η . Moreover*

$$\lambda_\epsilon \longrightarrow \rho^5/2 + 3i/10 \quad \text{as } \epsilon \downarrow 0.$$

For each finite set of sectors in $4\mathbb{Z}$ and each finite radial Sobolev order k , the corresponding propagator has the decomposition

$$e^{t\mathcal{L}} = e^{\lambda_\epsilon t} P_+ + e^{\overline{\lambda_\epsilon} t} P_- + \mathcal{R}_k(t), \quad \|\mathcal{R}_k(t)\|_{H^k \rightarrow H^k} \leq C_k e^{2\eta t}. \quad (7)$$

The growing terms are included only if their sectors are present.

Proof. First consider any eigenvalue λ with $\Re\lambda > 0$. The eigenvalue equation forces $f = 0$ off the derivative collars and gives

$$f(r)r dr = g(r)W'_\epsilon(r) dr, \quad \text{quad}g(r) = \frac{im\psi_f(r)}{\lambda + im\Omega_\epsilon(r)}.$$

There is no division by W'_ϵ here. In the Hilbert space $L^2(|W'_\epsilon| dr)$ the equation for g is

$$\lambda g = -im\Omega_\epsilon g - \frac{i}{2} \int h_m(r, s) g(s) W'_\epsilon(s) ds.$$

Each same-collar block is skew-adjoint, since the sign of W'_ϵ is constant there and $h_m(r, s)$ is real and symmetric. Angular multiplication is also skew-adjoint. The two opposite-collar blocks have operator norms at most their Hilbert–Schmidt norms, bounded by

$$Ca_*^m, \quad a_* = (\rho + \epsilon_*)/(1 - \epsilon_*) < 1,$$

because their two weighted masses are 1 and ρ^2 and their kernels are bounded by a_*^m . Consequently

$$\Re \lambda \leq Ca_*^m. \quad (8)$$

This fixes a finite angular cutoff in any half-plane $\Re \lambda \geq \eta > 0$, independently of ϵ .

For one of the finitely many positive sectors, eliminate f to obtain the equation on the two collars

$$\psi(r) = -\frac{i}{2} \int h_m(r, s) \frac{W'_\epsilon(s) \psi(s)}{\lambda + im\Omega_\epsilon(s)} ds =: \mathcal{T}_\epsilon(\lambda) \psi(r). \quad (9)$$

Work first in the sup-norm space of continuous functions on the two collars. Let P evaluate at r_1, r_2 and let E extend a vector to constants on the two collars, so $PE = I$. The receiving-radius derivative of the kernel is bounded there for fixed m . Hence

$$\|(I - EP)\mathcal{T}_\epsilon(\lambda)\| \leq C\epsilon$$

uniformly on compact subsets of $\Re \lambda \geq \eta$. For $\psi = Ep + w$, $Pw = 0$, solve the w equation by an analytic Neumann series. The remaining two-dimensional equation has a determinant converging uniformly on these compact sets to the determinant of

$$p_i = \frac{i}{2} \sum_j \frac{c_j h_m(r_i, r_j)}{\lambda + im\Omega_0(r_j)} p_j, \quad (c_1, c_2) = (-1, \rho^2).$$

The sign follows from $W'_0 = -\sum c_j \delta_{r_j}$. Using interface variables $f(r)r dr = \sum c_i r_i^2 q_i \delta_{r_i}$, this equation is equivalent to the matrix eigenvalue problem

$$M_m q = \lambda q, \quad M_m = \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

For $m = 4$ its roots are $\pm \rho^5/2 + 3i/10$, both simple. For $m = 8, 12, \dots$ they are imaginary, since the real-matrix discriminant is

$$\left(\frac{2(m-4)}{5}\right)^2 - 4\rho^{2m+2} > 0.$$

At $m = 8$ the first term is $64/25$ and the second is $4(3/5)^9 < 1$; thereafter the first increases and the second decreases. Large $|\lambda|$ gives no omitted roots: once $|\lambda| > C_m$, the operator in (??) has norm less than one, uniformly in ϵ , because the denominator then has size comparable with $|\lambda|$. Uniform determinant convergence and Rouché's theorem therefore give one simple root near the growing $m = 4$ root and no other roots in $\Re \lambda \geq \eta$. Combined with (??), this proves the claimed positive spectrum after one fixed ϵ is selected. Take η smaller first if necessary, so that $2\eta < \sigma_\epsilon$.

We give the operator-domain and multiplicity details needed for the propagator claim. Choose a closed interval $I \Subset (0, \infty)$ containing both collars strictly in its interior. On $H^k(I)$ define ψ_f by the Green integral against the zero extension of f from I . This extension need not be globally H^k : multiplication of the potential by W'_ϵ restricts output strictly inside I . The radial elliptic equation, or direct differentiation of its Green kernel, gives two local derivatives. Thus the coupling in (??) is compact on $H^k(I)$ and the angular multiplier is bounded there. Its spectrum is imaginary: off the multiplier's value range the inverse is multiplication by $(\lambda + im\Omega_\epsilon)^{-1}$. The compact-operator Fredholm alternative consequently reduces all spectrum in $\Re\lambda > 0$ to eigenfunctions. The equation bootstraps such eigenfunctions to smoothness and forces their support into the collars, so the roots already found are exactly the positive spectrum on every $H^k(I)$.

To check algebraic simplicity independently of a sup-norm determinant, start on $X_0 = L^2(I)$. Write $Af = -im\Omega_\epsilon f$, $Cf = \psi_f|_{\text{collars}}$ and $B\psi = imW'_\epsilon\psi/r$. Then $\mathcal{L}_m = A + BC$ and $\mathcal{T}_\epsilon = C(\lambda - A)^{-1}B$. Off the imaginary multiplier range, the resolvent identity is

$$(\lambda - \mathcal{L}_m)^{-1} = (\lambda - A)^{-1} + (\lambda - A)^{-1}B(I - \mathcal{T}_\epsilon)^{-1}C(\lambda - A)^{-1}. \quad (10)$$

Here $C : X_0 \rightarrow C(\text{collars})$ is bounded by its elliptic gain, and $B : C(\text{collars}) \rightarrow X_0$ is bounded. The invertible complementary Schur factor and the simple two-by-two determinant zero imply a rank-one simple pole in (??). The residue is nonzero, since an eigenfunction exists. On $X_k = H^k(I)$ the resolvent equation and its radial elliptic bootstrap give, locally uniformly for λ off the imaginary multiplier range,

$$\|v\|_{H^k(I)} \leq C_k(\|f\|_{H^k(I)} + \|v\|_{L^2(I)}), \quad (\lambda - \mathcal{L}_m)v = f.$$

The X_k resolvent is the restriction of the X_0 resolvent; the simple pole has no higher-order singularity by this estimate and has the same nonzero smooth one-dimensional range. This proves algebraic simplicity on every space used below. The $m = -4$ conclusion follows by conjugation; $m = 0$ is the identity propagator.

After removing the growing Riesz projections, the remaining bounded generator has spectral bound at most η . Its exponential is bounded by $Ce^{2\eta t}$. For clarity this last fact does not assume a spectral growth theorem for unbounded generators. For a bounded operator, integrate $e^{tz}(z - \mathcal{L})^{-1}$ on a finite rectangular contour enclosing its remaining spectrum and having right edge $\Re z = 2\eta$. Expansion of the resolvent on a large circle and termwise integration gives the exponential series; contour deformation gives the rectangle formula. The resolvent is bounded on this fixed contour, proving the estimate. The finitely many sectors and derivative orders give (??). Coefficients initially smooth and supported in the collars remain so under this propagator: multiplication preserves that support and the coupling has smooth output flat at the collar endpoints. This proves the lemma. $\square$

1.2 An exactly neutral smooth seed

Normalize the growing radial eigenfunction f_ϵ so that its first collar mass tends to $c_1 r_1^2 = -\rho^2$. The finite-dimensional reduction then implies

$$f_\epsilon(r)r \, dr \rightharpoonup -\rho^2 \delta_\rho - i\rho^3 \delta_1.$$

Indeed its interface vector tends to $(1, -i\rho)$, the growing eigenvector of M_4 . Define

$$b_\epsilon = 2\pi \int_0^\infty r^5 f_\epsilon(r) \, dr. \quad (11)$$

The weak convergence on the two compact collars gives

$$b_\epsilon \longrightarrow 2\pi(-\rho^6 - i\rho^3) \neq 0.$$

Reduce the fixed ϵ if necessary. Since $\mathcal{L}_4$ is i times a real operator, $\overline{f_\epsilon}$ is an eigenfunction of $\mathcal{L}_4$ with eigenvalue $-\overline{\lambda_\epsilon} = -\sigma_\epsilon + i\tau_\epsilon$. Define the real seed

$$\Phi_\vartheta(r, \theta) = e^{i\vartheta} \left(f_\epsilon(r) - \frac{b_\epsilon}{\overline{b_\epsilon}} \overline{f_\epsilon(r)} \right) e^{4i\theta} + \text{complex conjugate}. \quad (12)$$

Its circulation is zero and its only initial sectors are ± 4 . For the convention $B_4 = \int \bar{z}^4 \omega$, angular integration gives $B_4(f(r)e^{4i\theta}) = 2\pi \int r^5 f(r) dr$. Therefore $B_4(\Phi_\vartheta) = 0$ exactly, while the growing coefficient is $e^{i\vartheta}$. The stable coefficient has modulus one.

1.3 The finite seed–viscosity expansion

For the rest of the proof abbreviate the fixed profile to W , and write $\sigma = \sigma_\epsilon$, $\tau = \tau_\epsilon$, $W_\kappa(t) = e^{\kappa t \Delta} W$ and $U_\kappa = K * W_\kappa$. The latter is an exact radial heat solution. Put $X = L^1 \cap L^p$ with the sum norm. Splitting the Biot–Savart convolution at radius one gives

$$\|K * g\|_\infty \leq C_p \|g\|_X. \quad (13)$$

On the near part use Hölder and $|x|^{-1} \in L^{p'}(B_1)$, which holds because $p' < 2$. On the far part use its bounded kernel and the L^1 norm. This estimate requires no zero-circulation assumption.

The frozen full linear operator is $\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W$. Its sectors are those in (??). Choose integers $M, N \geq 1$, to be fixed below, and construct

$$f_{\text{app}} = \sum_{j=1}^M \sum_{\ell=0}^N \alpha^j \kappa^\ell f_{j,\ell}, \quad \omega_{\text{app}} = W_\kappa + f_{\text{app}}. \quad (14)$$

Set $f_{1,0}(0) = \Phi_\vartheta$ and all other initial coefficients zero. With $W_r(t) = t^r \Delta^r W / r!$, solve the triangular recursion

$$\begin{aligned} (\partial_t - \mathcal{L})f_{j,\ell} &= \mathbf{1}_{\ell \geq 1} \Delta f_{j,\ell-1} - \sum_{r=1}^{\ell} [(K * W_r) \cdot \nabla f_{j,\ell-r} + (K * f_{j,\ell-r}) \cdot \nabla W_r] \\ &\quad - \sum_{\substack{j_1+j_2=j, j_1, j_2 \geq 1 \\ \ell_1+\ell_2=\ell}} (K * f_{j_1,\ell_1}) \cdot \nabla f_{j_2,\ell_2}. \end{aligned} \quad (15)$$

Every right-hand side uses earlier coefficients in the order $j + \ell$. They are smooth and supported in the fixed collars: Laplacians preserve the support of smooth flat functions; $\Delta^r W$, $r \geq 1$, has that support; and every advective factor differentiates such a supported function. The linear source is supported where W' is supported. Each coefficient has zero circulation, because each forcing is a Laplacian or divergence of an integrable vector field. Its frequencies belong to $4\mathbb{Z}$ and satisfy $|m| \leq 4j$.

Lemma 1.3 (Coefficient bounds and seeded residual). *For every finite derivative order k , the finitely many coefficients in (??) satisfy*

$$\|f_{j,0}(t)\|_{H^k} \leq C_{j,k} e^{j\sigma t}, \quad (16)$$

$$\|f_{j,\ell}(t)\|_{H^k} \leq C_{j,\ell,k} (1+t)^D e^{j\sigma t}, \quad \ell > 0. \quad (17)$$

There is a finite D , enlarged as needed, such that if $a(t) = \alpha e^{\sigma t} \leq \delta_0 \leq 1$ on $[0, H]$ and $\kappa(1+H)^D \leq c$, the residual

$$R_{\text{app}} = \partial_t \omega_{\text{app}} + (K * \omega_{\text{app}}) \cdot \nabla \omega_{\text{app}} - \kappa \Delta \omega_{\text{app}}$$

satisfies

$$\|R_{\text{app}}(t)\|_X \leq C[a(t)^{M+1} + \kappa^{N+1}(1+t)^D a(t)]. \quad (18)$$

All constants are uniform in ϑ .

Proof. The first coefficient is the exact growing-plus-decaying solution from (??). For $j \geq 2$, $\ell = 0$, induction and Sobolev multiplication bound its forcing by $Ce^{j\sigma t}$. Convolution against the growing exponent σ , or the remaining exponent $2\eta < \sigma$, is bounded by a constant times $e^{j\sigma t}$, because $j\sigma - \sigma \geq \sigma$. This proves (??) with no polynomial factor. For positive ℓ the same induction applies, with finite polynomial losses from $W_r(t)$ and possible resonance with the first growing mode. This proves (??). Derivative loss causes no infinite iteration: the coefficient set is finite and the profile is smooth in every order. At each induction step apply the preceding coefficient estimates in a sufficiently higher fixed order and the propagator lemma in the requested order. Compact support and zero circulation also control $K * f$ and its needed derivatives, including low frequencies. More explicitly $|\hat{f}(\xi)| \leq C|\xi|$ near zero, from its zero mass and finite first absolute moment, while the high-frequency multiplier has order minus one.

Expand the radial heat base only in its couplings to f_{app} . The integral Taylor formula and heat contraction in L^1, L^p give

$$G_N(t) := W_\kappa(t) - \sum_{r=0}^N \kappa^r W_r(t), \quad \|G_N(t)\|_X + \|\nabla G_N(t)\|_X \leq C\kappa^{N+1}t^{N+1}.$$

Its residual couplings obey

$$\|(K * G_N) \cdot \nabla f_{\text{app}} + (K * f_{\text{app}}) \cdot \nabla G_N\|_X \leq C\kappa^{N+1}(1+t)^D a(t)$$

by (??) and the coefficient bounds. These are the only noncompact factors. Every other coefficient is supported in one fixed bounded set, where its high Sobolev norms control the required L^1, L^p product norms. The recursion cancels exactly all monomials with $1 \leq j \leq M$, $0 \leq \ell \leq N$. Terms of seed order $j > M$ are bounded by Ca^{M+1} : the pure-seed bounds contain no time polynomial, and each positive-viscosity polynomial is absorbed after enlarging D and using $\kappa(1+H)^D \leq c$. Terms of viscosity order $\ell > N$, including the diffusion of $f_{j,N}$, have the second bound in (??); $a^j \leq a$ for $a \leq 1$ absorbs their higher seed powers. Crucially the exact radial heat base is part of (??), so no residual term without a seed factor occurs. Finally rotations of (??) change only unit complex phases; all bounds are uniform in ϑ . $\square$

1.4 Nonlinear closure in the actual vorticity norm

Let ω^κ be the NS solution from $W + \alpha\Phi_\vartheta$ and put $\zeta = \omega^\kappa - \omega_{\text{app}}$. Its exact equation, with zero initial data, is

$$\partial_t \zeta + (K * \omega^\kappa) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta) \cdot \nabla \omega_{\text{app}} - R_{\text{app}}. \quad (19)$$

Kato's inequality gives its L^1 estimate, and multiplication by $|\zeta|^{p-2}\zeta$ gives its L^p estimate; transport disappears by divergence freedom and diffusion is nonpositive in both inequalities. Using (??) and adding gives

$$\frac{d^+}{dt} \|\zeta\|_X \leq C_p \|\nabla \omega_{\text{app}}\|_X \|\zeta\|_X + \|R_{\text{app}}\|_X. \quad (20)$$

These inequalities follow first with smooth convex regularizations of the norm and compact cutoffs, then by their limits. All solutions here are smooth at positive time, have finite X norm, and are continuous in that norm at time zero, so the integrated inequality starts at zero. No derivative bound on the actual NS vorticity enters its coefficient.

We choose parameters without circular dependence. Fix

$$C_1 = C_p(\|\nabla W\|_X + 1), \quad r_1 = (C_1/\sigma - 1)_+,$$

before choosing expansion orders. Choose M with $(M+1)\sigma > C_1$ and N with

$$\frac{3}{2}(N+1) > K_* r_1. \quad (21)$$

Construct the finite coefficients, then reduce a cap threshold δ_0 and the positive-viscosity threshold so that $\|\nabla f_{\text{app}}\|_X \leq 1$ whenever $a \leq \delta_0$ and $\kappa(1+H)^D$ is sufficiently small. This is possible by the finite coefficient bounds. The exact heat gradient obeys $\|\nabla W_\kappa\|_X \leq \|\nabla W\|_X$, so C_1 is the already fixed exponent in (??). The coefficient constants affect only the later cap threshold; they do not change the previous choice of M, N .

Convolving (??) against $e^{C_1(t-s)}$ gives the all-time estimate

$$\|\zeta(t)\|_X \leq Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)t}. \quad (22)$$

The first integral has denominator $(M+1)\sigma - C_1 > 0$. For the second use a bound by $e^{\max(C_1, \sigma)t}$ and an additional polynomial factor if the two exponents coincide. At $H = \sigma^{-1} \log(\delta/\alpha)$ this yields

$$\|\zeta(H)\|_X \leq C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \quad (23)$$

For the theorem's parameters, (??) implies that the second bound is $o(\delta_q)$: a strict positive power of q beats every fixed power of $\log(1/q)$. Also $\kappa_q(1+H_q)^D \rightarrow 0$, $\delta_q \rightarrow 0$, and $\alpha_q/\delta_q \rightarrow 0$. The coefficient estimates and (??) therefore prove (??), including uniformity through the entire interval. They also give the actual nonradial density bound

$$\|(I - P_{\text{rad}})\omega^\kappa(t)\|_1 \leq Ca(t) + Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)t}. \quad (24)$$

Indeed radial averaging contracts L^1 , and the heat base is radial. This estimate concerns the actual density; it is not inferred from a contour coordinate.

Similarly (??), the coefficient bounds and (??) imply

$$\int_0^H \|K * (\omega^\kappa - W_\kappa)\|_\infty dt \leq C\delta + C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \quad (25)$$

To integrate the approximate terms use $\int_0^H a(t)^j dt \leq \delta^j/(j\sigma)$. The error terms grow at a fixed positive exponential rate; their nondecreasing polynomial factor can be evaluated at H , so there is no additional factor H . This proves (??).

1.5 Whole-moment control and the cap

We verify the tails needed to remove the moment cutoff. Write $u^\kappa = U_\kappa + v^\kappa$, with U_κ tangential, and $B_H = \int_0^H \|v^\kappa\|_\infty dt$. By (??), $B_{H_q} = O(\delta_q)$. Let $d(x) = \sqrt{1+|x|^2}$. Then $|\nabla d| \leq 1$, $\Delta d \leq 2$, and $U_\kappa \cdot \nabla d = 0$. Kato's inequality with $e^{\lambda d}$ gives, for every $\lambda > 0$,

$$\int e^{\lambda d(x)} |\omega^\kappa(t, x)| dx \leq m \exp\{\lambda d_0 + \lambda B_H + 2\kappa\lambda t + \kappa\lambda^2 t\}, \quad (26)$$

where $m = \|\omega^\kappa(0)\|_1$ and $d_0 = \sqrt{1 + (1 + \epsilon)^2}$. No initial exponential tail is presupposed beyond compact support. To justify the unbounded test use

$$F_L(d) = \frac{e^{\lambda d}}{1 + e^{\lambda d}/L}.$$

It has $F'_L \leq \lambda F_L$ and $F''_L \leq \lambda^2 F_L$; therefore $\Delta F_L(d) \leq (\lambda^2 + 2\lambda)F_L(d)$. For fixed L add a compact cutoff and let its radius tend to infinity. Finite mass and bounded drift make the cutoff errors vanish. Gronwall then gives (??) uniformly in L , and monotone convergence removes L . The drift is bounded here by (??) and the scalar L^1, L^p contractions.

Choose a fixed $R > 1 + \epsilon$ with $d_R - d_0 > 2c_* > 0$. For small q , $B_{H_q} < c_*/2$ and $\kappa_q H_q \rightarrow 0$. Exponential Markov and optimization of (??) give

$$\int_{|x|>s} |\omega^\kappa(t, x)| dx \leq m \exp \left\{ -\frac{(d_s - d_0 - B_H - 2\kappa t)^2}{4\kappa t} \right\}, \quad s \geq R.$$

Integrating this mass bound in s gives

$$\sup_{0 \leq t \leq H_q} \int_{|x|>R} |x|^4 |\omega^{q, \vartheta}(t, x)| dx \leq C e^{-c/(\kappa_q H_q)}. \quad (27)$$

For example the weighted tail equals R^4 times the mass tail plus $4 \int_R^\infty s^3$ times the lower-radius mass tail. The displayed Gaussian estimate bounds both terms after reducing c . This also proves finiteness of the required absolute moments.

Let ξ be a smooth radial cutoff equal to one on B_R and supported in B_{R+1} . Every formal nonradial coefficient is supported in B_R ; the exact heat base pairs to zero with both $\bar{z}^4$ and $\xi \bar{z}^4$, since it is radial. At the cap the first coefficient has growing moment

$$\alpha e^{\lambda_\epsilon H} e^{i\vartheta} b_\epsilon = \delta e^{i(\vartheta + \tau_\epsilon H)} b_\epsilon$$

and stable moment of size $C\alpha^2/\delta$. The remaining pure-seed coefficients have compactly tested moment $O(\delta^2)$; positive-viscosity coefficients have moment $O(\kappa(1 + H)^D \delta)$. The actual error pairs against $\xi \bar{z}^4$ with bound $\|\xi \bar{z}^4\|_\infty \|\zeta(H)\|_1$. Using (??) and (??) proves

$$\begin{aligned} |B_4(\omega^\kappa(H)) - \delta e^{i(\vartheta + \tau_\epsilon H)} b_\epsilon| &\leq C\delta^2 + C\alpha^2/\delta + C\kappa(1 + H)^D \delta + C\delta^{M+1} \\ &\quad + C\kappa^{N+1}(1 + H)^{D+1} \delta(\delta/\alpha)^{r_1} + C e^{-c/(\kappa H)}. \end{aligned}$$

For $\delta = \delta_q$, every bound divided by δ_q tends to zero. This proves (??) and completes the theorem.

Remark 1.4 (Scaling and scope). For a physical colour $A\omega(x/R)$ and viscosity $\nu = \kappa_q AR^2$, the cap time is H_q/A , the density bound is $CAR^2\delta_q$, and the cap moment is $AR^6\delta_q e^{i(\vartheta + \tau_\epsilon H_q)} b_\epsilon + o(AR^6\delta_q)$. The integrated perturbation velocity bound is $CR\delta_q$. These normalized constants do not depend on the scale or amplitude. They apply to an isolated colour. A foreign-colour field introduces additional terms in the coefficient equations and the error equation; no estimate for those terms follows merely from this scaling. In particular, (??) cannot be asserted for an interacting stack by substituting its contour amplitudes into this isolated theorem.