

A fixed-data viscous cascade on the plane: isolated activation and conditional interaction estimates

Research continuation

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Abstract

We continue the investigation of one fixed zero-circulation datum in $L^1 \cap L^p(\mathbb{R}^2)$, $2 < p < \infty$, with more than one unforced Navier–Stokes vanishing-viscosity accumulation point. We prove a nonlinear isolated active-stage estimate for one fixed smooth radial annulus, with an exactly fourth-moment-neutral unstable seed. A finite expansion in the seed and viscosity preserves the seed factor in the error, and gives actual nonradial-density control and a nonzero whole fourth moment on a logarithmically long window. A higher Sobolev comparison identifies a canonical ancient isolated Euler profile and its nonzero bilateral moment transform and nonlinear transmission for a prescribed smooth realization of that canonical harmonic input. We prove persistence of constant cores under an explicit actual-density hypothesis, and a nonlinear full-window inner-tail bound under the resulting local harmonic field. An actual inactive-density row closes under an explicit packet-force input. A scalar split of seeded and forced prefix contributions and a protected first-cap phase functional give conditional tools for selecting distinct total Euler paths. Strong vanishing-viscosity precompactness for fixed compact zero-circulation $L^1 \cap L^p$ data is proved separately. These results do not prove a fixed-data nonunique vanishing-viscosity limit: the depth-uniform interacting active-stage theorem and its signed pulse and cap-protection estimates remain to be established. The conditions in each partial result are included in its statement.

1 Equation, candidate data, and logical scope

The equation is

$$\partial_t \omega^\nu + (K * \omega^\nu) \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu, \quad K(x) = \frac{x^\perp}{2\pi|x|^2}, \quad x^\perp = (-x_2, x_1). \quad (1)$$

Identify x with $z = x_1 + ix_2$, and write

$$B_4(f) = \int_{\mathbb{R}^2} \bar{z}^4 f(x) \, dx.$$

An entire moment is used only after its fourth absolute moment is proved finite. Fourfold symmetry is $f(ix) = f(x)$. The symbol P_{rad} denotes angular averaging; a local symbol P denotes the same operation. We use the standard global mild well-posedness and positive-time smoothing of (1) for integrable initial vorticity, as stated in [1, Theorem 2.2], after rescaling viscosity. The L^r scalar contractions follow directly from the convex energy inequalities. The Biot–Savart L^p gradient and HLS bounds used below are the usual singular-integral inequalities, also stated in [1, Lemma 2.1]. The new comparison, moment, core, and selection estimates are proved here; no periodic construction is used as an input.

Lemma 1.1 (Admissibility of a fixed infinite smooth stack). *Let W be a fixed real smooth compactly supported radial density with zero integral, and let Φ_θ be a bounded smooth family with common compact support, zero integral, and $B_4(\Phi_\theta) = 0$. Suppose Φ_θ is fourfold symmetric. Fix $\alpha > 0$, $A_j = A_1\beta^{j-1}$, $R_j = R_1q^{j-1}$, and $\beta q^{2/p} < 1$, where $2 < p < \infty$. For any phases θ_j , the single datum*

$$\omega_0(x) = \sum_{j \geq 1} A_j [W(x/R_j) + \alpha \Phi_{\theta_j}(x/R_j)]$$

converges absolutely in $L^1 \cap L^p$, has compact support and zero circulation, and satisfies $B_4(\omega_0) = 0$. Changing only phase θ_N changes its $L^1 \cap L^p$ norm by at most $C\alpha A_N(R_N^2 + R_N^{2/p})$, which tends to zero.

Proof. For either summand its L^r norm is at most $CA_j R_j^{2/r}$, uniformly in phase, for $r = 1, p$. The geometric ratio is less than one for both norms. All supports are in a common ball. Integral and phase claims follow by absolute convergence. The absolute fourth moment of each summand is bounded by $CA_j R_j^6$; this series converges since $\beta q^6 < 1$. Radial symmetry gives $B_4(W) = 0$, so termwise integration proves the whole-moment assertion. $\square$

The lemma verifies the datum class; it does not establish the desired behaviour of its solutions. For this construction q , α , the profile and all phases must be fixed before viscosity is sent to zero. In the isolated theorem below q varies to prove a quantitative estimate; that theorem by itself is not a fixed-data selection statement.

2 An isolated nonlinear active stage for a fixed smooth profile

The result of this section concerns one isolated annulus. It replaces the discontinuous reference vorticity by one fixed smooth profile, explicitly. The smoothing length is chosen once and is independent of viscosity and of the small parameter below. In particular, the theorem does not assert the analogous nonlinear estimate for a family of sharp patch boundaries.

Theorem 2.1 (Isolated smooth activation). *There exist a real, radial, compactly supported C^∞ vorticity W_ϵ , with zero circulation, a number $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with $\sigma_\epsilon > 0$, and a nonzero complex number b_ϵ , with the following property. For every $2 < p < \infty$ and $K_* > 0$ there is $q_0 > 0$ such that the conclusions below hold for $0 < q < q_0$. Put*

$$L_q = \log(1/q), \quad \alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad \delta_q = L_q^{-4}, \quad H_q = \frac{1}{\sigma_\epsilon} \log \frac{\delta_q}{\alpha_q}.$$

There is a 2π -periodic family of smooth real seeds Φ_ϑ , supported in two fixed annular collars, such that

$$\int \Phi_\vartheta = 0, \quad B_4(\Phi_\vartheta) = 0, \quad \Phi_\vartheta(ix) = \Phi_\vartheta(x).$$

Let $\omega^{q,\vartheta}$ be the unforced Navier–Stokes solution with viscosity κ_q and initial vorticity $W_\epsilon + \alpha_q \Phi_\vartheta$. Then, uniformly in $\vartheta \in \mathbb{R}$,

$$\sup_{0 \leq t \leq H_q} \|\omega^{q,\vartheta}(t) - e^{\kappa_q t \Delta} W_\epsilon\|_{L^1 \cap L^p} \leq C\delta_q, \tag{2}$$

$$\int_0^{H_q} \|K * (\omega^{q,\vartheta}(t) - e^{\kappa_q t \Delta} W_\epsilon)\|_\infty dt \leq C\delta_q, \tag{3}$$

$$B_4(\omega^{q,\vartheta}(H_q)) = \delta_q e^{i(\vartheta + \tau_\epsilon H_q)} b_\epsilon + o(\delta_q). \tag{4}$$

Here $\|f\|_{L^1 \cap L^p} = \|f\|_1 + \|f\|_p$; all fourth absolute moments in (4) are finite. In particular, (4) is a statement about the whole moment, rather than only a compactly tested moment. The constants depend on the fixed profile, p and K_* , but not on q or ϑ .

The positive cap δ_q will eventually be fixed by choosing one q . Its logarithmic decrease is useful for the simultaneous condition $H_q \sqrt{\delta_q} \rightarrow 0$. The theorem concerns a family of isolated initial data and does not claim nonuniqueness of any fixed-data vanishing-viscosity limit.

2.1 The fixed profile and its positive angular spectrum

Fix $\rho^2 = 3/5$ and choose a nonnegative, radial, radially decreasing $\chi \in C_c^\infty(B_1)$ with integral one. Define

$$\chi_\epsilon(x) = \epsilon^{-2} \chi(x/\epsilon), \quad W_\epsilon = \chi_\epsilon * (-\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}).$$

The convolution is smooth, radial, supported in $B_{1+\epsilon}$ and has zero integral. It is constant on $B_{\rho-\epsilon}$. Choose $\epsilon_* < (1 - \rho)/4$ and subsequently $0 < \epsilon < \epsilon_*$. Its radial derivative has the form

$$W'_\epsilon(r) = D_{\rho,\epsilon}(r) - \rho^2 D_{1,\epsilon}(r), \quad D_{a,\epsilon} \geq 0, \quad \text{supp } D_{a,\epsilon} \subset [a - \epsilon, a + \epsilon], \quad \int_0^\infty D_{a,\epsilon}(r) dr = 1. \quad (5)$$

To verify the sign, express the radial decreasing mollifier by its superlevel disks. The area of intersection of two disks is nonincreasing in their center separation: along each line parallel to their displacement, the overlap of the two interval sections is nonincreasing. Layer-cake integration gives radial monotonicity of the disk convolution. Its value is one at the origin and zero at infinity, giving the displayed derivative mass. The support assertion follows because the convolution is exactly constant at points whose distance to the disk boundary exceeds ϵ .

Set

$$\Omega_\epsilon(r) = r^{-2} \int_0^r s W_\epsilon(s) ds.$$

On the two derivative collars, respectively,

$$\Omega_\epsilon(r) = \Omega_0(r_i) + O(\epsilon), \quad (r_1, r_2) = (\rho, 1), \quad (\Omega_0(r_1), \Omega_0(r_2)) = (-1/5, 0). \quad (6)$$

Indeed disk translation changes its L^1 norm by at most a constant times the translation length. Thus $\|W_\epsilon - W_0\|_1 \leq C\epsilon$, where $W_0 = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}$. The radial circulation formula, on radii bounded away from zero, gives $|\Omega_\epsilon - \Omega_0| \leq C\epsilon$. The piecewise smooth Ω_0 is Lipschitz there. Direct integration gives the interface values.

For a positive angular mode $m \geq 1$ use the convention $f(r)e^{im\theta}$. Put

$$h_m(r, s) = \left(\frac{\min(r, s)}{\max(r, s)} \right)^m, \quad \psi_f(r) = -\frac{1}{2m} \int_0^\infty h_m(r, s) f(s) s ds.$$

The function $\psi_f e^{im\theta}$ solves $\Delta \psi = f e^{im\theta}$, is regular at the origin and decays at infinity. This follows either by solving the radial second-order equation on each side of $r = s$, or by checking the derivative jump $1/s$ of the displayed Green kernel. Since $u_r = -im\psi_f/r$, the frozen Euler linearization at W_ϵ is

$$\mathcal{L}_m f = -im\Omega_\epsilon f + im\psi_f W'_\epsilon/r. \quad (7)$$

For $m = 0$ the linearization is zero: two radial vorticities have tangential velocities and do not advect one another. Negative sectors are the complex conjugates of the positive sectors. From this point onward all angular sectors belong to $4\mathbb{Z}$. No spectral assertion is made for other symmetry classes.

Lemma 2.2 (The fixed smooth unstable pair). *One can choose a fixed sufficiently small $\epsilon > 0$ and $0 < \eta < \sigma_\epsilon/4$ so that in the quarter-turn symmetry class: the $m = 4$ generator has one algebraically simple eigenvalue $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with positive real part greater than η ; its growing radial eigenfunction f_ϵ is smooth and supported in the two derivative collars; and all other positive-real-part spectral values, except the conjugate growing eigenvalue in $m = -4$, have real part at most η . Moreover*

$$\lambda_\epsilon \longrightarrow \rho^5/2 + 3i/10 \quad \text{as } \epsilon \downarrow 0.$$

For each finite set of sectors in $4\mathbb{Z}$ and each finite radial Sobolev order k , the corresponding propagator has the decomposition

$$e^{t\mathcal{L}} = e^{\lambda_\epsilon t} P_+ + e^{\overline{\lambda}_\epsilon t} P_- + \mathcal{R}_k(t), \quad \|\mathcal{R}_k(t)\|_{H^k \rightarrow H^k} \leq C_k e^{2\eta t}. \quad (8)$$

The growing terms are included only if their sectors are present.

Proof. First consider any eigenvalue λ with $\Re \lambda > 0$. The eigenvalue equation forces $f = 0$ off the derivative collars and gives

$$f(r)r \, dr = g(r)W'_\epsilon(r) \, dr, \quad g(r) = \frac{im\psi_f(r)}{\lambda + im\Omega_\epsilon(r)}.$$

There is no division by W'_ϵ here. In the Hilbert space $L^2(|W'_\epsilon| \, dr)$ the equation for g is

$$\lambda g = -im\Omega_\epsilon g - \frac{i}{2} \int h_m(r, s) g(s) W'_\epsilon(s) \, ds.$$

Each same-collar block is skew-adjoint, since the sign of W'_ϵ is constant there and $h_m(r, s)$ is real and symmetric. Angular multiplication is also skew-adjoint. The two opposite-collar blocks have operator norms at most their Hilbert–Schmidt norms, bounded by

$$Ca_*^m, \quad a_* = (\rho + \epsilon_*)/(1 - \epsilon_*) < 1,$$

because their two weighted masses are 1 and ρ^2 and their kernels are bounded by a_*^m . Consequently

$$\Re \lambda \leq Ca_*^m. \quad (9)$$

This fixes a finite angular cutoff in any half-plane $\Re \lambda \geq \eta > 0$, independently of ϵ .

For one of the finitely many positive sectors, eliminate f to obtain the equation on the two collars

$$\psi(r) = -\frac{i}{2} \int h_m(r, s) \frac{W'_\epsilon(s)\psi(s)}{\lambda + im\Omega_\epsilon(s)} \, ds =: \mathcal{T}_\epsilon(\lambda)\psi(r). \quad (10)$$

Work first in the sup-norm space of continuous functions on the two collars. Let P evaluate at r_1, r_2 and let E extend a vector to constants on the two collars, so $PE = I$. The receiving-radius derivative of the kernel is bounded there for fixed m . Hence

$$\|(I - EP)\mathcal{T}_\epsilon(\lambda)\| \leq C\epsilon$$

uniformly on compact subsets of $\Re \lambda \geq \eta$. For $\psi = Ep + w$, $Pw = 0$, solve the w equation by an analytic Neumann series. The remaining two-dimensional equation has a determinant converging uniformly on these compact sets to the determinant of

$$p_i = \frac{i}{2} \sum_j \frac{c_j h_m(r_i, r_j)}{\lambda + im\Omega_0(r_j)} p_j, \quad (c_1, c_2) = (-1, \rho^2).$$

The sign follows from $W'_0 = -\sum c_j \delta_{r_j}$. Using interface variables $f(r)r dr = \sum c_i r_i^2 q_i \delta_{r_i}$, this equation is equivalent to the matrix eigenvalue problem

$$M_m q = \lambda q, \quad M_m = \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

For $m = 4$ its roots are $\pm \rho^5/2 + 3i/10$, both simple. For $m = 8, 12, \dots$ they are imaginary, since the real-matrix discriminant is

$$\left(\frac{2(m-4)}{5}\right)^2 - 4\rho^{2m+2} > 0.$$

At $m = 8$ the first term is $64/25$ and the second is $4(3/5)^9 < 1$; thereafter the first increases and the second decreases. Large $|\lambda|$ gives no omitted roots: once $|\lambda| > C_m$, the operator in (10) has norm less than one, uniformly in ϵ , because the denominator then has size comparable with $|\lambda|$. Uniform determinant convergence and Rouché's theorem therefore give one simple root near the growing $m = 4$ root and no other roots in $\Re \lambda \geq \eta$. Combined with (9), this proves the claimed positive spectrum after one fixed ϵ is selected. Take η smaller first if necessary, so that $2\eta < \sigma_\epsilon$.

We give the operator-domain and multiplicity details needed for the propagator claim. Choose a closed interval $I \Subset (0, \infty)$ containing both collars strictly in its interior. On $H^k(I)$ define ψ_f by the Green integral against the zero extension of f from I . This extension need not be globally H^k : multiplication of the potential by W'_ϵ restricts output strictly inside I . The radial elliptic equation, or direct differentiation of its Green kernel, gives two local derivatives. Thus the coupling in (7) is compact on $H^k(I)$ and the angular multiplier is bounded there. Its spectrum is imaginary: off the multiplier's value range the inverse is multiplication by $(\lambda + im\Omega_\epsilon)^{-1}$. The compact-operator Fredholm alternative consequently reduces all spectrum in $\Re \lambda > 0$ to eigenfunctions. The equation bootstraps such eigenfunctions to smoothness and forces their support into the collars, so the roots already found are exactly the positive spectrum on every $H^k(I)$.

To check algebraic simplicity independently of a sup-norm determinant, start on $X_0 = L^2(I)$. Write $Af = -im\Omega_\epsilon f$, $Cf = \psi_f|_{\text{collars}}$ and $B\psi = imW'_\epsilon \psi/r$. Then $\mathcal{L}_m = A + BC$ and $\mathcal{T}_\epsilon = C(\lambda - A)^{-1}B$. Off the imaginary multiplier range, the resolvent identity is

$$(\lambda - \mathcal{L}_m)^{-1} = (\lambda - A)^{-1} + (\lambda - A)^{-1}B(I - \mathcal{T}_\epsilon)^{-1}C(\lambda - A)^{-1}. \quad (11)$$

Here $C : X_0 \rightarrow C(\text{collars})$ is bounded by its elliptic gain, and $B : C(\text{collars}) \rightarrow X_0$ is bounded. The invertible complementary Schur factor and the simple two-by-two determinant zero imply a rank-one simple pole in (11). The residue is nonzero, since an eigenfunction exists. On $X_k = H^k(I)$ the resolvent equation and its radial elliptic bootstrap give, locally uniformly for λ off the imaginary multiplier range,

$$\|v\|_{H^k(I)} \leq C_k(\|f\|_{H^k(I)} + \|v\|_{L^2(I)}), \quad (\lambda - \mathcal{L}_m)v = f.$$

The X_k resolvent is the restriction of the X_0 resolvent; the simple pole has no higher-order singularity by this estimate and has the same nonzero smooth one-dimensional range. This proves algebraic simplicity on every space used below. The $m = -4$ conclusion follows by conjugation; $m = 0$ is the identity propagator.

After removing the growing Riesz projections, the remaining bounded generator has spectral bound at most η . Its exponential is bounded by $Ce^{2\eta t}$. For clarity this last fact does not assume a spectral growth theorem for unbounded generators. For a bounded operator, integrate $e^{tz}(z - \mathcal{L})^{-1}$

on a finite rectangular contour enclosing its remaining spectrum and having right edge $\Re z = 2\eta$. Expansion of the resolvent on a large circle and termwise integration gives the exponential series; contour deformation gives the rectangle formula. The resolvent is bounded on this fixed contour, proving the estimate. The finitely many sectors and derivative orders give (8). Coefficients initially smooth and supported in the collars remain so under this propagator: multiplication preserves that support and the coupling has smooth output flat at the collar endpoints. This proves the lemma. $\square$

2.2 An exactly neutral smooth seed

Normalize the growing radial eigenfunction f_ϵ so that its first collar mass tends to $c_1 r_1^2 = -\rho^2$. The finite-dimensional reduction then implies

$$f_\epsilon(r)r \, dr \rightarrow -\rho^2 \delta_\rho - i\rho^3 \delta_1.$$

Indeed its interface vector tends to $(1, -i\rho)$, the growing eigenvector of M_4 . Define

$$b_\epsilon = 2\pi \int_0^\infty r^5 f_\epsilon(r) \, dr. \quad (12)$$

The weak convergence on the two compact collars gives

$$b_\epsilon \rightarrow 2\pi(-\rho^6 - i\rho^3) \neq 0.$$

Reduce the fixed ϵ if necessary. Since $\mathcal{L}_4$ is i times a real operator, $\overline{f_\epsilon}$ is an eigenfunction of $\mathcal{L}_4$ with eigenvalue $-\overline{\lambda_\epsilon} = -\sigma_\epsilon + i\tau_\epsilon$. Define the real seed

$$\Phi_\vartheta(r, \theta) = e^{i\vartheta} \left(f_\epsilon(r) - \frac{b_\epsilon}{\overline{b_\epsilon}} \overline{f_\epsilon(r)} \right) e^{4i\theta} + \text{complex conjugate}. \quad (13)$$

Its circulation is zero and its only initial sectors are ± 4 . For the convention $B_4 = \int \bar{z}^4 \omega$, angular integration gives $B_4(f(r)e^{4i\theta}) = 2\pi \int r^5 f(r) \, dr$. Therefore $B_4(\Phi_\vartheta) = 0$ exactly, while the growing coefficient is $e^{i\vartheta}$. The stable coefficient has modulus one.

2.3 The finite seed–viscosity expansion

For the rest of the proof abbreviate the fixed profile to W , and write $\sigma = \sigma_\epsilon$, $\tau = \tau_\epsilon$, $W_\kappa(t) = e^{\kappa t \Delta} W$ and $U_\kappa = K * W_\kappa$. The latter is an exact radial heat solution. Put $X = L^1 \cap L^p$ with the sum norm. Splitting the Biot–Savart convolution at radius one gives

$$\|K * g\|_\infty \leq C_p \|g\|_X. \quad (14)$$

On the near part use Hölder and $|x|^{-1} \in L^{p'}(B_1)$, which holds because $p' < 2$. On the far part use its bounded kernel and the L^1 norm. This estimate requires no zero-circulation assumption.

The frozen full linear operator is $\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W$. Its sectors are those in (7). Choose integers $M, N \geq 1$, to be fixed below, and construct

$$f_{\text{app}} = \sum_{j=1}^M \sum_{\ell=0}^N \alpha^j \kappa^\ell f_{j,\ell}, \quad \omega_{\text{app}} = W_\kappa + f_{\text{app}}. \quad (15)$$

Set $f_{1,0}(0) = \Phi_\vartheta$ and all other initial coefficients zero. With $W_r(t) = t^r \Delta^r W / r!$, solve the triangular recursion

$$\begin{aligned} (\partial_t - \mathcal{L})f_{j,\ell} = & \mathbf{1}_{\ell \geq 1} \Delta f_{j,\ell-1} - \sum_{r=1}^{\ell} [(K * W_r) \cdot \nabla f_{j,\ell-r} + (K * f_{j,\ell-r}) \cdot \nabla W_r] \\ & - \sum_{\substack{j_1+j_2=j, j_1, j_2 \geq 1 \\ \ell_1+\ell_2=\ell}} (K * f_{j_1,\ell_1}) \cdot \nabla f_{j_2,\ell_2}. \end{aligned} \quad (16)$$

Every right-hand side uses earlier coefficients in the order $j + \ell$. They are smooth and supported in the fixed collars: Laplacians preserve the support of smooth flat functions; $\Delta^r W$, $r \geq 1$, has that support; and every advective factor differentiates such a supported function. The linear source is supported where W' is supported. Each coefficient has zero circulation, because each forcing is a Laplacian or divergence of an integrable vector field. Its frequencies belong to $4\mathbb{Z}$ and satisfy $|m| \leq 4j$.

Lemma 2.3 (Coefficient bounds and seeded residual). *For every finite derivative order k , the finitely many coefficients in (16) satisfy*

$$\|f_{j,0}(t)\|_{H^k} \leq C_{j,k} e^{j\sigma t}, \quad (17)$$

$$\|f_{j,\ell}(t)\|_{H^k} \leq C_{j,\ell,k} (1+t)^D e^{j\sigma t}, \quad \ell > 0. \quad (18)$$

There is a finite D , enlarged as needed, such that if $a(t) = \alpha e^{\sigma t} \leq \delta_0 \leq 1$ on $[0, H]$ and $\kappa(1+H)^D \leq c$, the residual

$$R_{\text{app}} = \partial_t \omega_{\text{app}} + (K * \omega_{\text{app}}) \cdot \nabla \omega_{\text{app}} - \kappa \Delta \omega_{\text{app}}$$

satisfies

$$\|R_{\text{app}}(t)\|_X \leq C[a(t)^{M+1} + \kappa^{N+1}(1+t)^D a(t)]. \quad (19)$$

All constants are uniform in ϑ .

Proof. The first coefficient is the exact growing-plus-decaying solution from (13). For $j \geq 2$, $\ell = 0$, induction and Sobolev multiplication bound its forcing by $Ce^{j\sigma t}$. Convolution against the growing exponent σ , or the remaining exponent $2\eta < \sigma$, is bounded by a constant times $e^{j\sigma t}$, because $j\sigma - \sigma \geq \sigma$. This proves (17) with no polynomial factor. For positive ℓ the same induction applies, with finite polynomial losses from $W_r(t)$ and possible resonance with the first growing mode. This proves (18). Derivative loss causes no infinite iteration: the coefficient set is finite and the profile is smooth in every order. At each induction step apply the preceding coefficient estimates in a sufficiently higher fixed order and the propagator lemma in the requested order. Compact support and zero circulation also control $K * f$ and its needed derivatives, including low frequencies. More explicitly $|\hat{f}(\xi)| \leq C|\xi|$ near zero, from its zero mass and finite first absolute moment, while the high-frequency multiplier has order minus one.

Expand the radial heat base only in its couplings to f_{app} . The integral Taylor formula and heat contraction in L^1, L^p give

$$G_N(t) := W_\kappa(t) - \sum_{r=0}^N \kappa^r W_r(t), \quad \|G_N(t)\|_X + \|\nabla G_N(t)\|_X \leq C\kappa^{N+1}t^{N+1}.$$

Its residual couplings obey

$$\|(K * G_N) \cdot \nabla f_{\text{app}} + (K * f_{\text{app}}) \cdot \nabla G_N\|_X \leq C\kappa^{N+1}(1+t)^D a(t)$$

by (14) and the coefficient bounds. These are the only noncompact factors. Every other coefficient is supported in one fixed bounded set, where its high Sobolev norms control the required L^1, L^p product norms. The recursion cancels exactly all monomials with $1 \leq j \leq M$, $0 \leq \ell \leq N$. Terms of seed order $j > M$ are bounded by Ca^{M+1} : the pure-seed bounds contain no time polynomial, and each positive-viscosity polynomial is absorbed after enlarging D and using $\kappa(1+H)^D \leq c$. Terms of viscosity order $\ell > N$, including the diffusion of $f_{j,N}$, have the second bound in (19); $a^j \leq a$ for $a \leq 1$ absorbs their higher seed powers. Crucially the exact radial heat base is part of (15), so no residual term without a seed factor occurs. Finally rotations of (13) change only unit complex phases; all bounds are uniform in ϑ . $\square$

2.4 Nonlinear closure in the actual vorticity norm

Let ω^κ be the NS solution from $W + \alpha\Phi_\vartheta$ and put $\zeta = \omega^\kappa - \omega_{\text{app}}$. Its exact equation, with zero initial data, is

$$\partial_t \zeta + (K * \omega^\kappa) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta) \cdot \nabla \omega_{\text{app}} - R_{\text{app}}. \quad (20)$$

Kato's inequality gives its L^1 estimate, and multiplication by $|\zeta|^{p-2}\zeta$ gives its L^p estimate; transport disappears by divergence freedom and diffusion is nonpositive in both inequalities. Using (14) and adding gives

$$\frac{d^+}{dt} \|\zeta\|_X \leq C_p \|\nabla \omega_{\text{app}}\|_X \|\zeta\|_X + \|R_{\text{app}}\|_X. \quad (21)$$

These inequalities follow first with smooth convex regularizations of the norm and compact cutoffs, then by their limits. All solutions here are smooth at positive time, have finite X norm, and are continuous in that norm at time zero, so the integrated inequality starts at zero. No derivative bound on the actual NS vorticity enters its coefficient.

We choose parameters without circular dependence. Fix

$$C_1 = C_p(\|\nabla W\|_X + 1), \quad r_1 = (C_1/\sigma - 1)_+,$$

before choosing expansion orders. Choose M with $(M+1)\sigma > C_1$ and N with

$$\frac{3}{2}(N+1) > K_* r_1. \quad (22)$$

Construct the finite coefficients, then fix a cap threshold δ_0 using the pure-seed coefficients alone and reduce the positive-viscosity threshold so that $\|\nabla f_{\text{app}}\|_X \leq 1$ whenever $a \leq \delta_0$ and $\kappa(1+H)^D$ is sufficiently small. This is possible by the finite coefficient bounds: make the pure-seed gradient at most $1/2$ by the cap choice, and the finitely many positive-viscosity contributions at most $1/2$ by the later viscosity threshold. Thus a later increase of N need not reduce a previously fixed pure-seed cap. The exact heat gradient obeys $\|\nabla W_\kappa\|_X \leq \|\nabla W\|_X$, so C_1 is the already fixed exponent in (21). The coefficient constants affect only the later cap threshold; they do not change the previous choice of M, N .

Convolving (19) against $e^{C_1(t-s)}$ gives the all-time estimate

$$\|\zeta(t)\|_X \leq Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)t}. \quad (23)$$

The first integral has denominator $(M+1)\sigma - C_1 > 0$. For the second use a bound by $e^{\max(C_1, \sigma)t}$ and an additional polynomial factor if the two exponents coincide. At $H = \sigma^{-1} \log(\delta/\alpha)$ this yields

$$\|\zeta(H)\|_X \leq C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \quad (24)$$

For the theorem's parameters, (22) implies that the second bound is $o(\delta_q)$: a strict positive power of q beats every fixed power of $\log(1/q)$. Also $\kappa_q(1 + H_q)^D \rightarrow 0$, $\delta_q \rightarrow 0$, and $\alpha_q/\delta_q \rightarrow 0$. The coefficient estimates and (23) therefore prove (2), including uniformity through the entire interval. They also give the actual nonradial density bound

$$\|(I - P_{\text{rad}})\omega^\kappa(t)\|_1 \leq Ca(t) + Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)t}. \quad (25)$$

Indeed radial averaging contracts L^1 , and the heat base is radial. This estimate concerns the actual density; it is not inferred from a contour coordinate.

Similarly (14), the coefficient bounds and (23) imply

$$\int_0^H \|K * (\omega^\kappa - W_\kappa)\|_\infty dt \leq C\delta + C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \quad (26)$$

To integrate the approximate terms use $\int_0^H a(t)^j dt \leq \delta^j/(j\sigma)$. The error terms grow at a fixed positive exponential rate; their nondecreasing polynomial factor can be evaluated at H , so there is no additional factor H . This proves (3).

2.5 Whole-moment control and the cap

We verify the tails needed to remove the moment cutoff. Write $u^\kappa = U_\kappa + v^\kappa$, with U_κ tangential, and $B_H = \int_0^H \|v^\kappa\|_\infty dt$. By (26), $B_{H_q} = O(\delta_q)$. Let $d(x) = \sqrt{1 + |x|^2}$. Then $|\nabla d| \leq 1$, $\Delta d \leq 2$, and $U_\kappa \cdot \nabla d = 0$. Kato's inequality with $e^{\lambda d}$ gives, for every $\lambda > 0$,

$$\int e^{\lambda d(x)} |\omega^\kappa(t, x)| dx \leq m \exp\{\lambda d_0 + \lambda B_H + 2\kappa\lambda t + \kappa\lambda^2 t\}, \quad (27)$$

where $m = \|\omega^\kappa(0)\|_1$ and $d_0 = \sqrt{1 + (1 + \epsilon)^2}$. No initial exponential tail is presupposed beyond compact support. To justify the unbounded test use

$$F_L(d) = \frac{e^{\lambda d}}{1 + e^{\lambda d}/L}.$$

It has $F'_L \leq \lambda F_L$ and $F''_L \leq \lambda^2 F_L$; therefore $\Delta F_L(d) \leq (\lambda^2 + 2\lambda)F_L(d)$. For fixed L add a compact cutoff and let its radius tend to infinity. Finite mass and bounded drift make the cutoff errors vanish. Gronwall then gives (27) uniformly in L , and monotone convergence removes L . The drift is bounded here by (14) and the scalar L^1, L^p contractions.

Choose a fixed $R > 1 + \epsilon$ with $d_R - d_0 > 2c_* > 0$. For small q , $B_{H_q} < c_*/2$ and $\kappa_q H_q \rightarrow 0$. Exponential Markov and optimization of (27) give

$$\int_{|x|>s} |\omega^\kappa(t, x)| dx \leq m \exp\left\{-\frac{(d_s - d_0 - B_H - 2\kappa t)^2}{4\kappa t}\right\}, \quad s \geq R.$$

Integrating this mass bound in s gives

$$\sup_{0 \leq t \leq H_q} \int_{|x|>R} |x|^4 |\omega^{q,\vartheta}(t, x)| dx \leq C e^{-c/(\kappa_q H_q)}. \quad (28)$$

For example the weighted tail equals R^4 times the mass tail plus $4 \int_R^\infty s^3$ times the lower-radius mass tail. The displayed Gaussian estimate bounds both terms after reducing c . This also proves finiteness of the required absolute moments.

Let ξ be a smooth radial cutoff equal to one on B_R and supported in B_{R+1} . Every formal nonradial coefficient is supported in B_R ; the exact heat base pairs to zero with both $\bar{z}^4$ and $\xi\bar{z}^4$, since it is radial. At the cap the first coefficient has growing moment

$$\alpha e^{\lambda_\epsilon H} e^{i\vartheta} b_\epsilon = \delta e^{i(\vartheta+\tau_\epsilon H)} b_\epsilon$$

and stable moment of size $C\alpha^2/\delta$. The remaining pure-seed coefficients have compactly tested moment $O(\delta^2)$; positive-viscosity coefficients have moment $O(\kappa(1+H)^D\delta)$. The actual error pairs against $\xi\bar{z}^4$ with bound $\|\xi\bar{z}^4\|_\infty\|\zeta(H)\|_1$. Using (24) and (28) proves

$$\begin{aligned} |B_4(\omega^\kappa(H)) - \delta e^{i(\vartheta+\tau_\epsilon H)} b_\epsilon| &\leq C\delta^2 + C\alpha^2/\delta + C\kappa(1+H)^D\delta + C\delta^{M+1} \\ &\quad + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1} + Ce^{-c/(\kappa H)}. \end{aligned}$$

For $\delta = \delta_q$, every bound divided by δ_q tends to zero. This proves (4) and completes the theorem.

Remark 2.4 (Scaling and scope). For a physical colour $A\omega(x/R)$ and viscosity $\nu = \kappa_q AR^2$, the cap time is H_q/A , the density bound is $CA R^2\delta_q$, and the cap moment is $AR^6\delta_q e^{i(\vartheta+\tau_\epsilon H_q)} b_\epsilon + o(AR^6\delta_q)$. The integrated perturbation velocity bound is $CR\delta_q$. These normalized constants do not depend on the scale or amplitude. They apply to an isolated colour. A foreign-colour field introduces additional terms in the coefficient equations and the error equation; no estimate for those terms follows merely from this scaling. In particular, (25) cannot be asserted for an interacting stack by substituting its contour amplitudes into this isolated theorem.

3 The canonical ancient profile of the isolated smooth activation

We retain the fixed profile, eigenvalue and seeds constructed in Section 2. Abbreviate them to $W = W_\epsilon$, $\lambda = \sigma + i\tau = \lambda_\epsilon$, $b = b_\epsilon \neq 0$, and Φ_ϑ . In this section the cap $\delta > 0$ is fixed and sufficiently small. It does not decrease with the vanishing parameter. An additional Sobolev estimate permits identification of a unique isolated ancient profile. This identification requires no convergence of an infinite formal series. Here δ specifies the ancient growing amplitude and the linear preparation level. It does not assert that $s = 0$ is the exact first hitting time of $|\mathcal{A}(\Omega_\delta)| = \delta$; at that time $\mathcal{A}(\Omega_\delta(0)) = \delta + O(\delta^2)$.

Theorem 3.1 (The isolated ancient limit). *Fix an integer $k \geq 3$ and $2 < p < \infty$. There exists $\delta_* > 0$ such that, for $0 < \delta < \delta_*$, there is a real quarter-turn invariant Euler solution $\Omega_\delta(s)$ on $-\infty < s < \infty$, with zero circulation, having the following properties. For every fixed $K_* > 0$, put*

$$\alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad H_q = \sigma^{-1} \log(\delta/\alpha_q), \quad \vartheta_q = -\tau H_q.$$

Let ω^q be the isolated Navier–Stokes solution with viscosity κ_q and initial vorticity $W + \alpha_q \Phi_{\vartheta_q}$. For every $J < \infty$ and every $0 < \varepsilon < k$,

$$\omega^q(H_q + \cdot) \longrightarrow \Omega_\delta \quad \text{in } C([-J, 0]; L^1 \cap L^p) \text{ and in } C([-J, 0]; H^{k-\varepsilon}) \quad \text{as } q \downarrow 0. \quad (29)$$

The parameter δ_ and the limiting profile can be chosen independently of K_* ; the threshold for small q may depend on K_* . On $s \leq 0$, Ω_δ is supported in one fixed ball and has a uniform H^k bound. Its whole fourth harmonic moment satisfies*

$$B_4(\Omega_\delta(s)) = \delta b e^{\lambda s} + O(\delta^2 e^{2\sigma s}) \quad (s \leq 0). \quad (30)$$

For $s \geq 0$ it satisfies

$$|B_4(\Omega_\delta(s))| \leq C_\delta (1+s)^4. \quad (31)$$

These assertions concern the isolated annulus.

3.1 A high Sobolev error estimate without derivative loss

Fix once and for all a radius $R > 1 + \epsilon$ with a positive gap in $d(x) = \sqrt{1 + |x|^2}$, for example $R = 3 + \epsilon$. This radius and the corresponding compact-support norm constants are fixed before the energy exponent or expansion orders. The later cap smallness condition will make the integrated perturbation drift smaller than this fixed gap. For this proof let

$$\|g\|_{Z_k} = \|g\|_1 + \|g\|_{H^k}.$$

The finite expansion from (15) and (16) can be constructed in every finite derivative order. In particular, the residual estimate holds in Z_k . A useful version separates its pure-seed constant from constants depending on the viscosity order:

$$\|R_{\text{app}}(t)\|_{Z_k} \leq [C_M + C_{M,N}\kappa(1+t)^D]a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^D a(t), \quad a(t) = \alpha e^{\sigma t} \leq 1. \quad (32)$$

Here C_M is independent of N . To verify the additional norm and this separation, the pure-seed products are estimated by their compactly supported coefficient Sobolev bounds. Every other omitted seed-order product contains at least one positive power of κ , and its finite time polynomial is incorporated in D . The noncompact heat remainder products are

$$(K * G_N) \cdot \nabla f_{\text{app}} + (K * f_{\text{app}}) \cdot \nabla G_N.$$

The heat Taylor formula gives

$$\|K * G_N\|_{W^{k,\infty}} + \|\nabla G_N\|_1 + \|\nabla G_N\|_{H^k} \leq C_{k,N}\kappa^{N+1}t^{N+1}.$$

For the first term use heat Taylor on the smooth initial velocity $K * W$ and contraction of each bounded derivative. For the other terms use heat contraction on $\nabla \Delta^{N+1}W$ in L^1 and H^k . The coefficient bounds give $\|K * f_{\text{app}}\|_{W^{k,\infty}} + \|\nabla f_{\text{app}}\|_1 + \|\nabla f_{\text{app}}\|_{H^k} \leq Ca(t)$ after the positive-viscosity threshold is made small. Leibniz's rule proves the claimed seeded remainder in both norms.

Let $\zeta = \omega^\kappa - \omega_{\text{app}}$ and $u_{\text{app}} = K * \omega_{\text{app}}$. Rewriting (20) gives

$$\partial_t \zeta + u_{\text{app}} \cdot \nabla \zeta + (K * \zeta) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta) \cdot \nabla \omega_{\text{app}} - R_{\text{app}}. \quad (33)$$

We record the estimates that avoid a quasilinear derivative loss. Sobolev embedding and the near-far kernel split give, for $k \geq 3$,

$$\|K * g\|_\infty \leq C_k(\|g\|_1 + \|g\|_{H^k}), \quad (34)$$

$$\|\nabla K * g\|_{H^j} \leq C_j \|g\|_{H^j}, \quad j \geq 0. \quad (35)$$

The second is Plancherel's theorem for a bounded order-zero Fourier multiplier. In particular $\|\nabla K * g\|_\infty \leq C_k \|g\|_{H^k}$.

For each integer derivative order up to k , the undifferentiated transport term has zero energy pairing. The commutator with $(K * \zeta) \cdot \nabla$ is bounded by

$$C_k(\|\nabla K * \zeta\|_\infty \|\zeta\|_{H^k} + \|\nabla \zeta\|_\infty \|\nabla K * \zeta\|_{H^{k-1}}) \leq C_k \|\zeta\|_{H^k}^2.$$

This commutator inequality follows from Leibniz's rule: the endpoint terms put the first derivative in L^∞ and the last derivative in L^2 , and the intermediate terms follow by Sobolev interpolation

in dimension two. The embeddings for both displayed endpoint terms hold because $k > 2$. No derivative beyond order k of ζ occurs. Similarly,

$$\|(K * \zeta) \cdot \nabla \omega_{\text{app}}\|_{H^k} \leq C \left(\|\nabla \omega_{\text{app}}\|_{H^k} + \sum_{j=1}^k \|\nabla^j \omega_{\text{app}}\|_{\infty} \right) \|\zeta\|_{Z_k}.$$

For the term where no derivative lands on $K * \zeta$, use (34); for every term with a positive number of such derivatives use (35) in L^2 and put the corresponding derivative of the smooth approximation in L^∞ . The H^k transport estimate with u_{app} uses only its fixed $W^{k,\infty}$ bounds. Its coefficients are bounded, independently of α, κ, H , by fixed norms of W plus one, once the approximate perturbation is made small in the finitely many required norms.

Together with Kato's L^1 inequality this gives, for $Y = \|\zeta\|_{Z_k}$,

$$\frac{d^+ Y}{dt} \leq C_k Y + C_k Y^2 + \|R_{\text{app}}\|_{Z_k}, \quad Y(0) = 0. \quad (36)$$

Diffusion has a nonpositive energy contribution. The constant C_k can be fixed before M, N : the pure-seed approximation is made small by choosing its cap after M , and each positive-viscosity coefficient is made small by choosing its viscosity threshold after N .

Set $E_k = 2C_k$, increasing it if necessary to dominate $C_p(\|\nabla W\|_{L^1 \cap L^p} + 1)$ and the compact-support $L^1 \cap L^p$ stability constant used below. Choose one finite M with

$$(M+1)\sigma > E_k. \quad (37)$$

After its pure-seed coefficients are known, choose $\delta_* > 0$ small enough for all their required norms and their pure-seed remainder. For each subsequently prescribed K_* choose N so that, with $r_k = (E_k/\sigma - 1)_+$,

$$\frac{3}{2}(N+1) > K_* r_k. \quad (38)$$

The N -dependent coefficients are controlled by decreasing the threshold for q , not by changing δ_* . In particular arrange $C_{M,N}\kappa(1+H)^D \leq 1$ in (32). On a bootstrap interval $Y \leq 1$, (36) now gives

$$Y(t) \leq C_M a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^{D+1} a(t) e^{(E_k - \sigma)t}. \quad (39)$$

At the cap, the second term is

$$C_{M,N}\kappa^{N+1}(1+H)^{D+1} \delta(\delta/\alpha)^{r_k} = o(1)$$

in the prescribed polynomial window. The first is smaller than $1/2$ after reducing δ_* . Continuity contradicts a first bootstrap exit, so the estimate holds through the full cap interval. It also yields a uniform H^k bound for the actual vorticity. This bound is obtained from the actual error equation, rather than assumed from the formal approximation.

3.2 Finite homological coefficients and compactness

For a real angle θ define the rotation

$$(\mathcal{T}_\theta f)(r, \varphi) = f(r, \varphi + \theta/4), \quad \mathcal{L}_{\text{rot}} = \mathcal{L} - (\tau/4)\partial_\varphi.$$

It acts isometrically on every radial or full-plane Sobolev norm, commutes with the radial linearization, and respects Biot–Savart advection. The pure-seed coefficient $f_{j,0}$ with seed phase zero has a limit

$$e^{-j\sigma t} \mathcal{T}_{-\tau t} f_{j,0}(t) \longrightarrow F_j \quad \text{in every finite } H^m \text{ as } t \rightarrow \infty. \quad (40)$$

For $j = 1$ the decaying part is $O(e^{-2\sigma t})$ after this normalization, so

$$F_1 = f_\epsilon(r)e^{4i\varphi} + \text{complex conjugate}.$$

For $j \geq 2$, put $g_j = e^{-j\sigma t}\mathcal{T}_{-\tau t}f_{j,0}$. Its equation is

$$\partial_t g_j = (\mathcal{L}_{\text{rot}} - j\sigma)g_j - \sum_{j_1+j_2=j} (K * g_{j_1}) \cdot \nabla g_{j_2}.$$

On its finite set of sectors, the homogeneous propagator decays at least as $Ce^{-(j-1)\sigma t}$ by (8). Induction, the coefficient bounds, and dominated convergence in Duhamel's integral prove (40) and the finite homological relations

$$F_j = -(j\sigma - \mathcal{L}_{\text{rot}})^{-1} \sum_{j_1+j_2=j} (K * F_{j_1}) \cdot \nabla F_{j_2}, \quad 2 \leq j \leq M. \quad (41)$$

The inverse is the integral of the exponentially decaying propagator. All these functions are smooth and supported in the fixed collars. Only the finitely many sectors in $4\mathbb{Z}$ with $|m| \leq 4M$ are used. No all-mode Sobolev spectral-growth assertion is needed.

Define the fixed polynomial ancient profile

$$P_M(s) = W + \sum_{j=1}^M \delta^j e^{j\sigma s} \mathcal{T}_{\tau s} F_j, \quad s \leq 0. \quad (42)$$

For the phase $\vartheta_q = -\tau H_q$, (40) shows that every pure-seed term of the cap-shifted approximation converges to the corresponding term in (42). The positive-viscosity terms tend to zero because $\kappa_q(1+H_q)^D \rightarrow 0$, while $e^{\kappa_q(H_q+s)\Delta} W \rightarrow W$. For each fixed $s \leq 0$, (39) gives the asymptotic error $C_M(\delta e^{\sigma s})^{M+1}$, since its viscosity remainder tends to zero by (38).

We justify actual compactness separately. The kernel bound and (39), integrated as in (26), give

$$\int_0^{H_q} \|K * (\omega^q(t) - W_{\kappa_q}(t))\|_\infty dt \leq C_M \delta + o(1).$$

Choose δ_* so this is less than the positive radial gap fixed before the expansion orders. The exponential Kato proof of (28), with this fixed cap and the already fixed radius R , gives

$$\sup_{0 \leq t \leq H_q} \int_{|x| > R} |x|^4 |\omega^q(t, x)| dx \leq C e^{-c/(\kappa_q H_q)}. \quad (43)$$

Initial velocities have uniformly bounded finite energy, since the initial vorticities are compactly supported and have zero circulation. The NS velocity energy inequality and the uniform vorticity H^k bound therefore bound the velocities in H^{k+1} . The equation bounds $\partial_t \omega^q$ uniformly in H^{k-2} on the cap interval.

On each shifted interval $[-J, 0]$, Rellich compactness and interpolation give local compactness in $C(H^{k-\varepsilon})$. The tail bound and the uniform H^k , hence L^∞ , bound give global L^2 tightness. Interpolation applied to a smooth exterior cutoff then gives global $H^{k-\varepsilon}$ tightness; the cutoff is chosen outside the same fixed B_R . Thus compactness holds globally in that norm. The mass tail gives global L^1 tightness, and local convergence plus uniform L^∞ bounds gives compactness in $C(L^1 \cap L^p)$ for every finite p . A diagonal subsequence over J gives an ancient Euler solution: the transport term passes by the stated strong convergence, and $\kappa_q \Delta \omega^q \rightarrow 0$ on every compact

shifted interval. The solution is supported in B_R on $s \leq 0$, since the right side of (43) tends to zero.

By lower semicontinuity of the H^k norm and the strong L^1 convergence, every such limit satisfies, for all $s \leq 0$,

$$\|\Omega(s) - P_M(s)\|_{Z_k} \leq C_M(\delta e^{\sigma s})^{M+1}. \quad (44)$$

One may first take rational times; continuity in the weaker compactness norm and weak H^k continuity extend the bound to every time.

3.3 Uniqueness of the ancient subsequential limit

We use a norm in which no extra derivative of a comparison orbit is required. If Ω_1, Ω_2 are two ancient subsequential limits, their compact supports and uniform H^k bounds imply a uniform bound for $\|\nabla \Omega_j\|_{L^1 \cap L^p}$. Indeed compact support controls the L^1 norm by the L^2 derivative norm, and H^k , $k \geq 3$, controls every finite L^p derivative norm. Their perturbations from W are small in these norms after reducing δ_* , so the stability coefficient is bounded by the fixed constant dominated by E_k above.

For $D = \Omega_1 - \Omega_2$ the exact Euler equation is

$$\partial_s D + (K * \Omega_1) \cdot \nabla D = -(K * D) \cdot \nabla \Omega_2.$$

The same Kato and L^p argument as (21) gives, for $s_0 \leq s_1 \leq 0$,

$$\|D(s_1)\|_{L^1 \cap L^p} \leq e^{E_k(s_1 - s_0)} \|D(s_0)\|_{L^1 \cap L^p}.$$

Both limits have the same polynomial (42). Using (44) at $s_0 = -T$ yields

$$\|D(0)\|_{L^1 \cap L^p} \leq C_M \delta^{M+1} e^{-(M+1)\sigma - E_k T} \longrightarrow 0.$$

The same comparison with any fixed endpoint $s_1 \leq 0$ proves equality throughout the ancient interval. The strict inequality (37) uses one finite M only. Thus every subsequence has the same limit, proving the full convergence in (29). Repeating the argument for any different K_* gives the same limit; only its permissible viscosity expansion order and small- q threshold change.

For clarity, a direct H^k difference stability estimate would require an H^{k+1} bound on the comparison vorticity because it differentiates $(K * D) \cdot \nabla \Omega_2$ k times. The $L^1 \cap L^p$ comparison above deliberately avoids this extra hypothesis. The high Sobolev estimate is used for compactness and for the available first derivatives.

3.4 Forward extension and the whole-moment asymptotics

The ancient limit has a smooth enough, compactly supported initial state at zero. The classical smooth two-dimensional Euler construction extends it uniquely to all $s \geq 0$. We recall the bounds behind global continuation. Its transported vorticity has conserved L^1 and L^∞ norms. The near-far Biot-Savart split gives a time-independent velocity bound. Splitting the gradient kernel at a small radius and using its cancellation gives the logarithmic estimate

$$\|\nabla u\|_\infty \leq C_{\|\omega\|_1, \|\omega\|_\infty} [1 + \log(2 + \|u\|_{H^{k+1}})].$$

The near part uses the Hölder norm of ω supplied by H^k ; the intermediate annuli give the logarithm, and the far part uses L^1 . Choosing the small radius inversely proportional to a fixed

positive power of $2 + \|u\|_{H^{k+1}}$ gives the displayed estimate. The velocity Sobolev energy inequality $\frac{d}{ds} \|u\|_{H^{k+1}} \leq C \|\nabla u\|_\infty \|u\|_{H^{k+1}}$ then prevents finite-time blowup. The local smooth solution is obtained by regularization and these same energy bounds; their difference energy bound gives uniqueness. Finite energy at zero follows from compact support and zero circulation, so these velocity norms are legitimate.

The bounded velocity transports its support into B_{R+Vs} . Consequently

$$\int |x|^4 |\Omega_\delta(s, x)| dx \leq \|\Omega_\delta(0)\|_1 (R + Vs)^4 \quad (s \geq 0),$$

which proves (31). On $s \leq 0$, its support is in the fixed B_R . Pairing (44) with $\bar{z}^4$ is therefore bounded by its L^1 error times R^4 . The first homological coefficient has moment b , rotation multiplies it by $e^{i\tau s}$, and every higher term is $O(\delta^2 e^{2\sigma s})$. This proves (30) and completes the theorem.

Corollary 3.2 (Every fixed source window, including after the cap). *For the isolated family in Theorem 3.1, every fixed $J < \infty$ satisfies*

$$\begin{aligned} \omega^q(H_q + \cdot) &\longrightarrow \Omega_\delta \quad \text{in } C([-J, J]; L^1 \cap L^p), \\ \sup_{|s| \leq J} |B_4(\omega^q(H_q + s)) - B_4(\Omega_\delta(s))| &\longrightarrow 0. \end{aligned}$$

Write $B_q(s) = B_4(\omega^q(H_q + s))$ and $B_\delta(s) = B_4(\Omega_\delta(s))$. For fixed $S, U < \infty$ and $a \in \mathbb{R}$, its source moment therefore satisfies

$$\int_{-S}^U e^{-as} |B_q(s) - B_\delta(s)| ds \longrightarrow 0.$$

Proof. The negative window follows from (29) and (43). On $0 \leq s \leq J$, let $\zeta_q(s) = \omega^q(H_q + s) - \Omega_\delta(s)$. Its exact equation is

$$\partial_s \zeta_q + (K * \omega^q(H_q + s)) \cdot \nabla \zeta_q = \kappa_q \Delta \zeta_q - (K * \zeta_q) \cdot \nabla \Omega_\delta + \kappa_q \Delta \Omega_\delta.$$

The global Euler extension has bounded H^k norm and compact support on this fixed interval. Since $k \geq 3$, its second derivatives belong to H^1 , which embeds in every finite L^p in dimension two; compact support also bounds their L^1 norms. Thus $\|\nabla \Omega_\delta\|_{L^1 \cap L^p} + \|\Delta \Omega_\delta\|_{L^1 \cap L^p} \leq C_J$. Kato's inequality, the L^p inequality and (14) give

$$\sup_{0 \leq s \leq J} \|\zeta_q(s)\|_{L^1 \cap L^p} \leq e^{C_J J} (\|\zeta_q(0)\|_{L^1 \cap L^p} + C_J \kappa_q) \longrightarrow 0.$$

No actual NS derivative bound is used in this comparison.

We remove the moment cutoff separately. Scalar norm contraction gives one uniform post-cap velocity bound V . Choose a fixed radius $R' > R + VJ + d$, also enclosing the Euler support on $[0, J]$, with $d > 0$. The positive post-cap propagator has a forward diffusion with $|X_s - y| \leq VJ + \sqrt{2\kappa_q} |B_s|$. For $|y| \leq R$, its terminal fourth moment outside $B_{R'}$ is at most $C_J e^{-c_J/\kappa_q}$ by Gaussian endpoint moments. For unrestricted y its full fourth moment is at most $C_J(1 + |y|^4)$. The positive domination of the signed solution by the propagation of $|\omega^q(H_q)|$ therefore gives

$$\sup_{0 \leq s \leq J} \int_{|x| > R'} |x|^4 |\omega^q(H_q + s, x)| dx \leq C_J e^{-c_J/\kappa_q} + C_J \int_{|y| > R} (1 + |y|^4) |\omega^q(H_q, y)| dy \longrightarrow 0$$

by (43). A compact radial cutoff of $\bar{z}^4$, the just-proved L^1 convergence, and this tail prove uniform whole-moment convergence. On a fixed weighted time window the weight is bounded, so its final assertion follows by integration. $\square$

Corollary 3.3 (A nonzero bilateral discounted moment). *Define*

$$\mathcal{F}_\delta(z) = \int_{-\infty}^{\infty} e^{-zs} B_4(\Omega_\delta(s)) ds.$$

It is analytic and is not identically zero on $0 < \Re z < \sigma$. More precisely it extends meromorphically to $0 < \Re z < 2\sigma$ as

$$\mathcal{F}_\delta(z) = \frac{\delta b}{\lambda - z} + \mathcal{G}_\delta(z), \quad (45)$$

where $\mathcal{G}_\delta$ is analytic on that larger strip. For every fixed real c , the exceptional set

$$\{\beta > 1 : \mathcal{F}_\delta((\sigma - ic)/\beta) = 0\}$$

is discrete in $(1, \infty)$. In particular one can select arbitrarily large β outside it, including for $c = 1/2$.

Proof. The negative-time estimate (30) and the positive-time polynomial bound give absolute, locally uniform convergence on $0 < \Re z < \sigma$, including after any fixed number of z derivatives. Subtract the leading negative-time exponential. Its integral is

$$\int_{-\infty}^0 \delta b e^{(\lambda - z)s} ds = \frac{\delta b}{\lambda - z}.$$

The remaining negative-time integral is analytic for $\Re z < 2\sigma$, and the positive-time integral is analytic for $\Re z > 0$. This proves (45). Its pole at $z = \lambda$ has nonzero coefficient, so the transform cannot be identically zero on the smaller strip. Its zeros there are isolated. The curve $\beta \mapsto (\sigma - ic)/\beta$ is nonconstant, lies in that strip for $\beta > 1$, and has no interior accumulation of its zeros. Any terminal interval of large β contains values outside the zero set, proving the last assertion. Accumulation as $\beta \rightarrow \infty$ is possible at the boundary $z = 0$ and is not excluded or needed. $\square$

Corollary 3.4 (Changing the canonical cap). *If $0 < \delta' < \delta < \delta_*$ and $s_0 = \sigma^{-1} \log(\delta'/\delta)$, then*

$$\Omega_{\delta'}(s) = \mathcal{T}_{-\tau s_0} \Omega_\delta(s + s_0) \quad (-\infty < s < \infty), \quad \mathcal{F}_{\delta'}(z) = e^{(z - i\tau)s_0} \mathcal{F}_\delta(z).$$

Thus the zero set of the transform is independent of the canonical cap. At $z = (\sigma - ic)/\beta$ its modulus scales by $(\delta'/\delta)^{1/\beta}$.

Proof. The time-translated and rotated orbit has the polynomial (42) with δ replaced by δ' , since $\delta e^{\sigma s_0} = \delta'$. It has the same order- $M + 1$ remainder. The ancient uniqueness argument therefore identifies it with $\Omega_{\delta'}$ on negative times, and forward Euler uniqueness extends the identity to every time. Rotation $\mathcal{T}_\theta$ multiplies B_4 by $e^{i\theta}$. Substituting $v = s + s_0$ into the absolutely convergent bilateral transform gives the stated exponential factor, which never vanishes. Its modulus at the displayed point is $e^{\sigma s_0/\beta}$. $\square$

Remark 3.5 (Scope). The ancient profile, its cap compactness and its nonzero transform have been proved for one isolated fixed smooth annulus. They can supply an isolated source profile and a possible discounted transfer coefficient. This corollary does not identify that coefficient in a coupled receiver, does not estimate the effect of all other colours, and does not construct distinct vanishing-viscosity accumulation points from one fixed datum.

4 The smooth receiver's protected coordinate and transfer coefficient

Throughout this section $W = W_\epsilon$ is the one fixed profile of Section 2, $\lambda = \sigma + i\tau$ its growing fourth-sector eigenvalue, and $f_+ = f_\epsilon$ its normalized radial eigenfunction. All angular conventions are unchanged.

Lemma 4.1 (A global growing-coordinate test). *There is a smooth bounded complex test*

$$T(x) = a(r)e^{-4i\theta}$$

whose derivatives of every fixed order are bounded, such that

$$\mathcal{A}(g) = \int_{\mathbb{R}^2} T(x)g(x) \, dx, \quad \mathcal{A}(f_+(r)e^{4i\theta}) = 1, \quad \mathcal{A}(\overline{f_+(r)}e^{4i\theta}) = 0,$$

and $\mathcal{A}(\mathcal{L}g) = \lambda\mathcal{A}(g)$ for smooth integrable perturbations for which the pairings are defined. Near the origin T is a constant multiple of $\bar{z}^4$; outside a fixed ball it is a constant multiple of z^{-4} . In particular, ΔT is compactly supported away from the origin.

Proof. On a fixed radial interval containing the derivative collars, the rank-one Riesz projection in the positive fourth sector has form $P_+g = f_+\ell(g)$ with $\ell(f_+) = 1$ and $\ell(\mathcal{L}_4g) = \lambda\ell(g)$. Represent this continuous L^2 functional by a bilinear density $2\pi a(r)r \, dr$; complex conjugation is included in the representing density rather than in the pairing. Substitution of (7) gives

$$(\lambda + 4i\Omega_\epsilon(s))a(s) = -\frac{i}{2} \int_0^\infty a(r)W'_\epsilon(r)h_4(r, s) \, dr. \quad (46)$$

On the collars, the radial Green equation and this nonvanishing denominator bootstrap a to smoothness. Extend it to every $s > 0$ by the right side of (46) divided by its denominator. This agrees with the original density where W' is supported, so it still satisfies the equation everywhere. Integration of the Green kernel and Fubini then verify the global eigenfunctional identity.

For s smaller than every derivative collar, the numerator is a constant times s^4 and Ω_ϵ is the constant core angular velocity $-1/5$. Thus $a(s) = Cs^4$ there and $T = C\bar{z}^4$. Beyond the profile support the numerator is a constant times s^{-4} and $\Omega_\epsilon = 0$ by zero circulation. Thus $a(s) = C's^{-4}$ and $T = C'z^{-4}$. These two extensions are harmonic; on the remaining fixed annulus the test is smooth. This proves all the regularity and Laplacian assertions. Normalize by $2\pi \int a f_+ r \, dr = 1$. Since the stable eigenvalue is $-\bar{\lambda} \neq \lambda$, the eigenfunctional identity annihilates its eigenfunction. The angular test kills every sector other than $+4$. $\square$

Lemma 4.2 (Exact inverse-quintic forcing coefficient). *Let a source be real, compactly supported, quarter-turn symmetric and of zero circulation. At points outside its support, its leading complex velocity is*

$$U(z) = \frac{iB_4}{2\pi\bar{z}^5} + \frac{iB_8}{2\pi\bar{z}^9} + \cdots.$$

The first term forces the receiver's positive fourth vorticity sector by $B_4F_\epsilon(r)e^{4i\theta}$, where

$$F_\epsilon(r) = -\frac{iW'_\epsilon(r)}{4\pi r^5}.$$

Its growing projection is $\gamma_\epsilon B_4$, with $\gamma_\epsilon = \mathcal{A}(F_\epsilon e^{4i\theta})$. For one fixed sufficiently small smoothing length,

$$\gamma_\epsilon \longrightarrow \gamma_0 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0 \quad (\epsilon \downarrow 0). \quad (47)$$

Proof. In complex notation Biot–Savart is $U(z) = \frac{i}{2\pi} \int (\bar{z} - \bar{y})^{-1} w(y) dy$. The convergent exterior geometric series gives the stated expansion; quarter-turn symmetry eliminates every exponent not divisible by four, and zero circulation eliminates the constant moment. For its first term,

$$u_r = \Re(Ue^{-i\theta}) = \frac{iB_4}{4\pi r^5} e^{4i\theta} + \text{complex conjugate}.$$

The linearized forcing is $-u_r W'$, giving F_ϵ .

We check nonvanishing without assuming an adjoint convergence theorem. The inhomogeneous resolvent Schur reduction in the proof of Lemma 2.2 applies to this forcing. Its natural radial measure has limit

$$F_\epsilon(r) r dr \rightarrow \sum_{i=1}^2 \frac{ic_i}{4\pi r_i^4} \delta_{r_i}.$$

Thus in interface variables its forcing vector tends to $h = (i/(4\pi\rho^6), i/(4\pi))$. The growing dual of M_4 , normalized on $(1, -i\rho)$, is $\ell_0(q) = (q_1 + iq_2/\rho)/2$. Its pairing with h is γ_0 in (47). More explicitly, on a small fixed contour around the growing root the inhomogeneous Schur equations converge uniformly to $(z - M_4)q = h$. Integrating their resolvent in z gives convergence of the rank-one projected forcing coefficient. The eigenfunction normalization agrees with that in Section 2. This proves the limit and nonvanishing for a fixed sufficiently small ϵ . $\square$

The direction of this interaction matters. A smaller source acts on the larger receiver by the inverse-quintic exterior field above. A larger annulus acts in its interior through a cubic harmonic field. They are different coefficients and cannot be interchanged.

For comparison, an exterior annulus's interior harmonic field $U(z) = c_3 \bar{z}^3$ has positive fourth radial coefficient $\bar{c}_3 r^3/2$. Its frozen-base forcing is $-\bar{c}_3 r^3 W'/2$, with growing projection $\gamma_\epsilon^{\text{out}} \bar{c}_3$. The same resolvent limit gives

$$\gamma_\epsilon^{\text{out}} \rightarrow (\rho^2 + i/\rho)/4 \neq 0.$$

Indeed its interface forcing vector is $(\rho^2/2, 1/2)$, paired against the same growing dual. Thus older cubic fields can genuinely seed the growing coordinate; their size must be bounded, rather than discarded by symmetry.

Lemma 4.3 (Duhamel coefficient for a child source). *Let $A_c = \beta A_r$, $R_c = q R_r$, and use receiver time $s = A_r t$. Suppose, in a frame removing all common solid rotation, that the source's normalized fourth moment is $B(u)$, with child time $u = \beta s - s_*$. The receiver core has angular velocity $-A_r/5$; this rotates the child's moment by $e^{4is/5}$ in the receiver frame. The ideal linear growing receiver coefficient obeys*

$$a'(s) = \lambda a(s) + \beta q^6 \gamma_\epsilon e^{4is/5} B(\beta s - s_*).$$

For the response with no independent growing coefficient at $s = -\infty$,

$$e^{-\lambda s} a(s) = q^6 \gamma_\epsilon e^{-\mu_\beta s_*} \int_{-\infty}^{\beta s - s_*} e^{-\mu_\beta u} B(u) du, \quad \mu_\beta = \frac{\sigma + i(\tau - 4/5)}{\beta}. \quad (48)$$

This integral exists whenever $B(u) = O(e^{\sigma u})$ at $-\infty$ and has polynomial growth at $+\infty$, provided $\beta > 1$.

Proof. The dimensional fourth moment is $A_c R_c^6 B$. Dividing the child velocity by $A_r R_r$ leaves the factor βq^6 , so Lemma 4.2 gives the forcing magnitude. Rotation through angle Ωt multiplies B_4 by $e^{-4i\Omega t}$. The parent constant core has vorticity $-2A_r/5$, hence angular velocity $-A_r/5$,

which gives the factor $e^{4is/5}$. Multiplication by the integrating factor $e^{-\lambda s}$ and the substitution $u = \beta s - s_*$ give (48). Its negative tail has exponential exponent $\sigma - \sigma/\beta > 0$; its positive tail has discount $\sigma/\beta > 0$. $\square$

Lemma 4.4 (A nonzero bilateral source transform from an ancient orbit). *Suppose a global Euler source trajectory has a fourth moment $B(u)$ with*

$$B(u) = cb_\epsilon e^{\lambda u} + O(e^{2\sigma u}) \quad (u \rightarrow -\infty), \quad c \neq 0,$$

and $|B(u)| \leq C(1+u)^4$ for $u \geq 0$. Define

$$\mathcal{F}(z) = \int_{\mathbb{R}} e^{-zu} B(u) du, \quad 0 < \Re z < \sigma.$$

Then $\mathcal{F}$ is analytic and not identically zero. For every finite interval $[B_, B_* + 1] \subset (1, \infty)$ there is a choice of β in that interval such that $\mathcal{F}(\mu_\beta) \neq 0$.*

Proof. The two explicit bounds give absolute locally uniform convergence of the integral and of all complex derivatives, proving analyticity. Split at zero and subtract the leading negative-time asymptotic. It gives

$$\mathcal{F}(z) = \frac{cb_\epsilon}{\lambda - z} + \mathcal{G}(z), \quad 0 < \Re z < \sigma,$$

where $\mathcal{G}$ is analytic in $0 < \Re z < 2\sigma$: the remainder on the negative half-line has rate 2σ , and the positive half-line has polynomial growth. Thus its meromorphic extension has a nonremovable simple pole at $z = \lambda$. In particular it cannot vanish identically on the original strip. The analytic nonconstant curve $\beta \mapsto \mu_\beta$ lies inside that strip for real $\beta > 1$. If the transform vanished along any real interval, its zeros would have an interior accumulation point and the identity theorem would force identical vanishing. Its zeros along the displayed compact interval are therefore discrete, and a nonzero choice exists. $\square$

The canonical ancient Euler orbit of Theorem 3.1 has precisely this negative-time estimate. It comes from the high-Sobolev construction and not merely from the isolated X -norm cap theorem. Its positive-time moment bound is elementary once a compactly supported bounded-vorticity orbit exists: L^1, L^p conservation bounds its velocity by (14), its support radius grows at most linearly, and its fourth absolute moment is therefore polynomial of degree at most four. Lemma 4.4 isolates the precise ancient-orbit input needed for the noncancellation assertion.

Lemma 4.5 (Quantitative window and source errors). *Let $\beta > 1$ be fixed and $\mu = \mu_\beta$. Suppose $\mathcal{F}(\mu) \neq 0$ and the source satisfies the bounds of Lemma 4.4. Truncating the source interval to $[-S, U]$ changes its transform by at most*

$$Ce^{-(\sigma - \Re \mu)S} + C_\beta(1+U)^4 e^{-\Re \mu U}.$$

Replacing its moment there by $\tilde{B}$ adds at most

$$\int_{-S}^U e^{-\Re \mu u} |\tilde{B}(u) - B(u)| du.$$

Consequently source windows and errors smaller than $|\mathcal{F}(\mu)|/4$ preserve a nonzero growing transmission. An independent old growing seed a_0 cannot cancel it if

$$|a_0| < \frac{1}{4} q^6 |\gamma_\epsilon| e^{-\sigma s_*/\beta} |\mathcal{F}(\mu)|.$$

If $s_* = \sigma^{-1} \log(\delta/\alpha)$ and old and child seed amplitudes both have size $\alpha = q^{K_*}$, this magnitude condition is algebraically feasible when

$$K_*(1 - 1/\beta) > 6.$$

Fixed logarithmic factors in δ do not change this strict power inequality.

Proof. Integrate the negative exponential bound and the polynomial positive bound against $e^{-\Re \mu u}$. For the positive half-line repeated integration by parts gives a fixed polynomial times $e^{-\Re \mu U}$. The error bound is the triangle inequality. Apply (48); its effective growing coefficient has modulus $q^6 |\gamma_\epsilon| e^{-\sigma s_*/\beta} |\mathcal{F}(\mu)|$. The two strict one-quarter bounds leave a nonzero coefficient. Finally $e^{-\sigma s_*/\beta} = (\alpha/\delta)^{1/\beta}$, so $\alpha/[q^6(\alpha/\delta)^{1/\beta}]$ tends to zero under the stated power inequality, including logarithmically decreasing caps. $\square$

4.1 Exact heat shielding and the remaining coupled terms

The neutral seed (13) has only sectors ± 4 and $B_4 = 0$. Its velocity is zero outside $B_{1+\epsilon}$: in the exterior angular Green formula its coefficient is exactly its fourth moment. Thus its velocity $V_\vartheta = K * \Phi_\vartheta$ is smooth and compactly supported. Heat commutes with Biot–Savart, so a passive physical seed of amplitude $A\alpha$ and radius r has exterior heat velocity bounded, for $|x| \geq R > 2(1 + \epsilon)r$, by

$$|e^{\nu t \Delta} [A\alpha r V_\vartheta(\cdot/r)](x)| \leq C A\alpha r \exp[-cR^2/(\nu t)]. \quad (49)$$

Indeed convolution integrates a bounded function supported in that ball, and the Gaussian mass at distance at least $R/2$ has the displayed bound. No damping estimate of a conserved harmonic moment is invoked. The bound concerns a passive heat evolution; its nonlinear moment generation is an additional term.

For precision, the actual foreign-colour equation yields an exact protected-coordinate identity. Suppose a receiver colour w solves

$$w_t + (K * w + b) \cdot \nabla w = \kappa \Delta w, \quad \operatorname{div} b = 0,$$

and put $f = w - W_\kappa$. Remove any common rigid rotation first. Testing with Lemma 4.1 gives

$$\frac{d}{dt} \mathcal{A}(f) = \lambda \mathcal{A}(f) - \int T b \cdot \nabla W \, dx + \mathcal{E}(t), \quad (50)$$

where the exact error is the sum of

$$\begin{aligned} & \kappa \int \Delta T f \, dx, \quad - \int T [(K * (W_\kappa - W)) \cdot \nabla f + (K * f) \cdot \nabla (W_\kappa - W)] \, dx, \\ & \int f (K * f) \cdot \nabla T \, dx, \quad \int f b \cdot \nabla T \, dx, \quad - \int T b \cdot \nabla (W_\kappa - W) \, dx. \end{aligned}$$

There is no derivative of the actual f after integrating its transport terms by parts. Smooth heat Taylor and (14) give

$$|\mathcal{E}(t)| \leq C[\kappa(1+t)\|f\|_X + \|f\|_X^2] + \int |f||b||\nabla T| \, dx + \left| \int T b \cdot \nabla (W_\kappa - W) \, dx \right|. \quad (51)$$

For the term with ∇f , integration by parts bounds it by $\|\nabla T\|_\infty \|K * (W_\kappa - W)\|_\infty \|f\|_1$; the other heat term uses $\|\nabla(W_\kappa - W)\|_1 = O(\kappa t)$. The quadratic self term uses (14). If $b = \Omega(t)Jx$ is a

rigid rotation, its base forcing vanishes and its perturbation contribution is exactly $-4i\Omega(t)\mathcal{A}(f)$; it can be removed by its integrating phase.

Equation (50) identifies quantitative coupled-stage requirements rather than treating the forcing as a free parameter. On the receiver collars one must approximate b by the inverse-quintic child field of Lemma 4.2; its projected error is bounded by the collar sup error times $\|T\nabla W\|_1$. One must also control the two weighted foreign terms in (51) and prove a nonlinear density bound for this actual colour. The canonical source transform must be approximated in the weighted norm of Lemma 4.5, including its post-cap history. Passive shielding (49) and the isolated activation theorem alone do not establish these facts for an infinite interacting stack. No nonlinear receiver transmission theorem is asserted without those estimates.

4.2 A nonlinear receiver theorem for the concrete canonical input

There is an intermediate statement that does close without assuming a foreign field has unspecified desirable properties. It uses the actual moment of the canonical isolated Euler orbit, but realizes its exterior harmonic field through a fixed smooth extension. This extension is an auxiliary prescribed drift, not an assertion that the source and receiver jointly solve the unforced Navier–Stokes system.

Fix an ancient orbit Ω_{δ_c} from Theorem 3.1, with fixed small $\delta_c > 0$, and put $B(u) = B_4(\Omega_{\delta_c}(u))$. Choose $\beta > 1$ for which

$$C_* := \gamma_\epsilon \mathcal{F}(\mu_\beta) \neq 0.$$

Such a choice is possible in every sufficiently large fixed interval by Lemma 4.4. Take a smooth radial cutoff ζ_0 equal to zero near the origin and equal to one on and outside the two receiver derivative collars. For $c \in \mathbb{C}$ set

$$\Psi_c(r, \theta) = -\frac{\zeta_0(r)}{16\pi r^4} (ce^{4i\theta} + \bar{c}e^{-4i\theta}), \quad \mathcal{V}_c = \nabla^\perp \Psi_c.$$

This is a smooth real divergence-free field with every fixed spatial derivative bounded by a constant times $|c|$. On the receiver collars it agrees exactly with the inverse-quintic field of a source moment c .

Theorem 4.6 (Transmission for a prescribed canonical harmonic input). *Fix $K_* > 0$ such that $K_*(1 - 1/\beta) > 6$. For small q put*

$$\alpha = q^{K_*}, \quad \varepsilon_q = q^6, \quad \delta_q = [\log(1/q)]^{-4}, \quad t_0 = \frac{1}{\beta\sigma} \log \frac{\delta_c}{\alpha}, \quad S_q = \frac{1}{\sigma} \log \frac{\delta_q}{\varepsilon_q |C_*|}, \quad H_q = t_0 + S_q.$$

For $0 < \kappa \leq q^{3/2}$ define the prescribed foreign drift

$$b_q(t, x) = \beta \varepsilon_q \mathcal{V}_{e^{4i(t-t_0)/5} B(\beta(t-t_0))}(x), \quad 0 \leq t \leq H_q.$$

Let w^q solve the coloured equation

$$\partial_t w^q + (K * w^q + b_q) \cdot \nabla w^q = \kappa \Delta w^q, \quad w^q(0) = W + \alpha \Phi_0.$$

Then uniformly in $0 < \kappa \leq q^{3/2}$,

$$\sup_{0 \leq t \leq H_q} \|w^q(t) - e^{\kappa t \Delta} W\|_X \leq C \delta_q,$$

$$\mathcal{A}(w^q(H_q)) = \delta_q e^{i(\tau S_q + \arg C_*)} + o(\delta_q),$$

$$B_4(w^q(H_q)) = b_\epsilon \delta_q e^{i(\tau S_q + \arg C_*)} + o(\delta_q).$$

The whole fourth moment is finite. Multiplying the source moment in b_q by a fixed unit complex phase multiplies the leading cap phase by the same phase; all bounds are uniform in that phase.

Proof. Use $s = t - t_0$ and write $b_q = \varepsilon_q b_1(s)$. The ancient and future moment bounds give

$$\|b_1(s)\|_{C^j} \leq \begin{cases} C_j e^{\beta \sigma s}, & s \leq 0, \\ C_j (1+s)^4, & s \geq 0. \end{cases}$$

In particular this is bounded by $C_j e^{\sigma s}$ for all s , and its growing-mode weighted time integral is finite, uniformly in the initial time $-t_0$. The first seed-order coefficient solves

$$(\partial_t - \mathcal{L})g_{1,0} = -b_1(t - t_0) \cdot \nabla W, \quad g_{1,0}(0) = (\alpha/\varepsilon_q)\Phi_0.$$

Its growing coordinate is

$$e^{-\lambda s} \mathcal{A}(g_{1,0}(t)) = \frac{\alpha}{\varepsilon_q} e^{\lambda t_0} + \gamma_\epsilon \int_{-\beta t_0}^{\beta s} e^{-\mu_\beta u} B(u) du.$$

The first term tends to zero in modulus because $K_*(1 - 1/\beta) > 6$. The integral tends to $\mathcal{F}(\mu_\beta)$ as $t_0, s \rightarrow \infty$. Thus its growing coefficient tends to C_* . The complementary propagator has exponent $2\eta < \sigma$; integrating its ancient exponential forcing and its future polynomial forcing gives a complementary bound $Ce^{2\eta s}$ for $s \geq 0$. For $s \leq 0$ the whole first coefficient is bounded by $Ce^{\sigma s}$. Indeed its forcing has ancient rate $\beta\sigma > \sigma$ and $(\alpha/\varepsilon_q)e^{\sigma t_0}$ is bounded. All bounds are uniform in q .

Expand in ε_q and κ exactly as in (15), replacing α there by ε_q . In the recursion add $-b_1 \cdot \nabla g_{j-1,\ell}$ for $j \geq 2$ and add $-b_1 \cdot \nabla W_\ell$ for $j = 1$. The initial first coefficient is the one above and the other initial coefficients vanish. The support and finite angular-sector properties are unchanged. Induction gives pure-seed bounds $Ce^{j\sigma s}$ and positive-viscosity bounds $C(1+t)^D e^{j\sigma s}$, uniformly in t_0 : the first coefficient has just been estimated, and for $j \geq 2$ the external forcing is at most $Ce^{\sigma s} e^{(j-1)\sigma s}$, leaving the same nonresonant gap $j\sigma - \sigma$. Positive-viscosity resonances produce only the finite absolute-time polynomial. Consequently the residual estimate (19) holds with

$$a(t) = \varepsilon_q e^{\sigma(t-t_0)}, \quad a(0) = q^{6+K_*/\beta} \delta_c^{-1/\beta}, \quad a(H_q) = \delta_q/|C_*|.$$

The residual contains no forcing-free term: the exact heat base is retained, and $b_1 \cdot \nabla$ of its heat Taylor remainder has the seed factor ε_q . Its bound is $C\kappa^{N+1}(1+t)^D a(t)$ by the displayed input estimate.

The prescribed drift is identical in the actual and approximate equations, so it disappears from the transport part of their Kato comparison. Choose M using the fixed exponent C_1 as before, and choose N with

$$\frac{3}{2}(N+1) > (6 + K_*/\beta)r_1.$$

The exact same X error calculation yields Ca^{M+1} plus a viscosity remainder $o(\delta_q)$ at the cap. It is uniform for $\kappa \leq q^{3/2}$. The higher pure-seed terms contribute $O(\delta_q^2)$ and the positive-viscosity correctors contribute $O(\kappa(1+H_q)^D \delta_q)$. The complementary first-coefficient response at S_q is $\varepsilon_q e^{2\eta S_q} = o(\delta_q)$. Its growing response gives exactly the asserted protected-coordinate cap. These estimates also give the density bound.

For the whole moment, the integrated self-perturbation velocity is $O(\delta_q)$ as in (26). The additional prescribed drift has integral at most $Cq^6(1+H_q)^5 = o(\delta_q)$, by its ancient and future

bounds. The radial exponential Kato tail argument therefore applies unchanged. All formal coefficients are supported in the fixed collars, and the leading growing coefficient has moment b_ϵ . The complementary, nonlinear and viscosity errors estimated above, followed by the Gaussian tail bound, prove the whole-moment cap. This completes the proof. $\square$

Theorem 4.6 proves nonlinear transmission for this concrete smooth extension of a canonical source. Substituting the actual Biot–Savart field of interacting viscous source colours still requires controlling its omitted higher multipoles, core leakage, and feedback. For a quantitative approximation over a logarithmic stage, keeping only the fourth moment is generally inadequate: its next q^{10} error may be amplified by the generic X exponent. One can instead retain arbitrarily many finite harmonic terms in the prescribed coefficient recursion. Their omitted remainder must be bounded to a sufficiently high power of q to beat that exponent. This identifies an extension of the method, not an estimate already proved for the actual infinite stack.

5 A nonlinear comparison on a fixed time window

This estimate retains the original annular patch profile. It is conditional on explicitly stated regularity of an Euler family. Its constants depend on the fixed window; it is not a sharp long-time cascade estimate. Put

$$W = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}, \quad \rho^2 = 3/5, \quad U = K * W, \quad W_\kappa = e^{\kappa t \Delta} W, \quad U_\kappa = K * W_\kappa.$$

The radial field U_κ is tangential, so W_κ is an exact Navier–Stokes solution.

Proposition 5.1 (Seed-dependent comparison, including the whole moment). *Fix $H < \infty$, $0 < s < 1/2$, and $0 < \alpha \leq \alpha_0 \leq 1$. Suppose ω_0^α is real, compactly supported in a common ball, has zero circulation and uniformly bounded L^1 and L^∞ norms, and*

$$\|\omega_0^\alpha - W\|_1 \leq C\alpha.$$

*Suppose its Euler solution $\omega^{E,\alpha}$ exists on $[0, H]$, has uniformly bounded vorticity, and, for $v = K * \omega^{E,\alpha} - U$, satisfies*

$$\sup_{t \leq H} (\|v\|_{H^s} + \|v\|_\infty) \leq C_H \alpha, \quad \sup_{t \leq H} \|\nabla v\|_\infty \leq C_H. \quad (52)$$

*Let $\omega^{\kappa,\alpha}$ be the unforced Navier–Stokes solution with the same initial data and $0 < \kappa \leq 1$, and put $v_\kappa = K * \omega^{\kappa,\alpha} - U_\kappa$. Then*

$$\sup_{t \leq H} \|v_\kappa - v\|_2 \leq C_H \alpha \kappa^{s/2}, \quad (53)$$

$$\sup_{t \leq H} \left| B_4(\omega^{\kappa,\alpha}) - B_4(\omega^{E,\alpha}) \right| \leq C_H \alpha \kappa^{s/2}. \quad (54)$$

The whole fourth moments exist. All hypothesis constants are independent of α and κ for the fixed H . No gradient estimate on the actual Navier–Stokes velocity is assumed.

Proof. The annular circulation formula gives

$$U(r, \theta) = \Omega(r) r e_\theta, \quad \Omega(r) = r^{-2} \int_0^r a W(a) da.$$

Consequently ∇U is bounded, compactly supported, and smooth on the pieces separated by the two circles. Its squared L^2 translation difference is at most $C|h|$ for $|h| \leq 1$: the strips crossing a circle have area $O(|h|)$, and on their complement the squared difference is $O(|h|^2)$. The fractional difference-quotient formula therefore gives $\nabla U \in H^s$ for $s < 1/2$. Heat convolution gives

$$\|U_\kappa - U\|_\infty \leq C\sqrt{\kappa t}, \quad \|\nabla U_\kappa\|_\infty \leq C, \quad \|\nabla(U_\kappa - U)\|_2 \leq C(\kappa t)^{1/4}. \quad (55)$$

The first uses the Lipschitz bound on U and the first heat-kernel moment; the last uses the translation estimate just proved and Minkowski's inequality.

Let J_h be convolution with a smooth compactly supported mollifier, $h = \sqrt{\kappa}$, and set $V = J_h v$. Fractional translation estimates give

$$\|V - v\|_2 \leq C_H \alpha h^s, \quad \|\nabla V\|_2 \leq C_H \alpha h^{s-1}, \quad \|\nabla V\|_\infty \leq C_H. \quad (56)$$

Also $\|v \cdot \nabla U\|_{H^s} \leq C_H \alpha$. Indeed, in a product difference the term with a difference of v is bounded by $\|\nabla U\|_\infty \|v\|_{H^s}$, and the other by $\|v\|_\infty \|\nabla U\|_{H^s}$.

Write $u^E = U + v$. The transport commutator satisfies

$$\left\| u^E \cdot \nabla J_h v - J_h(u^E \cdot \nabla v) \right\|_2 \leq C \left\| \nabla u^E \right\|_\infty h^s \|v\|_{H^s}. \quad (57)$$

To verify this, its integral kernel is

$$\nabla \eta_h(y) \cdot [u^E(x) - u^E(x-y)][v(x-y) - v(x)].$$

The term with $v(x)$ vanishes because $\operatorname{div} u^E = 0$. Use the Lipschitz difference of u^E and the fractional L^2 translation bound for v , then integrate in y .

Subtract the mollified Euler perturbation equation from the exact Navier–Stokes perturbation equation. For $z = v_\kappa - V$, after absorbing gradients into the pressure, the equation is

$$\partial_t z + u^{\kappa, \alpha} \cdot \nabla z + z \cdot \nabla(U_\kappa + V) + \nabla \pi = \kappa \Delta z + \kappa \Delta V + \mathcal{R}.$$

Up to signs irrelevant for estimates, $\mathcal{R}$ is the sum of the commutator in (57) and

$$(V - v) \cdot \nabla V, \quad (U_\kappa - U) \cdot \nabla V, \quad (V - v) \cdot \nabla U_\kappa, \quad v \cdot \nabla(U_\kappa - U), \quad (I - J_h)(v \cdot \nabla U).$$

Equations (55)–(57) bound their L^2 norms by

$$C_H \alpha \left[h^s + \sqrt{\kappa H} h^{s-1} + (\kappa H)^{1/4} \right] \leq C_H \alpha \kappa^{s/2}.$$

The L^2 energy identity uses only $\|\nabla(U_\kappa + V)\|_\infty \leq C_H$; the full viscous transport term cancels. Moreover $\kappa \|\nabla V\|_2^2 \leq C_H \alpha^2 \kappa^s$ and $\|z(0)\|_2 \leq C_H \alpha \kappa^{s/2}$. Young's inequality and Gronwall's inequality prove (53). The identities follow by mollifying in space and passing in the energy inequalities; all the displayed coefficients have the stated integrable bounds.

We now control the noncompact viscous tail with a seed factor. The exact perturbation equation around U_κ gives independently

$$\sup_{t \leq H} \|v_\kappa(t)\|_2 \leq C_H \alpha, \quad (58)$$

since its only energy coefficient is ∇U_κ and its initial velocity has L^2 norm at most $C_H \alpha$. Put $f_\kappa = \omega^{\kappa, \alpha} - W_\kappa$. Its scalar equation is

$$\partial_t f_\kappa + u^{\kappa, \alpha} \cdot \nabla f_\kappa = \kappa \Delta f_\kappa - S_\kappa, \quad S_\kappa = v_\kappa \cdot \nabla W_\kappa. \quad (59)$$

If $a = \kappa t$, direct convolution of the two circle measures with the heat kernel yields

$$\left\| \nabla e^{a\Delta} W \right\|_2 \leq C a^{-1/4}, \quad \left\| \mathbf{1}_{|x| > r_0 + d} (1 + |x|^4) \nabla e^{a\Delta} W \right\|_2 \leq C_d a^{-N_d} e^{-c_d/a} \quad (60)$$

for $0 < a \leq H$, where r_0 encloses both initial supports and $d > 0$. Here the distributional gradient of each disk indicator is minus its outward normal times boundary measure. Its squared convolved L^2 norm is bounded by the double boundary integral of the heat kernel at time $2a$. On a circle the integral in the second boundary variable is at most $C a^{-1/2}$, proving the first bound. Outside the enlarged ball every boundary point is at distance at least d ; the Gaussian kernel and its polynomially weighted integral prove the second. Thus

$$\begin{aligned} \|S_\kappa(t)\|_1 &\leq C_H \alpha(\kappa t)^{-1/4}, \\ \left\| \mathbf{1}_{|x| > r_0 + d} (1 + |x|^4) S_\kappa(t) \right\|_1 &\leq C_{H,d} \alpha(\kappa t)^{-N_d} e^{-c_d/(\kappa t)}. \end{aligned}$$

The maximum principle and the uniform initial L^1, L^∞ bounds give $\|u^{\kappa,\alpha}\|_\infty \leq V_H$ uniformly. The positive advection–diffusion propagator has the forward diffusion representation

$$X_t = y + \int_s^t u^{\kappa,\alpha}(a, X_a) da + \sqrt{2\kappa}(B_t - B_s).$$

In particular $|X_t - y| \leq V_H(t - s) + \sqrt{2\kappa}|B_t - B_s|$. Choose $R = r_0 + V_H H + 3d$. For $|y| \leq r_0 + d$, the event $|X_t| > R$ requires a Brownian displacement at least $2d$. Gaussian endpoint moments imply

$$\mathbb{E}[|X_t|^4 \mathbf{1}_{|X_t| > R}] \leq C_{H,d} e^{-c_{H,d}/\kappa}, \quad \mathbb{E}[1 + |X_t|^4] \leq C_H(1 + |y|^4)$$

in the respective inside and unrestricted cases, uniformly for $0 \leq s \leq t \leq H$. Apply Kato's inequality and Duhamel's formula to (59). The initial absolute mass is $O(\alpha)$ inside B_{r_0} . The inside source has time-integrated mass at most $C_H \alpha \kappa^{-1/4}$; its propagation still has the Gaussian displacement factor. The outside source is controlled by (60) and the unrestricted endpoint moment. Absorbing finite powers of κ^{-1} into a slightly weaker Gaussian gives

$$\sup_{t \leq H} \int_{|x| > R} |x|^4 |f_\kappa(t, x)| dx \leq C_{H,d} \alpha e^{-c_{H,d}/\kappa}. \quad (61)$$

The forward representation can equivalently be obtained by duality from the scalar propagator. Its use with signed sources is justified by positivity and Kato's inequality. Smoothness for positive time and bounded drift allow passage to the initial endpoint.

The Euler perturbation $f^E = \omega^{E,\alpha} - W$ is supported in a common ball inside B_R , after increasing V_H if necessary. Finite propagation follows from the same bounded-velocity estimate and a moving radial cutoff. Let χ be radial, compactly supported, and equal to one on B_R . Radial densities have zero pairing with $\bar{z}^4$ and with $\chi \bar{z}^4$. Since $f_\kappa - f^E = \text{curl}(v_\kappa - v)$, integration by parts and (53) give

$$\left| \int \chi \bar{z}^4 (f_\kappa - f^E) dx \right| \leq \left\| \nabla(\chi \bar{z}^4) \right\|_2 \|v_\kappa - v\|_2 \leq C_H \alpha \kappa^{s/2}.$$

Equation (61) controls the omitted part and proves (54). $\square$

Remark 5.2. The hypotheses (52) are part of this proposition. They have not been deduced here for an interacting stack. Nor is a sufficiently sharp dependence of C_H on a logarithmically long preparation window asserted. The value of the proposition is that both its nonlinear comparison and its entire viscous tail retain the factor α .

6 Actual viscous outer cores and the local harmonic field

This section proves a conditional estimate for the actual coloured Navier–Stokes evolution. Its nonradial-density hypothesis is explicit; it is not deduced from material contours or from named unstable coordinates. We identify $x = (x_1, x_2)$ with $z = x_1 + ix_2$ and identify a velocity with $u_1 + iu_2$. Thus

$$K(z) = \frac{i}{2\pi\bar{z}}, \quad Jz = iz.$$

Let P denote circular averaging. For $f \in L^1 \cap L^\infty$, splitting the Biot–Savart integral at distance d and optimizing d gives

$$\|K * f\|_\infty \leq C\sqrt{\|f\|_1 \|f\|_\infty}. \quad (62)$$

Indeed the near integral is at most $Cd\|f\|_\infty$ and the far integral is at most $C\|f\|_1/d$.

Lemma 6.1 (Fourfold growth at the origin). *If $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is invariant under $z \mapsto iz$, then*

$$|K * f(z)| \leq C\|f\|_\infty |z|.$$

Neither zero circulation nor regularity of f is required.

Proof. For $r = |z| > 0$, the integral over $|y| \leq 2r$ is at most $C\|f\|_\infty r$, by integrating $|z - y|^{-1}$ over $B_{3r}(z)$. For $|y| > 2r$ use the absolutely convergent expansion

$$\frac{1}{\bar{z} - \bar{y}} = -\sum_{n \geq 0} \frac{\bar{z}^n}{\bar{y}^{n+1}}.$$

The inverse moment vanishes unless $n + 1$ is a multiple of four, by changing variables under a quarter turn. The surviving degrees are 3, 7, 11, ... Their absolute sum is at most

$$Cr^3 \int_{|y| > 2r} \frac{|f(y)|}{|y|^4} dy \leq C\|f\|_\infty r^3 \int_{2r}^\infty s^{-3} ds \leq C\|f\|_\infty r.$$

Termwise integration is justified by the integrable majorant $C|f(y)|/|y|$ on the far region. At $z = 0$ the velocity vanishes by rotational covariance and continuity. $\square$

Lemma 6.2 (Exit controlled only by radial drift). *Let a diffusion satisfy*

$$dX_s = v(s, X_s) ds + \sqrt{2\nu} dB_s, \quad X_0 = x,$$

and let σ be its first exit from B_b , where $|x| < b$. Suppose, before exit,

$$|v(s, y) \cdot y/|y|| \leq V(s) \quad (0 < |y| < b), \quad V \geq 0, \quad D := \int_0^T V(s) ds < b - |x|.$$

Then

$$\mathbb{P}(\sigma \leq T) \leq 2 \exp\left(-\frac{(b - |x| - D)^2}{8\nu T}\right). \quad (63)$$

There is no bound on the tangential component of v .

Proof. For $\lambda > 0$ set $\phi(y) = \cosh(\lambda|y|)$. Its even power series shows that it is smooth at zero. In dimension two,

$$\Delta\phi = \lambda^2 \cosh(\lambda r) + \frac{\lambda}{r} \sinh(\lambda r) \leq 2\lambda^2 \phi, \quad v \cdot \nabla \phi \leq \lambda V(s)\phi,$$

using $\sinh a \leq a \cosh a$ for $a \geq 0$. Therefore Itô's formula shows that

$$e^{-2\nu\lambda^2(s\wedge\sigma) - \lambda \int_0^{s\wedge\sigma} V(\tau) d\tau} \phi(X_{s\wedge\sigma})$$

is a nonnegative supermartingale. Stopping at T gives

$$\mathbb{P}(\sigma \leq T) \leq \frac{\cosh(\lambda|x|)}{\cosh(\lambda b)} e^{2\nu\lambda^2 T + \lambda D} \leq 2e^{-\lambda(b-|x|-D) + 2\nu\lambda^2 T}.$$

Choose $\lambda = (b - |x| - D)/(4\nu T)$. The stopped calculation remains valid for bounded drifts smooth for positive times: stop first before their time endpoint and then pass to that endpoint. No approximation of a radial drift bound is used. $\square$

Consider real, fourfold symmetric colours solving

$$\partial_t w_l + u^\nu \cdot \nabla w_l = \nu \Delta w_l, \quad u^\nu = K * \sum_l w_l, \quad (64)$$

with geometric scales

$$A_l = A_1 \beta^{l-1}, \quad R_l = R_1 q^{l-1}, \quad 1 < \beta_{\min} \leq \beta \leq \beta_{\max}, \quad q \leq \frac{1}{4}, \quad \beta q \leq \frac{1}{2}.$$

Assume

$$\|w_l(0)\|_1 \leq C_0 A_l R_l^2, \quad \|w_l(0)\|_\infty \leq C_0 A_l. \quad (65)$$

In the countable case assume that the initial series converges absolutely in $L^1 \cap L^p$, $p > 2$, and use its mild Navier–Stokes solution and the associated linear coloured evolutions. The common velocity is bounded and smooth for positive times. Norm contraction identifies its vorticity with the sum of the colours, and (62) implies uniform convergence of the velocity series because $\sum_l A_l R_l < \infty$.

Fix an index i , a window $T = H/A_i$, and

$$\kappa_i = \frac{\nu}{A_i R_i^2}.$$

Suppose that for some fixed $a_0 > 0$ every $l \leq i$ has an initial constant core,

$$w_l(0) = c_l \quad \text{on } B_{a_0 R_l}, \quad |c_l| \leq C_0 A_l. \quad (66)$$

Write $h_l = (I - P)w_l$ and impose the actual density hypothesis

$$\sup_{0 \leq t \leq T} \|h_l(t)\|_1 \leq \delta A_l R_l^2 \quad (l \leq i), \quad 0 < \delta \leq 1. \quad (67)$$

No nonradial hypothesis is imposed on $l > i$.

The scalar stochastic representation used below is

$$w_l(t, x) = \mathbb{E}[w_l(0, X_t)], \quad dX_s = -u^\nu(t-s, X_s) ds + \sqrt{2\nu} dB_s, \quad X_0 = x. \quad (68)$$

Its sign is verified by applying Itô's formula to $w_l(t-s, X_s)$: the drift is $-\partial_t w_l - u^\nu \cdot \nabla w_l + \nu \Delta w_l = 0$. For completeness, first stop at $t - \varepsilon$, where the coefficients are smooth. Boundedness of the drift gives a continuous limiting process at t . Its transition densities away from starting time are bounded by $C/[\nu s]$. This bound follows by L^1 contraction, the Nash inequality $\|f\|_2^4 \leq C \|f\|_1^2 \|\nabla f\|_2^2$, the L^2 energy identity, and the corresponding adjoint estimate, composed at the midpoint. Thus $w_l(\varepsilon) \rightarrow w_l(0)$ in L^1 suffices to pass to the endpoint in expectation. Approximation of the bounded initial colour in L^1 by continuous functions also justifies evaluation at the limiting endpoint. This verifies (68) for the actual colours without smoothing an asserted radial bound.

Proposition 6.3 (Uniform persistence of actual constant cores). *There is $C \geq 1$, depending only on $C_0, a_0, \beta_{\min}, \beta_{\max}$, with the following property. Set*

$$\mathcal{D} = \sqrt{\delta} + \frac{\delta q^2}{\beta - q^2} + \frac{\beta q}{1 - \beta q}, \quad c_* := \frac{9}{512}, \quad \varepsilon_* := \frac{a_0(\beta - 1)}{16CH}.$$

If

$$CH\mathcal{D} \leq \frac{a_0}{16}, \quad 4C_0 e^{-c_* a_0^2 / (\kappa_i H)} \leq \frac{\varepsilon_*}{2}, \quad (69)$$

then, with $a = a_0/2$, for $l \leq i$ and $0 < t \leq T$,

$$\|w_l(t) - c_l\|_{L^\infty(B_{aR_l})} \leq 4C_0 A_l e^{-c_* a_0^2 R_l^2 / (\nu t)}. \quad (70)$$

All constants are independent of i , viscosity, and the number of colours.

Proof. The maximum principle gives $\|h_l\|_\infty \leq 2C_0 A_l$. The radial average induces a purely tangential velocity, so (62) and (67) imply

$$|(K * w_l) \cdot e_r| \leq \|K * h_l\|_\infty \leq C\sqrt{\delta} A_l R_l \quad (l \leq i).$$

Every unrestricted inner colour has $\|K * w_k\|_\infty \leq C A_k R_k$, whence

$$\sum_{k>l} \|K * w_k\|_\infty \leq C A_l R_l \frac{\beta q}{1 - \beta q}. \quad (71)$$

Temporarily assume that all probe-core defects satisfy

$$\|w_k - c_k\|_{L^\infty(B_{aR_k})} \leq \varepsilon A_k.$$

For $k < l$ and $|z| \leq a_0 R_l$, one has $|z| \leq a R_k/2$. Decompose the older colour into

$$w_k = c_k \mathbf{1}_{B_{aR_k}} + (w_k - c_k) \mathbf{1}_{B_{aR_k}} + w_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}}.$$

The first term gives exactly $(c_k/2)Jz$. The second gives a velocity bounded by $C\varepsilon A_k |z|$ by Lemma 6.1. The radial part of the third has zero velocity inside the disk. Its remaining density is $h_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}}$, whose L^1 norm is at most $\delta A_k R_k^2$. Its exterior series starts at degree three, so

$$|K * (w_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}})(z)| \leq C\delta A_k |z|^3 / R_k^2. \quad (72)$$

Here and below the density is evaluated at its actual viscous time. Summing the self, inner and older bounds gives

$$|u^\nu(t, z) \cdot e_r| \leq C A_l R_l \left(\mathcal{D} + \frac{\varepsilon}{\beta - 1} \right) \quad (|z| \leq a_0 R_l, l \leq i). \quad (73)$$

We used the geometric sums

$$\sum_{k < l} \frac{A_k}{R_k^2} \leq \frac{A_l}{R_l^2} \frac{q^2}{\beta - q^2}, \quad \sum_{k < l} A_k \leq \frac{A_l}{\beta - 1}.$$

Fix C large enough for (73).

On a bootstrap interval with $\varepsilon \leq \varepsilon_*$, the integrated radial drift in the backward representation is at most $a_0 R_l / 8$, because $A_l T \leq H$ and (69) controls both terms. Start at $|x| \leq a_0 R_l / 2$ and stop on exiting $B_{a_0 R_l}$. If the backward path does not exit, its endpoint belongs to the initial constant core. The possible discrepancy outside it is at most $2C_0 A_l$. Lemma 6.2 therefore gives

$$|w_l(t, x) - c_l| \leq 4C_0 A_l \exp\left(-\frac{(3a_0 R_l / 8)^2}{8\nu t}\right),$$

which is (70). Subtracting c_l is legitimate: a constant is itself a bounded scalar solution, and bounded stochastic representation does not require its integrability.

The maximum normalized probe defect over the finite set $l \leq i$ is continuous for positive time. It tends to zero at time zero, since the probe has a fixed margin inside the initial constant disk and the common drift is bounded. If it first attained ε_* , the preceding estimate, $R_l \geq R_i$, $t \leq H/A_i$, and (69) would instead bound it by $\varepsilon_*/2$. This closes the bootstrap. The finite maximum adds no dependence on i ; every infinite inner sum was already controlled by (71). $\square$

Proposition 6.4 (Harmonic exterior field and actual density leak). *Under Proposition 6.3, set*

$$b_i = \sum_{l \leq i} K * w_l, \quad \Omega_i = \frac{1}{2} \sum_{l \leq i} c_l, \quad H_i = \sum_{l \leq i} K * (w_l \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_l}}), \quad L_i = \sum_{l \leq i} K * ((w_l - c_l) \mathbf{1}_{B_{aR_l}}).$$

For $|z| \leq aR_i/2$ one has the exact decomposition

$$b_i(t, z) = \Omega_i Jz + H_i(t, z) + L_i(t, z). \quad (74)$$

The field H_i is divergence free and curl free on B_{aR_i} , and

$$|H_i(t, z)| \leq C\delta A_i |z|^3 / R_i^2, \quad |DH_i(t, z)| \leq C\delta A_i |z|^2 / R_i^2, \quad (75)$$

$$|L_i(t, z)| \leq CA_i e^{-c_* a_0^2 / (\kappa_i H)} |z|. \quad (76)$$

Its cubic coefficient is exactly

$$c_{3,i}(t) = -\frac{i}{2\pi} \sum_{l \leq i} \int_{|y| \geq aR_l} \frac{w_l(t, y)}{\bar{y}^4} dy. \quad (77)$$

With $\eta_l(t) = \|h_l(t)\|_1 / (A_l R_l^2)$,

$$|c_{3,i}(t)| \leq C \frac{A_i}{R_i^2} \sum_{m=0}^{i-1} (q^2/\beta)^m \eta_{i-m}(t) \leq C\delta A_i / R_i^2. \quad (78)$$

Furthermore,

$$|H_i(t, z) - c_{3,i}(t) \bar{z}^3| \leq C\delta A_i |z|^7 / R_i^6, \quad (79)$$

$$|D(H_i(t, z) - c_{3,i}(t) \bar{z}^3)| \leq C\delta A_i |z|^6 / R_i^6. \quad (80)$$

The derivative denotes the real spatial Jacobian. No derivative bound on L_i is asserted.

Proof. The constant disks induce $(c_l/2)Jz$ in the specified region, proving (74). Exterior densities induce curl free velocities away from their supports, and all Biot–Savart velocities are divergence free. Radial averaging commutes with radial truncation, and an exterior radial density induces zero velocity inside its truncation radius. Consequently, on this region,

$$H_i(t, z) = -\frac{i}{2\pi} \sum_{l \leq i} \sum_{m \geq 0} \bar{z}^{4m+3} \int_{|y| \geq aR_l} \frac{h_l(t, y)}{\bar{y}^{4m+4}} dy.$$

The inverse radial moment is zero, so this gives exactly (77). The coefficient integral is at most

$$\frac{\|h_l\|_1}{(aR_l)^4}.$$

This proves (78).

The series and its first spatial derivatives converge uniformly on $|z| \leq aR_i/2$. Indeed each series ratio is at most $1/16$, and the derivative majorant adds only the factor $4m + 3$. Thus, for each colour, its harmonic field and derivative are bounded by $C\delta A_l |z|^3/R_l^2$ and $C\delta A_l |z|^2/R_l^2$, respectively. Removing $m = 0$ gives $C\delta A_l |z|^7/R_l^6$ and $C\delta A_l |z|^6/R_l^6$. Summing uses

$$\sum_{l \leq i} \frac{A_l}{R_l^d} \leq \frac{A_i/R_i^d}{1 - q^d/\beta} \quad (d = 2, 6).$$

These prove (75), (79), and (80). Finally Lemma 6.1, (70), and $\sum_{l \leq i} A_l \leq A_i \beta / (\beta - 1)$ prove (76). $\square$

Scope for the localized tail problem. If $R_* = \kappa_i^{1/4} R_i \leq aR_i/2$, then throughout B_{R_*} ,

$$|b_i - \Omega_i Jz| \leq C\delta A_i |z|^3/R_i^2 + CA_i e^{-c_* a_0^2/(\kappa_i H)} |z|.$$

The integrated linear-leak coefficient is at most $CH e^{-c_* a_0^2/(\kappa_i H)}$. The actual density leak is therefore quantified; no material hole has been retained by assumption. The harmonic derivative estimate (75) is available for a coupled transfer or energy calculation, while the core leak has only the growth estimate (76). The parameter margins are compatible with $H \asymp \log(1/q)$ and $\kappa_i = q^{3/2}$ if $\sqrt{\delta}H$ is sufficiently small. This compatibility does not prove (67); obtaining that hypothesis and controlling the coupled inner aggregate over its whole receiver window remain separate nonlinear tasks. In particular no fixed-data nonunique vanishing-viscosity theorem is concluded here.

7 A quantitative parabolic localization estimate

The next estimate concerns an actual convection–diffusion equation, and gives a quantitative substitute for compact material support. Its local cubic drift hypothesis must still be verified for a coupled Navier–Stokes cascade.

Lemma 7.1 (Localized tail exit). *Let f solve*

$$\partial_t f + U \cdot \nabla f = \nu \Delta f, \quad U(t, x) = \Omega(t)x^\perp + V(t, x) + b(t, x), \quad \operatorname{div} U = 0,$$

on $0 \leq t \leq T$, with $\|f(0)\|_1 \leq m$ and $\text{supp } f(0) \subset B_r$. Suppose

$$\begin{aligned} \|V(t)\|_\infty &\leq C_0 m (\nu t)^{-1/2}, \\ |b(t, x)| &\leq L|x| \quad \text{for all } x, \\ |b(t, x)| &\leq C_3 |x|^3 + \ell(t)|x| \quad \text{for } |x| \leq R_*. \end{aligned}$$

Assume $\ell \geq 0$, Ω is integrable, and

$$r \leq R_*/4, \quad C_3 R_*^2 T + \int_0^T \ell(t) dt \leq \frac{1}{4}, \quad C_0 \frac{m}{\nu} \sqrt{\nu T} \leq R_*/8.$$

There are absolute $c, C > 0$ such that

$$\sup_{t \leq T} \int_{|x| \geq R_*} |f(t, x)| dx \leq C m \exp\left(-\frac{c R_*^2}{\nu T}\right).$$

For each fixed integer $k \geq 0$ there are constants $c, C_k > 0$ such that

$$\int_{|x| \geq R_*} |x|^k |f(T, x)| dx \leq C_k m e^{C_k L T} R_*^k \exp\left(-\frac{c R_*^2}{\nu T}\right),$$

provided additionally $\sqrt{\nu T} \leq c R_*$. The conclusion holds for smooth drifts and for limits for which the parabolic solution and the corresponding diffusion representation are obtained by smooth approximation.

Proof. Kato's inequality bounds $|f|$ by the positive scalar solution from $|f(0)|$. Normalize this initial mass and represent the positive solution by the diffusion

$$dX_t = U(t, X_t) dt + \sqrt{2\nu} dB_t.$$

A time-dependent rotation removes $\Omega(t)x^\perp$; the rotated Brownian motion remains standard and all norm assumptions are unchanged. Let σ be the first exit from B_{R_*} . Before that exit, deterministic Gronwall applied to the integral stochastic equation gives

$$\sup_{s \leq t \wedge \sigma} |X_s| \leq e^{C_3 R_*^2 t + \int_0^t \ell} \left[r + 2C_0 \frac{m}{\nu} \sqrt{\nu t} + \sqrt{2\nu} \sup_{s \leq t} |B_s| \right].$$

The stated margins imply that $\sigma \leq T$ requires $\sup_{s \leq T} |B_s| \geq c R_*/\sqrt{\nu}$. The coordinate reflection bound for planar Brownian motion yields

$$\mathbb{P}(\sigma \leq T) \leq C \exp\left(-\frac{c R_*^2}{\nu T}\right).$$

An endpoint outside B_{R_*} requires this exit, proving the mass bound.

For the moment assertion, condition at σ . The remaining drift has linear growth bounded by $L|x|$ plus $\|V(t)\|_\infty$. Its integrated bounded contribution is at most $2C_0(m/\nu)\sqrt{\nu T} \leq R_*/4$. Gronwall and Brownian moments therefore give, uniformly in the exit time,

$$\mathbb{E}[|X_T|^k \mid \mathcal{F}_\sigma] \leq C_k e^{C_k L T} R_*^k \quad \text{on } \{\sigma \leq T\}.$$

Multiply by the indicator of this event and use its probability bound. Finally restore the initial mass, which is at most m . $\square$

For a receiver with $\kappa_i = \nu/(A_i R_i^2)$ and stage duration $T = H/A_i$, the choice $R_* = \kappa_i^{1/4} R_i$ gives

$$\frac{R_*^2}{\nu T} = \frac{1}{\kappa_i^{1/2} H}, \quad C_3 R_*^2 T \leq C x_i \kappa_i^{1/2} H \quad \text{if } C_3 \leq C A_i x_i / R_i^2.$$

A global bound $L \leq C A_i$ costs only $e^{C_k H}$, independent of depth. The exponential exit factor dominates that cost as $\kappa_i \downarrow 0$ for logarithmic stage lengths. This parameter calculation does not verify the required local cubic hypothesis for the actual foreign field.

8 Nonlinear inner-tail suppression through a receiver window

The estimate in this section is conditional on quantitative information about the actual foreign velocity. It closes a longer-time nonlinear subproblem than the isolated-tail and first-diffusion-time estimates. It does not establish the required nonradial-density bound for the active and older outer colours.

Write $J(x_1, x_2) = (-x_2, x_1)$, $z = x_1 + ix_2$, and

$$K(x) = \frac{Jx}{2\pi|x|^2}, \quad B_4(f) = \int_{\mathbb{R}^2} \bar{z}^4 f(z) \, dz.$$

A scalar is fourfold symmetric if $f(Jx) = f(x)$; a vector field has the corresponding covariance if $b(t, Jx) = Jb(t, x)$. Constants denoted by C can change from line to line.

8.1 A Gaussian-energy estimate for a globally weak foreign field

Proposition 8.1 (Weak foreign field through a finite window). *There are universal constants $c, \lambda_0, C > 0$ with the following property. Let $p > 2$, and let $f_0 \in L^1 \cap L^p$ be real, fourfold symmetric, supported in B_1 , and satisfy*

$$\int f_0 = 0, \quad B_4(f_0) = 0, \quad m = \|f_0\|_1.$$

Suppose b is divergence free, has fourfold covariance, and, after removal of a time-dependent solid rotation, satisfies

$$|b(s, x)| \leq \ell(s)|x|, \quad |b(s, x)| \leq d|x|^3, \quad (1+s)\ell(s) \leq \lambda_0.$$

Let f solve

$$\partial_s f + (K * f + b) \cdot \nabla f = \Delta f.$$

For $S \geq 2$, assume the displayed hypotheses on $0 \leq s \leq S-1$ and

$$m S^{1/25} \sqrt{1 + \log S} \leq c. \tag{81}$$

Then

$$\sup_{0 \leq s \leq S-1} |B_4(f(s))| \leq C \left[m^2 S^{7/50} (1 + \log S) + m d S^{102/25} \right]. \tag{82}$$

The constants are independent of a decomposition of the datum into smaller scales. The assertion holds for smooth drifts and for the parabolic limits obtained by approximation with the stated bounds.

Proof. Scalar contraction and the divergence-free Nash estimate give

$$\|f(s)\|_1 \leq m, \quad \|f(s)\|_\infty \leq Cm/s, \quad \|K * f(s)\|_\infty \leq Cm/\sqrt{s}.$$

The last estimate follows by splitting Biot–Savart at a radius and optimizing. Take $s_0 = 1/100$. The positive scalar solution from $|f_0|$ dominates $|f|$. Its diffusion representation, after removing solid rotation, obeys

$$|X_s| \leq e^{\lambda_0 s} \left[1 + Cm\sqrt{s} + \sqrt{2} \sup_{a \leq s} |B_a| \right] \quad (0 \leq s \leq s_0).$$

The Gaussian exponential moment of the Brownian supremum is finite at this fixed small time. With $\beta = 3/10$ it follows that

$$\int e^{\beta|x|^2/(1+s_0)} |f(s_0, x)| \, dx \leq Cm.$$

Multiplication by the Nash L^∞ bound gives a weighted L^2 norm $O(m)$. Fixed polynomial absolute moments are $O(m)$ on $[0, s_0]$. The exact harmonic-moment equation and symmetrization of self-advection therefore yield

$$|B_4(f(s_0))| \leq C(m^2 + md). \quad (83)$$

For clarity, the self contribution is bounded by $Cm \int |x|^2 |f|$, because $|\bar{z}^3 - \bar{y}^3|/|z - y| \leq C(|z|^2 + |y|^2)$; the foreign contribution is bounded by $d \int |x|^6 |f|$. These bounds also prove the result directly if the final time is smaller than s_0 .

For $s \geq s_0$ put

$$\tau = \log(1 + s), \quad y = x/\sqrt{1 + s}, \quad w(\tau, y) = (1 + s)f(s, x), \quad E(\tau, y) = \sqrt{1 + s}b(s, \sqrt{1 + s}y).$$

Then

$$w_\tau = \mathcal{L}w - (K * w + E) \cdot \nabla w, \quad \mathcal{L} = \Delta + \frac{1}{2}y \cdot \nabla + 1,$$

with $|E| \leq \lambda(\tau)|y|$, $\lambda \leq \lambda_0$, and $|E| \leq a(\tau)|y|^3$, where $a(\tau) = de^{2\tau}$. Let

$$F_\beta = \|e^{\beta|y|^2/2}w\|_2, \quad D_\beta = \|e^{\beta|y|^2/2}\nabla w\|_2^2, \quad \beta = 3/10.$$

Writing $g = e^{\beta|y|^2/2}w$ gives the exact identities

$$\begin{aligned} \langle w, \mathcal{L}w \rangle_\beta &= \frac{1}{2} \|g\|_2^2 - \|\nabla g\|_2^2 - \beta(1/2 - \beta) \| |y|g \|_2^2, \\ D_\beta &= \|\nabla g\|_2^2 + \beta^2 \| |y|g \|_2^2 + 2\beta \|g\|_2^2. \end{aligned}$$

For $a_\beta = \sqrt{\beta(1/2 - \beta)}$, completing squares in $\sum_j \|\partial_j g + a_\beta y_j g\|_2^2$ proves

$$\mathcal{H}_\beta := \|\nabla g\|_2^2 + \beta(1/2 - \beta) \| |y|g \|_2^2 \geq 2a_\beta \|g\|_2^2 = \frac{\sqrt{6}}{5} \|g\|_2^2.$$

Moreover $D_\beta \leq 3\mathcal{H}_\beta$: one can first bound it by $\frac{3}{2}\mathcal{H}_\beta + \frac{3}{5}\|g\|_2^2$ and then use the last inequality. Splitting $\mathcal{H}_\beta$ into $\frac{3}{50}\mathcal{H}_\beta + \frac{47}{50}\mathcal{H}_\beta$ yields

$$\langle w, \mathcal{L}w \rangle_\beta \leq -\frac{1}{50}D_\beta + \frac{1}{25}F_\beta^2. \quad (84)$$

Here $\frac{1}{2} - \frac{47}{50} \frac{\sqrt{6}}{5} < \frac{1}{25}$.

The Hardy–Littlewood–Sobolev estimate and Gaussian weighted Hölder give

$$\|K * w\|_4 \leq C \|w\|_{4/3} \leq C_\beta F_\beta.$$

The two-dimensional Gagliardo–Nirenberg inequality gives

$$\|g\|_4 \leq C \|g\|_2^{1/2} \|\nabla g\|_2^{1/2}.$$

Since $\|y|g\|_2 + \|\nabla g\|_2 \leq C_\beta \sqrt{D_\beta}$, self-advection contributes at most

$$CF_\beta^{3/2} D_\beta^{3/4} \leq C_\beta F_\beta D_\beta$$

to the weighted energy. The foreign contribution is $\beta \int E \cdot y e^{\beta|y|^2} w^2$, bounded by $C_\beta \lambda D_\beta$. Thus, while F_β is sufficiently small,

$$\frac{1}{2}(F_\beta^2)' \leq -\frac{1}{100} D_\beta + \frac{1}{25} F_\beta^2.$$

The initial weighted norm and a continuity bootstrap using (81) prove

$$F_\beta(\tau) \leq C m e^{\tau/25}, \quad \int_{\log(1+s_0)}^{\log S} D_\beta(\tau) d\tau \leq C m^2 S^{2/25}. \quad (85)$$

Let P be angular averaging, $G = Pw$, and $h = (I - P)w$. Every ordinary moment of h of degree at most three vanishes: angular averaging removes radial moments, and fourfold symmetry removes all other moments in these degrees. In

$$X = L^2(e^{|y|^2/4} dy), \quad H = \|h\|_X, \quad D_h = \|\nabla h\|_X^2,$$

one consequently has

$$D_h \geq 3H^2. \quad (86)$$

Indeed $\mathcal{L}$ is self-adjoint in X , and its Hermite basis is $\partial^\alpha \mathcal{G}$, with $\mathcal{G} = (4\pi)^{-1} e^{-|y|^2/4}$ and eigenvalue $-|\alpha|/2$. Orthogonality to degrees at most three is equivalent to the stated ordinary-moment cancellations, because the corresponding dual Hermite polynomials span these polynomial degrees. Expansion in this basis gives $\langle h, \mathcal{L}h \rangle_X \leq -2H^2$; since the left side is $-D_h + H^2$, (86) follows. This elementary Hermite expansion also follows from the tensor one-dimensional harmonic-oscillator factorization.

Angular averaging is orthogonal for all the radial weights used here, and commutes with $\mathcal{L}$. Writing $V_a = K * a$, the angular equation is

$$h_\tau = \mathcal{L}h - (I - P) \operatorname{div}[V_G h + V_h G + V_h h + EG + Eh].$$

The radial field V_G is tangential, so its weighted energy contribution is zero. The h – h and E – h contributions are bounded by CHD_h and $C\lambda D_h$. Put $D_G = \|\nabla G\|_X^2 \leq D_\beta$. Weighted integration by parts, unweighted HLS, and GN give

$$|\langle h, -\operatorname{div}(V_h G) \rangle_X| \leq CHF_\beta^{1/2} D_G^{1/4} D_h^{1/2} \leq \eta D_h + C_\eta H^2 F_\beta D_G^{1/2}.$$

The strict Gaussian margin $\beta - 1/4 = 1/20$ absorbs cubic multiplication:

$$\left\| e^{|y|^2/8} EG \right\|_2 \leq Ca(\tau) F_\beta.$$

Consequently the E - G contribution is bounded by $\eta D_h + C_\eta a(\tau)^2 F_\beta^2$. Take η and the bootstrap constants sufficiently small. Using (86) yields

$$(H^2)' \leq -\frac{19}{5}H^2 + CF_\beta D_G^{1/2} H^2 + Ca(\tau)^2 F_\beta^2.$$

By (85) and Cauchy–Schwarz,

$$\int F_\beta D_G^{1/2} d\tau \leq Cm^2 S^{2/25} \sqrt{1 + \log S} \leq Cc^2.$$

Variation of constants therefore gives

$$H(\tau) \leq Cm \left[e^{-19\tau/10} + de^{51\tau/25} \right]. \quad (87)$$

Factors depending on the fixed initial time are absorbed in C .

Finally $B_4(f(s)) = e^{2\tau} B_4(w(\tau))$. Harmonicity cancels diffusion, and differentiating $e^{2\tau}$ cancels the dilation term. Radial self-advection vanishes. HLS and Gaussian weighted Hölder show that

$$\left| \frac{d}{d\tau} B_4(f(e^\tau - 1)) \right| \leq Ce^{2\tau} [F_\beta H + a(\tau) F_\beta].$$

Substitution of (85) and (87), followed by integration, gives a first term $Cm^2 S^{1/25+1/10}(1+\log S)$ and remaining terms at most $CmdS^{102/25}$, using $m \leq 1$. Together with (83), this proves the proposition.

All energy tests can first be made with truncated weights and smooth data. The short-time exponential moment starts the weighted evolution; the uniform weighted estimates justify removal of the truncations and passage to the parabolic limit. No initial L^∞ norm enters a constant. $\square$

8.2 Comparison with the actual local foreign field

Proposition 8.2 (Local harmonic field and a rough small leak). *Let $\nu, r, T, R_* > 0$, $R_* \geq 4r$. Let f_0 have the properties of the preceding proposition at support radius r , with physical mass $M = \|f_0\|_1$. Let f solve*

$$f_t + (K * f + b) \cdot \nabla f = \nu \Delta f.$$

After removal of a solid rotation, suppose b is divergence free, has fourfold covariance, and admits, on B_{2R_} , the decomposition*

$$\begin{aligned} b &= b_h + e, & |b(t, x)| &\leq L|x| \quad \text{globally,} \\ |b_h(t, x)| &\leq C_3|x|^3, & |Db_h(t, x)| &\leq C_3|x|^2, \\ |e(t, x)| &\leq \epsilon L|x|. \end{aligned} \quad (88)$$

where b_h is divergence free with fourfold covariance and $0 \leq \epsilon \leq 1$. Its streamfunction is defined on B_{2R_} . Assume*

$$C_3 R_*^2 T + \epsilon LT \leq \frac{1}{4}, \quad C(M/\nu) \sqrt{\nu T} \leq R_*/8, \quad \sqrt{\nu T} \leq cR_*. \quad (89)$$

Take a fixed radial cutoff of that streamfunction, equal to one in B_{R_} and zero outside B_{2R_*} , and denote its velocity by $\tilde{b}$. Let g solve the same self-consistent equation as f , with foreign field $\tilde{b}$ and the same initial datum. Set $\ell = CC_3 R_*^2$. If M/ν and $\ell(r^2/\nu + T)$ are sufficiently small, then*

$$\begin{aligned} \sup_{t \leq T} |B_4(f(t)) - B_4(g(t))| &\leq CM R_*^4 e^{CLT} \left[\epsilon L(T + r\sqrt{T/\nu}) \right. \\ &\quad \left. + (1 + LR_* \sqrt{T/\nu}) e^{-cR_*^2/(\nu T)} \right]. \end{aligned} \quad (90)$$

No derivative bound on the leak e is required.

Proof. Choose the streamfunction ψ_h zero at the origin. Integration along radial segments gives $|\psi_h(x)| \leq CC_3|x|^4$. A streamfunction cutoff gives

$$|\tilde{b}| \leq CC_3|x|^3, \quad |\tilde{b}| \leq \ell|x|, \quad \|D\tilde{b}\|_\infty \leq \ell.$$

It is divergence free, has fourfold covariance, and equals b_h in B_{R_*} . A second useful bound is $|\tilde{b}(x)| \leq CL|x|$: from $\epsilon \leq 1$ and (88), $|b_h| \leq 2L|x|$ on B_{2R_*} , so $|\psi_h| \leq CL|x|^2$ there, and cutoff derivatives preserve this linear growth. Thus this bound does not require $C_3R_*^2 \leq L$.

We first verify the velocity-gradient estimate needed for comparison. Normalize $r = \nu = 1$ and write $m = M/\nu$; the cutoff growth and Lipschitz constant are normalized in the same way. Nash gives

$$\|g(t)\|_\infty \leq Cm/t, \quad \|u_g(t)\|_\infty \leq Cm/\sqrt{t}, \quad u_g = K * g.$$

For $M_k(t) = \int |x|^k |g(t, x)| \, dx$, Kato's inequality gives

$$\begin{aligned} M_2' &\leq 4m + Cmt^{-1/2}\sqrt{mM_2} + 2\ell M_2, \\ M_4' &\leq 16M_2 + Cmt^{-1/2}m^{1/4}M_4^{3/4} + 4\ell M_4. \end{aligned}$$

Since $M_2(0), M_4(0) \leq m$, a continuity bootstrap gives

$$M_2(t) \leq Cm(1+t), \quad M_4(t) \leq Cm(1+t)^2$$

when m and $\ell(1+T)$ are small. On these trial bounds the integrated self terms are respectively $O(m^2(1+t))$ and $O(m^2(1+t)^2)$; $\ell T \ll 1$ absorbs the foreign terms. Hence

$$\|xg(t)\|_1 \leq Cm\sqrt{1+t}, \quad \|xg(t)\|_4 \leq Cmt^{-3/4}\sqrt{1+t}.$$

The second follows from $\|xg\|_4^4 \leq \|g\|_\infty^3 M_4$.

With H_t the heat kernel, scaling and the smooth kernels' exterior decay give

$$\begin{aligned} \|\nabla K * H_t\|_\infty &\leq Ct^{-1}, \\ \|\nabla K * \operatorname{div} H_t\|_\infty &\leq Ct^{-3/2}, \quad \|\nabla K * \operatorname{div} H_t\|_1 \leq Ct^{-1/2}, \quad \|\nabla K * \operatorname{div} H_t\|_{4/3} \leq Ct^{-3/4}. \end{aligned}$$

For the nonlinear differentiated kernel the tail is $O(|x|^{-3})$, so the last three norms are finite before scaling. The self flux $u_g g$ has L^1 norm $Cm^2t^{-1/2}$ and L^∞ norm $Cm^2t^{-3/2}$. Split the heat-mild time integral at $t/2$, using the kernel's L^∞ norm first and its L^1 norm second; its contribution to ∇u_g is Cm^2/t . The foreign flux has L^1 norm $C\ell m\sqrt{1+t}$ and L^4 norm $C\ell m t^{-3/4}\sqrt{1+t}$. Use the kernel's L^∞ norm first and its $L^{4/3}$ norm second. This contribution is

$$C\ell m t^{-1/2}\sqrt{1+t} \leq Cm/t,$$

because $\ell(1+T)$ is small. The initial heat contribution is Cm/t . Undoing normalization proves

$$\|\nabla u_g(t)\|_\infty \leq C(M/\nu)/t. \tag{91}$$

Put $\delta = f - g$, $v = K * \delta$, and $F = (b - \tilde{b})f$. Then

$$\delta_t - \nu \Delta \delta + (u_f + \tilde{b}) \cdot \nabla \delta + v \cdot \nabla g = -\operatorname{div} F.$$

Let ψ be a streamfunction of δ . Testing in H^{-1} gives

$$\frac{1}{2}(\|v\|_2^2)' + \nu \|\delta\|_2^2 \leq (\|\nabla u_g\|_\infty + \|D\tilde{b}\|_\infty) \|v\|_2^2 + \|v\|_2 \|F\|_2. \tag{92}$$

Here $v \cdot \nabla \psi = 0$. Thus the term $v \cdot \nabla g$ vanishes when paired with ψ , and the same identity replaces $u_f = u_g + v$ by u_g in the other self transport term. Integration by parts bounds that term by $\|\nabla u_g\|_\infty \|v\|_2^2$. The identical calculation applies to $\tilde{b}$. The circulation of δ is zero, and its polynomial moments are finite, so $v \in L^2$ and these tests are justified by approximation.

The true coefficient in (92) is integrable at time zero, although the upper bound (91) is singular there. Indeed $g_0 \in L^p$, $p > 2$, gives from the same mild formula

$$\|\nabla u_g(t)\|_\infty \leq C_0(t^{-1/p} + t^{1/2-1/p}), \quad 0 < t \leq 1.$$

The constant may depend on the actual initial L^p norm and on the bounded cutoff drift. The kernel norms are $Ct^{-1/p}$ and $Ct^{-1/2-1/p}$ in L^p ; the latter time singularity is integrable, and $(u_g + \tilde{b})g$ has bounded L^p norm near zero by contraction and the initial L^1 - L^p Biot-Savart bound. Therefore the integrating factor in (92) starts from $v(0) = 0$. Only after writing its source integral do we use (91) on $[s, t]$, obtaining

$$\|v(t)\|_2 \leq \int_0^t (t/s)^{CM/\nu} e^{\ell(t-s)} \|F(s)\|_2 \, ds. \quad (93)$$

Common smooth approximations have bounded initial L^p norms, so no arbitrary initial term at a positive time is discarded in this argument.

The localization estimate, applied at each time $s \leq T$, gives

$$\int_{|x| \geq R_*} |x|^2 |f(s, x)| \, dx \leq CM e^{CLs} R_*^2 e^{-cR_*^2/(\nu s)}.$$

The cutoff field and the actual field have global growth $CL|x|$. Combining this tail estimate with Nash yields

$$\|(b - \tilde{b})f \mathbf{1}_{|x| \geq R_*}\|_2 \leq CLM \frac{R_*}{\sqrt{\nu s}} e^{CLs} e^{-cR_*^2/(\nu s)}.$$

Inside B_{R_*} the difference of drifts is exactly e . A weighted Kato estimate gives $\int |x|^2 |f(s)| \leq CM(r^2 + \nu s)e^{CLs}$. Consequently

$$\|ef \mathbf{1}_{|x| < R_*}\|_2 \leq C\epsilon LM \sqrt{\frac{r^2 + \nu s}{\nu s}} e^{CLs}.$$

Substitute these bounds into (93). Since $CM/\nu < 1/4$, integration of $s^{-1/2-CM/\nu}$ is legitimate, and yields, uniformly for $t \leq T$,

$$\|v(t)\|_2 \leq CM e^{CLT} \left[\epsilon L(T + r\sqrt{T/\nu}) + LR_* \sqrt{T/\nu} e^{-cR_*^2/(\nu T)} \right]. \quad (94)$$

The bounded factor $e^{\ell T}$ is absorbed into C .

For a radial cutoff χ_* equal to one in B_{R_*} and zero outside B_{2R_*} ,

$$\|\nabla(\chi_* \bar{z}^4)\|_2 \leq CR_*^4, \quad \left| \int \chi_* \bar{z}^4 \delta \right| \leq CR_*^4 \|v\|_2.$$

Both f and g satisfy the localized exit estimate, with the margins (89). Their exterior absolute fourth moments are at most $CMR_*^4 e^{CLT} e^{-cR_*^2/(\nu T)}$. Combining this with (94) proves (90). $\square$

8.3 Physical normalization, the available power margin, and scope

Set

$$m = M/\nu, \quad d = CC_3r^4/\nu, \quad S = 2 + \nu T/r^2.$$

The space-time change $x = ry$, $t = r^2s/\nu$, and $f(t, x) = \nu r^{-2}F(s, y)$ gives these quantities exactly; moreover $B_4(f) = \nu r^4 B_4(F)$. Applying Proposition 8.1 to the cutoff-field solution and Proposition 8.2 to the actual solution gives

$$\begin{aligned} \sup_{t \leq T} |B_4(f(t))| &\leq CMr^4 \left[\frac{M}{\nu} S^{7/50} (1 + \log S) + \frac{C_3 r^4}{\nu} S^{102/25} \right] \\ &\quad + CMR_*^4 e^{CLT} \left[\epsilon L(T + r\sqrt{T/\nu}) + (1 + LR_*\sqrt{T/\nu}) e^{-cR_*^2/(\nu T)} \right]. \end{aligned} \quad (95)$$

This requires (81) and the small normalized linear-growth bound of the cutoff field in addition to the local hypotheses. No cutoff has been applied to the actual nonlinear solution.

For the geometric cascade budget take a receiver (A_i, R_i) and

$$r = qR_i, \quad M \leq C\beta A_i q^2 R_i^2, \quad \kappa = \frac{\nu}{A_i R_i^2} = q^{3/2}, \quad T = H/A_i, \quad R_* = \kappa^{1/4} R_i.$$

Suppose the actual outer-colour estimates furnish

$$L \leq CA_i, \quad C_3 \leq C\delta A_i/R_i^2, \quad \epsilon \leq Ce^{-c/(\kappa H)}.$$

Then $m = O(\beta q^{1/2})$, $d = O(\delta q^{5/2})$, and $S = O(q^{-1/2}H)$. Dividing the first line of (95) by $A_i R_i^6$ gives

$$O\left(\beta^2 q^{643/100} H^{7/50} (1 + \log S) + \beta \delta q^{323/50} H^{102/25}\right). \quad (96)$$

The powers are 6.43 and 6.46. The localization terms are smaller than every q -power for logarithmic H , since

$$\frac{R_*^2}{\nu T} = \frac{1}{\sqrt{\kappa H}}, \quad LT = O(H), \quad C_3 R_*^2 T = O(\delta \sqrt{\kappa H}).$$

The other margins, including $(M/\nu)\sqrt{\nu T} \ll R_*$ and $mS^{1/25}\sqrt{1 + \log S} \ll 1$, hold for sufficiently small q . Both terms in (96) are consequently $o(q^6)$. A fixed prefactor in the support radius changes only the constants.

The actual outer-core argument supplies the required harmonic-plus-leak hypothesis conditionally on the density bound

$$\sup_{t \leq T} \|(I - P)w_l(t)\|_1 \leq \delta A_l R_l^2, \quad l \leq i.$$

Its harmonic exterior multipole series can be differentiated to give $|Db_h(x)| \leq CC_3|x|^2$ throughout B_{2R_*} when this disk lies strictly inside the constant-core probe region; the exponentially small core density defect contributes the rough leak. Thus the present section closes full-window nonlinear inner-tail suppression under this explicit outer-density input. It does not prove that input for the active colour and older seeds, a signed outward transfer, or distinct vanishing-viscosity accumulation paths.

Remark 8.3 (Selected-cutoff scope). The displayed $q^{6.43}$ and $q^{6.46}$ budget uses $\kappa_i = q^{3/2}$ at the selected cutoff level. At later, larger receivers the same physical viscosity has a smaller normalized value; their aggregate of smaller colours includes the intended active child. The budget is not an erasure estimate for that entire aggregate at every later stage. After a source leaves its small cap neighbourhood, its constant-core probe on a canonical source window requires a separate comparison; it is not a consequence of the small-density hypothesis over that post-cap window.

9 Fixed seeds: separate initial and forced prefix contributions

The result in this section is a majorant lemma. Its coupling inequality is an explicit hypothesis; no assertion that diffusive colours already satisfy that inequality is made.

Lemma 9.1 (Two prefix norms). *Let $\beta \geq 4$, $A_i = \beta^{i-1}$, and fix a receiving index $j \geq 2$. Suppose nonnegative continuous functions x_i , $1 \leq i < j$, satisfy, on $0 \leq \theta \leq S$,*

$$x_i(\theta) \leq Mbe^{\lambda_0\theta} + M \int_0^\theta e^{\lambda_0(\theta-s)} \left(\sum_{l < j} K_{il} x_l(s) + f_i(s) \right) ds, \quad \lambda_0 = \frac{\sigma + \eta}{\beta}, \quad (97)$$

where $M, C, \sigma > 0$, $\eta, r_* > 0$, $f_i \geq 0$ are integrable, and the nonnegative matrix K obeys

$$K_{ii} \leq Cr_*/\beta, \quad K_{il} \leq \begin{cases} C\beta^{l-j}q^{2(i-l)}, & l < i, \\ C\beta^{l-j}q^{6(l-i)}, & i < l < j. \end{cases}$$

Assume $\beta q^6 < 1$. Define

$$r_B = \frac{C}{\beta} \left(r_* + \frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right), \quad \lambda_B = \lambda_0 + Mr_B.$$

Then $x_i \leq h_i + p_i$, where the initial contribution satisfies

$$\max_{i < j} h_i(\theta) \leq Mbe^{\lambda_B\theta}. \quad (98)$$

For $D > \beta$ with $Dq^8 < \beta$, put $w_i = (Dq^6)^{j-i}$ and

$$r_P = \frac{C}{\beta} \left(r_* + \frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right), \quad \lambda_P = \lambda_0 + Mr_P.$$

The forced contribution satisfies

$$P(\theta) := \max_{i < j} \frac{p_i(\theta)}{w_i} \leq M \int_0^\theta e^{\lambda_P(\theta-s)} \max_{i < j} \frac{f_i(s)}{w_i} ds. \quad (99)$$

All constants are independent of j . One can choose η, r_* small, then D large and q small, so both λ_B and λ_P are strictly smaller than $\sigma/2$.

Proof. The row sums of K are bounded by r_B : for $l = i + d$ the inner coefficient is at most $C\beta^{i-j}(\beta q^6)^d$, and for $l = i - d$ the outer coefficient is at most $C\beta^{i-j}(q^2/\beta)^d$. Use $\beta^{i-j} \leq 1/\beta$ and sum the geometric series.

Solve the positive comparison system

$$Y' = (\lambda_0 I + MK)Y + Mf, \quad Y(0) = Mb\mathbf{1}.$$

Iteration of (97), or its integral comparison principle, gives $x \leq Y$. The variation-of-constants formula splits $Y = h + p$, with

$$h = e^{(\lambda_0 I + MK)\theta} Mb\mathbf{1}, \quad p = M \int_0^\theta e^{(\lambda_0 I + MK)(\theta-s)} f(s) ds.$$

The unweighted maximum norm gives (98). For the weighted maximum norm, the inner row acquires $(\beta/D)^d$ and the outer row acquires $(Dq^8/\beta)^d$. Their geometric sums give r_P and hence (99). Since λ_0 is close to $\sigma/\beta \leq \sigma/4$, first make the diagonal term small. Increasing D makes the remaining q -independent term in r_P small; subsequently decreasing q makes all other terms small. $\square$

Corollary 9.2 (The fixed-seed baseline is small in its actual field). *Assume the hypotheses of Lemma 9.1, with $\lambda_B \leq \lambda_* < \sigma/2$, in the clock $\theta = A_j t$. Suppose the completed cap clocks satisfy*

$$T_{j,N} \leq S/A_N + H \sum_{m=j}^{N-1} A_m^{-1}, \quad N \geq j, \quad S \geq H\beta/(\beta - 1).$$

Let $b = \delta e^{-\sigma S}$. Then through $T_{j,N}$ the initial contribution is bounded uniformly by

$$B = M\delta e^{-(\sigma - \lambda_*)S}.$$

Its normalized inward feedback on the receiver j is at most

$$C \sum_{i < j} \beta^{i-j} q^{2(j-i)} h_i \leq CB \frac{q^2/\beta}{1 - q^2/\beta}. \quad (100)$$

The forced contribution has feedback at most

$$CP \frac{Dq^8/\beta}{1 - Dq^8/\beta}.$$

In particular, with $H = h_0 \log(1/q)$ one can take $S = C_0 \log(1/q)$ so that $B = O(q^6)$ and its feedback is $O(q^8)$, with constants independent of depth.

Proof. Writing $a = \beta^{j-N} \in (0, 1]$ gives

$$A_j T_{j,N} \leq Sa + \frac{H\beta}{\beta - 1} (1 - a) \leq S.$$

Apply (98). The actual inward row in (100) is a geometric series with ratio q^2/β . For $p_i \leq w_i P$ its ratio is Dq^8/β . Choose $C_0 \geq h_0\beta/(\beta - 1)$ and $C_0(\sigma - \lambda_*) > 6$. No weighted norm of the initial seeds is used. If $Dq^6 < 1$, the local chart bound follows from $x_i \leq B + w_i P \leq B + Dq^6 P$ whenever the latter quantity is below r_* . $\square$

The corollary repairs a specific scalar obstruction. To use it for Navier–Stokes one must prove (97), including its spatial q^2 and q^6 rows for diffusive colours, and bound the forced term P . Those remain coupled PDE hypotheses.

Remark 9.3 (A generic smooth-profile self rate). The same positive-matrix proof works with $\lambda_0 = \gamma/\beta$ for any fixed $\gamma > 0$. Choose $\beta > 4\gamma/\sigma$, then choose the foreign-row parameters so Mr_B and Mr_P are less than $\sigma/4 - \gamma/\beta$. For a differential inequality whose diagonal rate is at most $\gamma\beta^{i-j}$, use γ/β as its common upper bound. This permits a generic fixed-profile Kato rate; it does not identify that rate with the sharp unstable eigenvalue. The transform selection in Lemma 4.4 permits choosing a good β in any fixed sufficiently large interval, so this parameter change causes no conflict with the isolated source transform.

10 A fixed passive colour is determined by the total Euler path

The next result allows signed smooth colours as well as indicator colours. It therefore does not require a patch reference profile.

Proposition 10.1 (Recovery of fixed passive colours). *Let u be bounded and divergence free on $[0, T] \times \mathbb{R}^2$, with $u \in L^1([0, T]; W_{\text{loc}}^{1,p})$ for some $p > 1$. For each compactly supported $f_0 \in L^1 \cap L^\infty$ there is at most one solution of*

$$\partial_t f + u \cdot \nabla f = 0, \quad f(0) = f_0,$$

in $L^\infty([0, T]; L^1 \cap L^\infty)$ with its weak initial trace. It is renormalized and belongs to $C([0, T]; L^2)$.

Suppose $u_n \rightarrow u$ uniformly, $\sup_n \|u_n\|_\infty \leq V$, $\nu_n \rightarrow 0$, and $0 \leq \nu_n \leq 1$. Let f_n solve

$$\partial_t f_n + u_n \cdot \nabla f_n = \nu_n \Delta f_n, \quad f_n(0) = f_0,$$

with the L^1 , L^2 , and L^∞ contraction bounds and the weak-form equation. Then their transport limit f is unique and

$$f_n \rightarrow f \quad \text{in } C([0, T]; L^1 \cap L^2).$$

The same conclusion holds for transported colours with $\nu_n = 0$.

Proof. We give the renormalization argument needed for uniqueness. Let $f_\varepsilon = f * \eta_\varepsilon$ be a spatial mollification. It satisfies

$$\partial_t f_\varepsilon + u \cdot \nabla f_\varepsilon = r_\varepsilon, \quad r_\varepsilon(x) = \int \nabla \eta_\varepsilon(y) \cdot [u(x) - u(x - y)] f(x - y) dy.$$

On a compact spatial set the difference-quotient estimate bounds its L^1 norm by $C \|\nabla u\|_p \|f\|_{p'}$, on a slightly larger set. This follows by writing the velocity difference as the integral of its gradient along the segment and then applying Hölder's inequality. Approximate u in $L_t^1 W_{\text{loc}}^{1,p}$ by divergence-free spatial convolutions. For a smooth approximating velocity, divergence freedom allows subtraction of $f(x)$ in the commutator integral. Translation continuity of f then makes the commutator tend to zero in $L_{t,x}^1$. The uniform difference-quotient bound handles the velocity approximation. Thus $r_\varepsilon \rightarrow 0$ locally in L^1 .

Multiplication by a bounded derivative of a C^1 function and passage to the limit shows that bounded renormalizations satisfy the transport equation, including their initial trace. For the difference of two bounded solutions, use a renormalization equal to s^2 on their bounded value range. Integration against expanding cutoffs has a flux bounded by $C \|u\|_\infty R^{-1} \int |f|^2$. Since bounded integrable functions are square integrable, that flux tends to zero. The initial square norm is zero, which proves uniqueness. The same argument preserves the square norm of one solution. Weak time continuity and preservation of that norm give strong L^2 time continuity.

For convergence, the weak-form identity gives, for compact smooth φ ,

$$\left| \frac{d}{dt} \int \varphi f_n \right| \leq \|f_0\|_1 (V \|\nabla \varphi\|_\infty + \|\Delta \varphi\|_\infty).$$

The contraction bounds and this equicontinuity give weak L^2 convergence uniformly in time along subsequences. The Kato inequality and a radial cutoff also give uniform tightness:

$$\sup_{t \leq T} \int_{|x| > 2R} |f_n(t, x)| dx \leq \int_{|x| > R} |f_0(x)| dx + CT \|f_0\|_1 (V/R + 1/R^2).$$

Passing to the distributional equation identifies every limit as a bounded integrable transported solution with the given initial trace. Uniqueness identifies all subsequences.

To prove uniform strong L^2 convergence, suppose otherwise and choose bad times $t_n \rightarrow t$. Weak time equicontinuity gives $f_n(t_n) \rightharpoonup f(t)$ in L^2 , while contraction and renormalization give

$$\limsup_n \|f_n(t_n)\|_2 \leq \|f_0\|_2 = \|f(t)\|_2.$$

Hilbert-space uniform convexity gives strong convergence, a contradiction. Local L^1 convergence follows from L^2 , and tightness gives global uniform L^1 convergence. $\square$

For $p > 2$ the kernel split gives the explicit useful bound

$$\|K * h\|_\infty \leq C_p(\|h\|_1 + \|h\|_p) : \quad (101)$$

the near kernel belongs to $L^{p'}(B_1)$ because $p' < 2$, and the far kernel is bounded. Thus strong convergence of total vorticity paths in $C_t(L^1 \cap L^p)$ implies uniform convergence of their velocities. Whenever the limiting velocities have the Sobolev regularity in Proposition 10.1, the proposition applies to every fixed bounded initial colour. In particular, if two total Euler paths coincide, their recovered fixed colours coincide. No uniqueness of the total nonlinear Euler equation is used.

Let $f_{1,0}, \dots, f_{k,0}$ be finitely many fixed bounded compact colours and choose complex constants c_l . A suitable observable is

$$a_u(t) = \sum_{l=1}^k c_l \int \Phi_l(x) f_l[u](t, x) dx, \quad \Phi_l(x) = \chi_l(|x|) \bar{z}^4, \quad (102)$$

where each χ_l is a fixed smooth radial compact cutoff. The weak-form estimate makes a_u uniformly Lipschitz for a bounded family of drifts and $\nu \leq 1$, and the proposition makes it continuous under the strong total-path convergence just specified. These tests cover one smooth outer colour and also a moment coordinate made from several patch colours. If uncut moments are wanted in an Euler limit, the cutoffs can equal one on a common support ball: bounded velocity gives finite propagation. Indeed test $|f|$ against a nondecreasing smooth function of $|x| - R_0 - Vt$; its material derivative is nonpositive and its initial pairing is zero.

11 Protected first caps and phase shooting

A phase evaluated at an arbitrary cap time need not distinguish trajectories. The following first-cap functional does distinguish them and requires no common cap clock.

Lemma 11.1 (Continuity of a protected first-cap phase). *Fix $\delta, c, e, h, L > 0$, with $2e < T$. Let $a : [0, T] \rightarrow \mathbb{C}$ be L -Lipschitz, let*

$$\tau(a) = \inf\{t \geq 0 : |a(t)| \geq \delta\}, \quad \mathcal{P}(a) = a(\tau(a))/\delta,$$

and suppose

$$\begin{aligned} |a(0)| &\leq \delta - h, & e &\leq \tau(a) \leq T - e, \\ |a(t)| &\leq \delta - h & (0 \leq t \leq \tau(a) - e), \\ |a(t)| - |a(s)| &\geq c(t - s) & (\tau(a) - e \leq s < t \leq \tau(a) + e). \end{aligned}$$

For any continuous b with $\varepsilon = \|a - b\|_\infty < \min(h/2, ce/4)$, its first cap exists,

$$|\tau(a) - \tau(b)| \leq 2\varepsilon/c, \quad |\mathcal{P}(a) - \mathcal{P}(b)| \leq \delta^{-1}(1 + 2L/c)\varepsilon.$$

Families satisfying the displayed conditions with common constants retain a protected first cap under uniform convergence. Hence $\mathcal{P}$ is a continuous functional on that protected class.

Proof. Before $\tau(a) - \varepsilon$ the strict margin excludes any cap of b . On $[\tau(a) - \varepsilon, \tau(a)]$ the radial lower-slope inequality gives $|a(t)| \leq \delta - c(\tau(a) - t)$. Thus b cannot cap before $\tau(a) - 2\varepsilon/c$. At $\tau(a) + 2\varepsilon/c$, its modulus is greater than or equal to $\delta + \varepsilon$, so it has capped by that time. This proves the time estimate. Since both first-cap values have modulus δ , the phase estimate follows from

$$|a(\tau(a)) - b(\tau(b))| \leq L|\tau(a) - \tau(b)| + \|a - b\|_\infty.$$

For a uniformly convergent protected sequence, take a convergent subsequence of its cap times. The common Lipschitz bound, the precap margin, and the integrated radial lower-slope inequality pass to the limit. They identify the limiting time as the unique first cap. This argument applies to every subsequence and proves the final assertion. $\square$

Corollary 11.2 (Separation of total inviscid paths). *Consider one fixed total initial datum and fixed bounded initial colours. Suppose two Navier–Stokes subsequences converge strongly in $C_t(L^1 \cap L^p)$, their limits satisfy the velocity regularity of Proposition 10.1, and their observables (102) have uniformly protected first caps. If their limiting first-cap phases are different, the two total Euler paths are different. The cap times may differ.*

Proof. Proposition 10.1 and Lemma 11.1 pass the colours, observables, and phases to the limit. Equal total paths would have equal velocities and recovered colours, hence equal first-cap phases, a contradiction. $\square$

Proposition 11.3 (Conditional degree shooting). *Fix $\nu > 0$ and an active index $N \geq 2$. Suppose there is a continuous initial-data family indexed by $(r, \theta) \in [0, 1] \times \mathbb{S}^1$: the active seed at N has phase θ , and all other seed amplitudes are multiplied by r . Assume the total and passive-colour solution paths at this fixed viscosity depend continuously on these parameters. Assume their outer observables have first caps protected by the common constants of Lemma 11.1.*

If at $r = 0$ all other colours are radial and the active seed family is obtained by spatial rotation through angle $-\theta/4$, then the output phase map at $r = 1$ is surjective onto $\mathbb{S}^1$.

Proof. The first-cap map is continuous by Lemma 11.1. At $r = 0$, rotational covariance and the radial cutoffs in (102) give

$$\mathcal{P}(a_{0,\theta}) = e^{i\theta} \mathcal{P}(a_{0,0}).$$

Its degree is one. The degree is constant along the homotopy in r : a continuous argument lifted on the rectangle $[0, 1] \times [0, 2\pi]$ has endpoint difference $2\pi d(r)$, and the continuous integer $d(r)$ is constant. A circle map omitting one point admits a periodic continuous argument and has degree zero. The map at $r = 1$ therefore cannot omit a point. $\square$

The continuity hypothesis at fixed positive viscosity is routine and does not hide a uniform-in-viscosity estimate. For completeness, in $X = L^1 \cap L^p$, (101) makes $(K * w)w$ locally Lipschitz on bounded X balls. The mild vorticity equation gives

$$d(t) \leq d(0) + \frac{C_p M}{\sqrt{\nu}} \int_0^t (t-s)^{-1/2} d(s) ds.$$

Short intervals on which the integral coefficient is at most $1/2$, followed by iteration, prove continuity for each fixed ν . The passive mild equation has the same kernel and an additional source controlled by the uniform drift difference. It gives continuity of colours, including when their initial bounded domains vary continuously. Uniform cap protection across the homotopy, by contrast, is a substantive coupled PDE hypothesis.

Proposition 11.4 (Conditional sparse-phase construction). *Fix $p > 2$ and $\beta q^{2/p} < 1$. Let a family of infinitely seeded, zero-circulation compact initial data be indexed by a phase at every depth. Assume its series converges absolutely in $X = L^1 \cap L^p$, that changing only the phase at depth N changes the datum by at most $\eta_N \rightarrow 0$, uniformly in all phases, and that depths $N \geq 2$ leave the fixed outer initial colours unchanged.*

Suppose $\nu_N \downarrow 0$, and for every depth and every choice of all other phases, Proposition 11.3 applies with common cap-protection constants. Finally suppose that for every datum in this family its Navier–Stokes family is relatively compact in $C_t X$, and every limit satisfies the velocity hypothesis of Proposition 10.1. Then some single datum in this family has two distinct vanishing-viscosity accumulation paths.

Proof. Start with arbitrary default phases at all depths. At step k , choose a new depth $N_k > N_{k-1}$ with $\nu_{N_k} < \nu_{N_{k-1}}/2$ and

$$\eta_{N_k} < 2^{-k} \min(1, r_1, \dots, r_{k-1}).$$

Shoot its phase to give first-cap output $(-1)^k$ at ν_{N_k} , using Proposition 11.3, and call the resulting datum g_k . Fixed-positive-viscosity continuity gives $r_k > 0$ such that an admissible datum within r_k of g_k changes that output by less than $1/4$. All admissible future phase choices already have protected caps by hypothesis; openness of a derivative lower bound in the uniform topology is not asserted.

The summable phase changes give one limiting datum g_∞ ; each coordinate is changed at most once and the absolutely convergent series identifies the limit with that phase sequence. For each k ,

$$\|g_\infty - g_k\|_X \leq \sum_{l>k} 2^{-l} r_k < r_k.$$

Hence its selected-viscosity output lies within $1/4$ of $(-1)^k$. Relative compactness supplies limiting paths along each parity. Their first-cap phases lie in the disjoint closed $1/4$ -neighbourhoods of $+1$ and -1 . Corollary 11.2 proves that the total paths are different. Zero circulation and compact support pass by strong L^1 convergence. $\square$

The proposition is an implication with explicit uniform cap and coupling hypotheses. It is not an unconditional Navier–Stokes counterexample. Every intermediate datum retains the full unbounded infinite background; the proof does not assert arbitrary-small macroscopic controllability around a bounded finite prefix.

12 Actual inactive densities and a large amplitude ratio

Fix the smooth radial reference W and $p > 2$. The estimate below uses the actual coloured Navier–Stokes equation and requires no derivative bound on an inactive colour. Its input is a weighted history bound for the current and deeper packet. That input remains part of the interacting active theorem, and is not proved in this section.

For clarity, the amplitude ratio can be chosen large after W and p are fixed. The interval $[4, 5]$ used in the earlier parameter discussion is not imposed here. A large fixed ratio compensates for the ordinary transport-stability constant of the smooth reference; it is not asserted that this constant equals its unstable eigenvalue.

Let P denote angular averaging. Use

$$\begin{aligned} A_l &= A_1 \beta^{l-1}, & R_l &= R_1 q^{l-1}, \\ G_l(t, x) &= A_l \left[e^{\nu t R_l^{-2} \Delta} W \right] (x/R_l), \\ \zeta_l &= w_l - G_l, & h_l &= (I - P)w_l = (I - P)\zeta_l. \end{aligned}$$

For any density v put

$$\|v\|_{X_l} = \frac{\|v\|_1}{A_l R_l^2} + \frac{\|v\|_p}{A_l R_l^{2/p}}, \quad x_l(t) = \|\zeta_l(t)\|_{X_l}.$$

Assume the initial colours are real, fourfold symmetric, have zero circulation, are supported in $B_{c_0 R_l}$, equal a constant in $B_{a_0 R_l}$, and satisfy

$$\|w_l(0)\|_1 \leq C_0 A_l R_l^2, \quad \|w_l(0)\|_\infty \leq C_0 A_l.$$

Here a_0, c_0, C_0 are fixed. For the smooth seeded family these facts follow from the fixed compact derivative collars. The series of colours is absolutely convergent in $L^1 \cap L^p$, and $\beta q < 1$, so its common velocity and coloured evolution are well defined as before.

Choose a fixed broad annulus $\mathcal{A}_l$, containing both derivative collars of $W(x/R_l)$, whose two radii are fixed multiples of R_l .

Proposition 12.1 (Stopped inactive row with explicit parabolic errors). *There are constants $\gamma, C, c > 0$, depending only on the fixed smooth profile, p , and the displayed initial prefactors, with the following property. Fix $j \geq 2$ and*

$$T \leq S/A_j, \quad \kappa_j = \frac{\nu}{A_j R_j^2} \leq q^{3/2}, \quad 1 \leq S \leq C_1 \log(1/q).$$

Stop at the first time any $i < j$ reaches $x_i = \delta$. Assume $\beta \geq 4$, $\beta q \leq 1/2$, q sufficiently small, and

$$CS(\sqrt{\delta} + \beta q) \leq c. \quad (103)$$

In the clock $\theta = A_j t$, every inactive $i < j$ obeys

$$\frac{d^+ x_i}{d\theta} \leq \gamma \beta^{i-j} x_i + C \sum_{l < i} \beta^{l-j} q^{2(i-l)} x_l + C \sum_{i < l < j} \beta^{l-j} q^{6(l-i)} x_l + f_i(\theta) + E_{j-i}, \quad (104)$$

where

$$f_i(\theta) = \frac{1}{A_j} \left\| \mathbf{1}_{\mathcal{A}_i} \left(\sum_{l \geq j} K * h_l \right) \cdot \nabla G_i \right\|_{X_i} \quad \text{at } t = \theta/A_j$$

is the actual current/deep nonradial forcing. With

$$e_m = C e^{CS} \exp \left[-\frac{cq^{-2m}}{\kappa_j S} \right], \quad m \geq 1,$$

one can take

$$E_k = C \left[\beta^{-k} e_k + \sum_{r > k} \beta^{-r} e_r + \sum_{1 \leq r < k} \beta^{-r} q^{6(k-r)} e_r + e_k \sum_{1 \leq r < k} \beta^{-r} q^{k-r} \right]. \quad (105)$$

The sums may be enlarged to infinite sums of nonnegative terms. No small nonradial-density assumption is imposed on colours $l \geq j$.

Proof. The exact radial heat profile is an isolated Navier–Stokes solution. For the actual common velocity u , subtraction gives

$$(\partial_t + u \cdot \nabla - \nu \Delta) \zeta_i = - \left[K * \zeta_i + \sum_{l \neq i} K * w_l \right] \cdot \nabla G_i. \quad (106)$$

Every radial foreign velocity is tangential and annihilates ∇G_i , so the foreign fields on the right can be replaced by $K * h_l$. In particular diffusion produces no separate angular source. If every initial colour is radial, $\zeta_i = h_i = 0$ solves this system.

The scaled Biot–Savart split gives $\|K * \zeta_i\|_\infty \leq C_p A_i R_i x_i$. Heat contraction of the fixed reference gradient in L^1 and L^p , and the scalar Kato inequalities, bound the self term by $\gamma A_i x_i$. For example γ can be chosen as a constant multiple of $C_p(\|\nabla W\|_1 + \|\nabla W\|_p)$. No derivative of ζ_i enters this estimate; the actual nonlinear transport term disappears from the norm inequalities.

We verify the localized foreign-field estimates under the stop. For any fixed index l , the global radial component obeys

$$|u(t, x) \cdot e_r| \leq C A_l |x| + C A_l R_l \beta q. \quad (107)$$

Indeed every radial average has tangential velocity. For $k \leq l$, the fourfold origin-growth bound gives $|K * h_k(x)| \leq C \|h_k\|_\infty |x| \leq C A_k |x|$, and $\sum_{k \leq l} A_k \leq C A_l$. For $k > l$ use $\|K * w_k\|_\infty \leq C A_k R_k$ and $\sum_{k > l} A_k R_k \leq C A_l R_l \beta q$. The same estimate holds for only the foreign radial field. Thus no divergent sum of amplitudes is used, even when κ_j is much smaller than $q^{3/2}$.

Apply the actual outer-core bootstrap to the stopped finite prefix $l \leq j - 1$. Its nonradial-density hypothesis follows from $\|h_l\|_1 \leq 2 A_l R_l^2 x_l \leq 2 \delta A_l R_l^2$. All colours $k \geq j$ enter that bootstrap through the unrestricted, convergent velocity sum $\sum_{k > l} A_k R_k$. The parameter margin (103) and small q therefore give the individual core estimate

$$\|w_l(t) - c_l\|_{L^\infty(B_{a R_l})} \leq C A_l e^{-c R_l^2 / (\nu t)}, \quad a = a_0/2. \quad (108)$$

The bootstrap error threshold is of order S^{-1} ; its improvement follows from $\kappa_j S \ll 1$ just as in the outer-core proof. Retaining the individual radius in that proof, rather than its maximum over l , gives (108).

Fix $c_2 > c_0$. On $B_{c_2 R_l}$ the own nonradial velocity is bounded by $C \sqrt{\delta} A_l R_l$. Older constant cores contribute only rotation; their harmonic remainders contribute $C \delta A_l R_l q^2 / \beta$ and their core leaks are exponentially small. All inner velocities contribute at most $C A_l R_l \beta q$. Choose q so that this ball is strictly inside every older core probe disk. By (103), the integrated radial drift is less than a fixed fraction of $(c_2 - c_0) R_l$. The radial exit estimate for the positive scalar evolution from $|w_l(0)|$ gives

$$\int_{|x| > c_2 R_l} |w_l(t, x)| \, dx \leq C A_l R_l^2 e^{-c R_l^2 / (\nu t)}. \quad (109)$$

For its post-exit fourth moment use $\phi(x) = (R_l^2 + |x|^2)^2$. Equation (107) gives $u \cdot \nabla \phi \leq C A_l \phi$, while $\nu \Delta \phi \leq C \nu R_l^{-2} \phi$. Conditioning at exit from $B_{c_2 R_l}$ consequently gives $\mathbb{E}[\phi(X_t) \mid \mathcal{F}_{\text{exit}}] \leq C R_l^4 e^{C(A_l + \nu/R_l^2)T} \leq C R_l^4 e^{CS}$. Multiplication by the exit probability proves

$$\int_{|x| > c_2 R_l} |x|^4 |w_l(t, x)| \, dx \leq C A_l R_l^6 e^{CS} e^{-c R_l^2 / (\nu t)}. \quad (110)$$

Angular averaging preserves radial integrals. Inside $B_{c_2 R_l}$ use $\|h_l\|_1 \leq 2 A_l R_l^2 x_l$; outside use (110). It follows that

$$\int |x|^4 |h_l(t, x)| \, dx \leq C A_l R_l^6 [x_l(t) + e_{j-l}]. \quad (111)$$

For an inner colour $i < l < j$, take a fixed small $\eta > 0$ such that $B_{\eta R_i}$ is inside the initial constant disk of colour i and $c_0 q < \eta/4$. In this ball the radial velocity estimate based on index i has the same small integrated bound: the own nonradial term is $C\sqrt{\delta}A_i R_i$, older remainders are controlled by (108), and all inner colours enter by $CA_i R_i \beta q$. Starting the l th positive colour in $B_{c_0 R_l}$, radial exit from $B_{\eta R_i}$ therefore gives

$$\int_{|x| > \eta R_i} |w_l(t, x)| \, dx \leq CA_l R_l^2 e^{-cR_i^2/(\nu t)}. \quad (112)$$

This receiver-radius estimate is essential.

On $\mathcal{A}_i$, for $l < i$, split h_l into its core and exterior. Equation (108) bounds the core density by $CA_l e^{-cR_i^2/(\nu t)}$; the origin-growth estimate bounds its velocity by $CA_l |x| e^{-cR_i^2/(\nu t)}$. The exterior part has mass at most $2A_l R_l^2 x_l$ and a fourfold inverse-moment expansion starting at degree three. Thus

$$\frac{|K * h_l(x) \cdot e_r|}{R_i} \leq CA_l [q^{2(i-l)} x_l + e_{j-l}]. \quad (113)$$

For $i < l < j$, truncate h_l at radius ηR_i . Every radial truncation has zero circulation because $Ph_l = 0$, and fourfold symmetry cancels harmonic moments of degrees one to three. Its exterior field is bounded by $CR_i^{-5} \int |y|^4 |h_l(y)| \, dy$. For the remaining density use (112), $\|h_l\|_\infty \leq CA_l$, and Biot–Savart interpolation. This gives

$$\frac{|K * h_l(x) \cdot e_r|}{R_i} \leq CA_l [q^{6(l-i)}(x_l + e_{j-l}) + q^{l-i} e_{j-i}]. \quad (114)$$

The fixed exponent c in e_m has been reduced to absorb square roots and polynomial factors.

Away from $\mathcal{A}_i$, ∇G_i is the Gaussian convolution of the compactly supported reference gradient. Its scaled L^1 and L^p tails after multiplication by $|x| + R_i$ are bounded by $Ce^{-cR_i^2/(\nu t)}$. The foreign version of (107) consequently bounds the total omitted forcing by $CA_i e_{j-i}$. Combining this with the self estimate, (113), (114), and the explicitly retained f_i proves (104)–(105). $\square$

Corollary 12.2 (Inactive density closure under packet history). *Let $\sigma > 0$ be the growth rate of the fixed smooth reference. Choose β sufficiently large that $\gamma/\beta < \sigma/4$, then a fixed $D > \beta$ sufficiently large, then q sufficiently small. Set $w_i = (Dq^6)^{j-i}$ and $b = \delta e^{-\sigma S}$. Assume $x_i(0) \leq Mb$, and the packet-history bound*

$$\sup_{\theta \leq A_j T} \int_0^\theta e^{\lambda_P(\theta-s)} \max_{i < j} \frac{f_i(s)}{w_i} \, ds \leq C_* \delta, \quad (115)$$

where $\lambda_P < \sigma/2$ is the weighted comparison rate below. Take $S = C_2 \log(1/q)$, with C_2 sufficiently large and fixed, and δ small enough for (103). Then the stopped solution remains strictly below its stopping threshold:

$$x_i(t) \leq B + w_i [CC_* \delta + o(q^K)] < \delta/4, \quad B \leq M\delta e^{-(\sigma - \lambda_B)S},$$

for every $i < j$ and $t \leq T$. Here K is any fixed prescribed power, and the constants are independent of depth. This is an actual inactive-density conclusion conditional on (115); that packet input is not established by the corollary.

Proof. Bound each self coefficient in (104) by $\lambda_0 = \gamma/\beta$. This replaces, rather than identifies with, the earlier sharp spectral rate $(\sigma + \varepsilon)/\beta$. The unweighted off-diagonal row sum is at most

$$r_B = \frac{C}{\beta} \left[\frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right].$$

The row sum in the forced weights w_i is at most

$$r_P = \frac{C}{\beta} \left[\frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right].$$

Thus $\lambda_B = \lambda_0 + r_B$ and $\lambda_P = \lambda_0 + r_P$ are both below $\sigma/2$ in the prescribed parameter order. The positive comparison equation splits exactly into initial, packet, and parabolic-error contributions. Its initial part is bounded by $Mbe^{\lambda_B \theta}$, giving the stated B .

For the explicit error put

$$\eta(q) = \sup_{m \geq 1} \frac{e_m}{(\beta D q^6)^m}.$$

This is smaller than every power of q , uniformly when $\kappa_j \leq q^{3/2}$ and $S = O(\log(1/q))$. Indeed successive ratios in this supremum are less than $1/2$ for small q , and its first term has Gaussian exponent of order $q^{-7/2}/S$. A direct summation of the four terms of (105) gives

$$\sup_{k \geq 1} \frac{E_k}{(Dq^6)^k} \leq C\eta(q) \left[1 + \frac{Dq^6}{1 - Dq^6} + \frac{1}{D - 1} + \frac{\beta q}{1 - \beta q} \right].$$

For example the third sum becomes $D^{-k} \sum_{r < k} \beta^{-r} q^{-6r} e_r \leq \eta(q) D^{-k} \sum_{r < k} D^r$. The fourth uses $\sum_{r < k} \beta^{-r} q^{k-r} \leq \beta^{-k} \beta q / (1 - \beta q)$. Its weighted convolution is therefore at most $C\eta(q) S e^{\lambda_P S}$, still smaller than every q -power.

The positive-matrix variation of constants and (115) yield the asserted bound. Choose C_2 so $C_2(\sigma - \lambda_B) > 6$; then $B \ll \delta q^6$. Finally $\max_{i < j} w_i = Dq^6$, so small q makes the complete right side less than $\delta/4$, contradicting a first stopping time. The absolute leak envelope has not been called a physical angular source; the exact equation (106) is zero on the all-radial family. In the fixed nonzero seeded family, its error envelope is much smaller even than the prescribed seed b . $\square$

Corollary 12.3 (A static packet bound suffices for inactive density). *For the inactive-density conclusion alone, the packet-history hypothesis (115) may be replaced by*

$$\sup_{\theta \leq A_j T} \max_{i < j} \frac{f_i(\theta)}{w_i} \leq C_* \delta.$$

Fix $C_2 > 6/\sigma$. Choose β, D , and then q , so that

$$C_2(\sigma - \lambda_B) > 6, \quad C_2 \lambda_P < 1.$$

With $S = C_2 \log(1/q)$, the actual inactive densities still remain below $\delta/4$. Their normalized inward nonradial velocity on the current receiver's fixed collars is

$$O(\delta S q^{8 - C_2 \lambda_P}) + o(q^K) = o(\delta q^6).$$

The static packet bound is an explicit unproved input here. This corollary does not supply a signed transfer or preserve the sharper pre-cap histories needed for phase protection.

Proof. The stated parameter choices are possible since λ_B, λ_P can be made arbitrarily small by taking the fixed β, D large and then q small. The packet contribution to the forced weighted norm is at most $CC_*\delta S e^{\lambda_P S}$. Thus

$$x_i \leq B + (Dq^6)^{j-i} \left[CC_*\delta S q^{-C_2\lambda_P} + o(q^K) \right].$$

The baseline $B = o(\delta q^6)$ and the nearest forced bound $C\delta S q^{6-C_2\lambda_P}$ are both below the stopping threshold. The parabolic errors remain smaller than every power of q . For the inward field use the actual older-core multipole estimate:

$$\sum_{i < j} \beta^{i-j} q^{2(j-i)} x_i \leq B \frac{q^2}{\beta - q^2} + CC_*\delta S q^{-C_2\lambda_P} \frac{Dq^8}{\beta - Dq^8} + o(q^K).$$

Since $C_2\lambda_P < 1$, this is $o(\delta q^6)$. The proof controls magnitudes; it asserts no lower bound on a complex unstable-mode impulse. $\square$

13 Compactness for one fixed compact zero-circulation datum

The compactness needed for phase separation does not require a new cascade hypothesis. We prove it for the initial-data class used in the construction. Only standard existence of the planar Navier–Stokes solution, its vorticity contraction bounds, and its finite-energy inequality are inputs. These inputs are obtained by smooth approximation of the initial datum; no Euler uniqueness or viscosity-selection principle is used.

Lemma 13.1 (The velocity estimates used below). *Let $p > 2$, $\alpha = 1 - 2/p$, and $w \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Then*

$$\|K * w\|_\infty + \|K * w\|_p \leq C_p(\|w\|_1 + \|w\|_p), \quad [K * w]_{C^\alpha} \leq C_p\|w\|_p, \quad \|\nabla K * w\|_p \leq C_p\|w\|_p.$$

*If w has compact support and zero integral, then $K * w \in L^2$.*

Proof. For the supremum norm split K at radius one: its near part is in $L^{p'}$ since $p' < 2$, and its far part is bounded. For the L^p norm the near part is in L^1 and the far part is in L^p ; apply Young's inequality. For a velocity difference at points a distance d apart, the two radius- $2d$ disks contribute at most $C_p\|w\|_p d^\alpha$. On their complement use $|K(x - z) - K(y - z)| \leq Cd|x - z|^{-2}$ and Hölder's inequality; the same power d^α follows.

We recall a direct proof of the derivative bound to specify the analytic input. Each component T of $\nabla K *$ has a Fourier multiplier bounded in absolute value, and hence is bounded on L^2 by Plancherel. Its off-origin kernel satisfies $|t(x)| \leq C|x|^{-2}$, $|\nabla t(x)| \leq C|x|^{-3}$ and has zero angular average; a possible multiple of the identity is harmless. For $f \in L^1 \cap L^2$ decompose at height λ into $g + \sum b_Q$ using maximal dyadic squares with average of $|f|$ larger than λ . The squares are disjoint, and

$$\|g\|_\infty \leq 4\lambda, \quad \|g\|_1 \leq \|f\|_1, \quad \sum |Q| \leq \|f\|_1/\lambda, \quad \int b_Q = 0, \quad \sum \|b_Q\|_1 \leq 2\|f\|_1.$$

The L^2 bound controls the level set of Tg by $C\|f\|_1/\lambda$. Cancellation and the kernel derivative estimate give

$$\int_{\mathbb{R}^2 \setminus 5Q} |Tb_Q(x)| dx \leq \int_Q |b_Q(y)| \int_{\mathbb{R}^2 \setminus 5Q} |t(x - y) - t(x - c_Q)| dx dy \leq C\|b_Q\|_1.$$

The last inner integral is bounded by $C|y - c_Q| \int_{C\ell(Q)}^\infty r^{-3} r \, dr$. This proves weak type $(1, 1)$, first for finite sums and then by their L^2 convergence. Splitting f above and below a height proportional to λ consequently gives

$$|\{|Tf| > \lambda\}| \leq C \left[\lambda^{-1} \int_{|f| > c\lambda} |f| + \lambda^{-2} \int_{|f| \leq c\lambda} |f|^2 \right].$$

Integration against $s\lambda^{s-1} d\lambda$, with $1 < s < 2$, and Fubini bound the L^s norm: the two integrals produce finite factors $(s-1)^{-1}$ and $(2-s)^{-1}$. The adjoint has the same bounds, so duality with $s = p'$ proves the required L^p estimate. Density identifies the result with the distributional derivative of $K * w$.

Finally, compact support gives a finite first absolute moment. At low frequencies zero circulation implies $|\widehat{w}(\xi)| \leq |\xi| \int |x| |w(x)| \, dx$. At high frequencies use $w \in L^2$, which follows by interpolation between L^1 and L^p . The Fourier multiplier $i\xi^\perp/|\xi|^2$ and Plancherel prove $K * w \in L^2$. $\square$

Theorem 13.2 (Strong vanishing-viscosity precompactness). *Let $2 < p < \infty$ and let one fixed real initial vorticity satisfy*

$$\omega_0 \in L^1 \cap L^p, \quad \text{supp } \omega_0 \subset B_{R_0}, \quad \int \omega_0 = 0.$$

For every sequence $\nu_n \downarrow 0$, the unforced planar Navier–Stokes vorticities with this same initial datum have a subsequence converging strongly in $C([0, T]; L^1 \cap L^p)$ for each fixed T . Every limit is a renormalized Euler solution with initial vorticity ω_0 . Its velocity is bounded, belongs to $L_t^\infty W^{1,p}$, and its vorticity preserves the L^1 and L^p norms and circulation at every time. In particular the Sobolev-velocity and compactness hypotheses needed for fixed-colour first-cap separation hold for this class.

Proof. Write $m = \|\omega_0\|_1$, $P = \|\omega_0\|_p$ and $E = \|K * \omega_0\|_2$. The last quantity is finite by Lemma 13.1. Contraction and the kinetic-energy inequality give uniformly, for $0 < \nu_n \leq 1$,

$$\|\omega_n(t)\|_1 \leq m, \quad \|\omega_n(t)\|_p \leq P, \quad \|u_n(t)\|_2 \leq E, \quad \|u_n(t)\|_\infty \leq V, \quad V = C_p(m + P). \quad (116)$$

The velocity lemma also bounds $[u_n(t)]_{C^\alpha}$ and $\|\nabla u_n(t)\|_p$, where $\alpha = 1 - 2/p$.

Local compactness of velocity. Apply the Fourier projection onto divergence-free fields to the velocity equation. It is bounded on every H^s by its bounded matrix multiplier. Since $\|u_n \otimes u_n\|_2 \leq VE$, the equation gives

$$\|\partial_t u_n\|_{H^{-2}} \leq CE(V + 1).$$

For a spatial mollifier η_h , $0 < h \leq 1$, $\|\eta_h\|_{H^2} \leq Ch^{-3}$ in two dimensions. Hence

$$\|u_n(t) - u_n(s)\|_\infty \leq Ch^\alpha + Ch^{-3}|t - s|.$$

Here the first term comes from the spatial Hölder bound and the second from duality of H^{-2} and H^2 after convolution. Taking $h = |t - s|^{1/(3+\alpha)}$ proves uniform time equicontinuity. Arzelà–Ascoli on compact cylinders and a diagonal extraction give

$$u_n \longrightarrow u \quad \text{locally uniformly on } [0, T] \times \mathbb{R}^2.$$

The uniform gradient bound passes weakly to $\nabla u \in L_t^\infty L^p$. Also u is bounded and divergence free. The L^p velocity bound in Lemma 13.1 passes by lower semicontinuity, so $u \in L_t^\infty W^{1,p}$.

Weak vorticity compactness and its tightness. For compact smooth φ the scalar equation gives

$$\left| \frac{d}{dt} \int \varphi \omega_n \right| \leq m(V \|\nabla \varphi\|_\infty + \|\Delta \varphi\|_\infty).$$

A countable dense set of tests in $L^{p'}$, the uniform L^p bound, and Arzelà–Ascoli provide a weakly continuous L^p path ω such that

$$\sup_{t \leq T} \left| \int \varphi(\omega_n(t) - \omega(t)) \right| \rightarrow 0 \quad \text{for every } \varphi \in L^{p'}. \quad (117)$$

Its initial value is ω_0 . Each weak value also has L^1 norm at most m , by testing on finite balls and then enlarging them.

Let χ_R be zero on B_R , one outside B_{2R} , with first and second derivatives bounded by C/R and C/R^2 . Kato's inequality and (116) give

$$\sup_{t \leq T} \int_{|x| > 2R} |\omega_n(t, x)| dx \leq \int_{|x| > R} |\omega_0(x)| dx + CTm(V/R + 1/R^2). \quad (118)$$

The same bound passes to $\omega(t)$ by local lower semicontinuity and monotone enlargement of the outer integration region.

The local uniform convergence of u_n and weak vorticity convergence pass the nonlinear drift product in compactly supported tests. The diffusion term is bounded by $\nu_n P \|\Delta \varphi\|_{p'}$ and vanishes. Thus

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \omega(0) = \omega_0.$$

We identify $u = K * \omega$ after establishing strong convergence below; this avoids requiring a preliminary global compactness argument for velocities.

Renormalization and traces at every time. Spatial mollification gives

$$\partial_t \omega_\varepsilon + u \cdot \nabla \omega_\varepsilon = r_\varepsilon,$$

where, using divergence freedom to subtract $\omega(x)$,

$$r_\varepsilon(x) = \int \nabla \eta_\varepsilon(y) \cdot [u(x) - u(x - y)] [\omega(x - y) - \omega(x)] dy.$$

The Sobolev difference estimate and Hölder's inequality yield

$$\|r_\varepsilon(t)\|_{p/2} \leq C \|\nabla u(t)\|_p \sup_{|y| \leq \varepsilon} \|\omega(t, \cdot - y) - \omega(t)\|_p.$$

Translation continuity and dominated convergence imply $r_\varepsilon \rightarrow 0$ in $L_t^1 L_x^{p/2}$. Since $p/2 > 1$, this gives local space-time L^1 convergence. For every C^1 function β with bounded derivative, the chain rule therefore passes to the renormalized transport equation.

There is no ambiguity about endpoint traces in this argument. For fixed ε the mollified equation is absolutely continuous in time in local smooth spatial norms. Integrate its renormalized equation between *any fixed* times s, t . At each of these fixed times, $\omega_\varepsilon(s) = \eta_\varepsilon * \omega(s) \rightarrow \omega(s)$ and $\omega_\varepsilon(t) = \eta_\varepsilon * \omega(t) \rightarrow \omega(t)$ strongly in L^p , by the spatial approximate-identity property. The Lipschitz function β passes these endpoint terms in local L^1 , while the commutator error vanishes. Consequently the renormalized identity has traces $\beta(\omega(s))$ and $\beta(\omega(t))$ for every s, t , including zero.

For $\beta(0) = 0$, integrate against cutoffs equal to one on B_R and vanishing outside B_{2R} . The flux error between any two times is at most $CTVm\|\beta'\|_\infty/R$. Letting $R \rightarrow \infty$ proves

$$\int \beta(\omega(t)) = \int \beta(\omega_0) \quad (0 \leq t \leq T).$$

To recover the L^p norm, use the convex tangent truncations

$$\beta_M(s) = \begin{cases} |s|^p, & |s| \leq M, \\ pM^{p-1}|s| - (p-1)M^p, & |s| > M. \end{cases}$$

They are C^1 with bounded derivative, nonnegative, and increase to $|s|^p$. For $|s| > M$ their derivative with respect to M is $p(p-1)M^{p-2}(|s| - M) \geq 0$. Monotone convergence therefore proves

$$\|\omega(t)\|_p = \|\omega_0\|_p \quad \text{at every } t. \quad (119)$$

Using $\sqrt{s^2 + \varepsilon^2} - \varepsilon$ and then $\varepsilon \downarrow 0$ proves conservation of the L^1 norm. Using $\beta(s) = s$ preserves circulation.

Strong convergence, uniformly in time. For $p \geq 2$ the pointwise Clarkson inequality gives

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{\|f\|_p^p + \|g\|_p^p}{2}.$$

Weak time continuity and the constant norm (119) imply $\omega \in C_t L^p$ by this inequality and lower semicontinuity. If $\omega_n \rightarrow \omega$ were not uniform strongly in L^p , choose bad times $t_n \rightarrow t$. Equation (117) gives $\omega_n(t_n) \rightharpoonup \omega(t)$, while contraction gives

$$\limsup_n \|\omega_n(t_n)\|_p \leq P = \|\omega(t)\|_p.$$

The same inequality proves strong convergence to $\omega(t)$; continuity of ω contradicts the bad times. Thus convergence is strong in $C_t L^p$. It is strong in $C_t L^1$ as well: on a fixed ball use Hölder's inequality, and outside it use (118) for both paths.

Finally Lemma 13.1 gives uniform global convergence $u_n = K * \omega_n \rightarrow K * \omega$. Their already obtained local limit is therefore exactly $u = K * \omega$. This proves the Euler equation and all claimed regularity and compactness statements. If a common subsequence on all finite windows is desired, repeat the extraction on $T = 1, 2, \dots$ and take the diagonal subsequence. $\square$

Corollary 13.3. *For the compact zero-circulation initial-data class of Theorem 13.2, the relative compactness as $\nu \downarrow 0$ and Sobolev-velocity assumptions in Proposition 11.4 are automatic. The substantive remaining hypotheses there are uniform protected caps and phase shooting for the actual coupled system.*

14 What remains for a nonunique limit

The estimates above give a verified conditional chain: small actual nonradial densities imply actual constant-core persistence; the resulting harmonic field and exponentially small density leak imply inner-tail suppression throughout a receiver window. The isolated smooth theorem supplies density control for one colour. It does not supply that control in the common velocity of an infinite stack.

The actual inactive coupling row and its density closure have been proved under an explicit packet-force input. The prescribed canonical harmonic input also gives a nonlinear receiving estimate, but its drift is an auxiliary smooth extension rather than the actual field of jointly evolving colours.

To finish the construction one still needs a simultaneous, depth-uniform theorem for the actual coloured evolution. It must establish the packet force and localization bounds, and transfer the isolated canonical moment history to a nonzero signed growing-mode impulse for each actual receiver, with first caps protected uniformly over the phase homotopies used for shooting. In particular, a nonzero instantaneous coupling coefficient does not exclude cancellation by the later source history. A finite-multipole forced approximation would have to retain the actual moment histories in its leading coefficients; placing the first neglected harmonic in a generic exponentially growing stability error can lose the required scale margin.

Strong subsequential compactness for the fixed compact initial-data class has been proved, so compactness is not an additional cascade hypothesis. The degree and sparse-phase arguments still require their stated uniform cap and shooting hypotheses. Those hypotheses have not been proved for the candidate stack. Accordingly this continuation asserts no theorem of nonunique fixed-data vanishing-viscosity limits.

References

- [1] T. Gallay and C. E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier–Stokes and vorticity equations on $\mathbb{R}^2$* , Arch. Rational Mech. Anal. 163 (2002), 209–258. <https://arxiv.org/abs/math/0102197>.
- [2] G. Ciampa, G. Crippa, and S. Spirito, *Strong convergence of the vorticity for the 2D Euler equations in the inviscid limit*, <https://arxiv.org/abs/2008.12133>. Theorem 3.2 concerns strong convergence to an identified renormalized limit; it does not assert uniqueness of that limit.