

Vanishing-viscosity selection for a radial planar Euler datum

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Abstract

A compactly supported radial vorticity selects its stationary Euler solution under the fixed-data Navier–Stokes inviscid limit. A second Euler trajectory that is strongly approximated by bounded exact Euler trajectories can also be selected by vanishing viscosity, with suitably perturbed initial data. The perturbations can be smooth, compactly supported, and have zero total circulation. The proof uses a diagonal choice and an explicit global $L^1 \cap L^p$ estimate against smooth Euler solutions.

Fix $2 < p < \infty$ and $0 < T < \infty$, and write

$$E_p = L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2), \quad \|f\|_{E_p} = \|f\|_1 + \|f\|_p, \quad K(x) = \frac{x^\perp}{2\pi|x|^2}.$$

The Euler and Navier–Stokes equations below are unforced and have velocity $u = K * \omega$:

$$\partial_t \omega + \operatorname{div}(u\omega) = 0, \tag{1}$$

$$\partial_t \omega^\nu + \operatorname{div}(u^\nu \omega^\nu) = \nu \Delta \omega^\nu. \tag{2}$$

Navier–Stokes solutions mean the usual unique mild vorticity solutions, equivalently the Leray–Hopf solutions in $L_t^\infty L_x^2 \cap L_t^2 H_x^1$ with the prescribed initial trace for the zero-circulation compactly supported initial data used here. We use the standard global smooth planar Euler theory and bounded-vorticity Euler well-posedness. For the stability input we use the precisely stated whole-plane result of Crippa–Semenova–Spirito [1, Theorem A]; the short reduction needed here is given below. Navier–Stokes well-posedness in the larger class of finite initial vorticity measures is established in [2, Theorems 1.1–1.2]. Its viscosity-one statement applies to any $\nu > 0$ by writing $\omega(t) = \nu \zeta(\nu t)$. The estimates required in this note are proved explicitly.

Theorem 1. *Let $\omega_0 \in E_p$ be radial, compactly supported, and satisfy $\int_{\mathbb{R}^2} \omega_0 = 0$. Suppose that an Euler solution $\Omega \in C([0, T]; E_p)$ is the strong limit*

$$\Omega_n \longrightarrow \Omega \quad \text{in } C([0, T]; E_p), \quad \Omega(0) = \omega_0, \tag{3}$$

where each Ω_n is the unique bounded-vorticity Euler solution with initial datum $f_n \in L_c^\infty(\mathbb{R}^2)$ and $\int f_n = 0$. Then:

1. *For every $\nu > 0$, the Navier–Stokes solution with the fixed initial vorticity ω_0 is*

$$\text{NS}_\nu(\omega_0)(t) = e^{\nu t \Delta} \omega_0.$$

As $\nu \downarrow 0$, the entire family converges in $C([0, T]; E_p)$ to the stationary Euler solution $\omega^{\text{st}}(t) = \omega_0$.

2. There exist $\nu_n \downarrow 0$ and $g_n \in C_c^\infty(\mathbb{R}^2)$ such that

$$\int g_n = 0, \quad g_n \longrightarrow \omega_0 \text{ in } E_p, \quad \text{NS}_{\nu_n}(g_n) \longrightarrow \Omega \text{ in } C([0, T]; E_p).$$

If the supports of f_n lie in a common ball, the same is true of the supports of g_n .

The hypothesis (3) is the construction-specific input. For the nonstationary branch obtained as a strong limit of the finite bounded Euler stacks, it is exactly the approximation already used to define that branch. The selection theorem itself does not need any estimates for the cascade.

1 Kernel bounds and a global viscous comparison

For $h \in E_p$, split the Biot–Savart convolution into $|z| < 1$ and $|z| \geq 1$. Since $p' = p/(p-1) < 2$, Hölder's inequality gives

$$\|K * h\|_\infty \leq \|K \mathbf{1}_{B_1}\|_{p'} \|h\|_p + \|K \mathbf{1}_{\mathbb{R}^2 \setminus B_1}\|_\infty \|h\|_1 \leq C_p \|h\|_{E_p}. \quad (4)$$

For $h \in L^1 \cap L^\infty$, splitting at radius $a > 0$ instead gives $\|K * h\|_\infty \leq C(a \|h\|_\infty + a^{-1} \|h\|_1)$. Optimizing, with the zero function handled separately, gives

$$\|K * h\|_\infty \leq C \|h\|_1^{1/2} \|h\|_\infty^{1/2}. \quad (5)$$

For a bounded compactly supported g with $\int g = 0$ and $\text{supp } g \subset B_R$, subtraction of $K(x) \int g$ gives

$$|K * g(x)| \leq C |x|^{-2} \int |y| |g(y)| dy \quad (|x| \geq 2R).$$

Together with (5), this proves $K * g \in L^2$. The transport maximum principle preserves Euler vorticity norms; the parabolic maximum principle makes Navier–Stokes vorticity norms nonincreasing, for $1 \leq r \leq \infty$ whenever finite. These statements follow by testing with convex approximations of $|\omega|^r$, and for Euler also follow from its incompressible transport flow. In particular, the velocities used in the next lemma are bounded on the whole plane.

Lemma 2 (Explicit comparison with a smooth Euler flow). *Let $g \in C_c^\infty(\mathbb{R}^2)$ have zero integral, and let Z be its global smooth Euler solution. For $w^\nu = \text{NS}_\nu(g)$ define*

$$\begin{aligned} a_Z(t) &= C_p (\|\nabla Z(t)\|_1 + \|\nabla Z(t)\|_p), \\ b_Z(t) &= \|\Delta Z(t)\|_1 + \|\Delta Z(t)\|_p, \\ D_Z &= \int_0^T b_Z(s) \exp \left(\int_s^T a_Z(\tau) d\tau \right) ds. \end{aligned}$$

Then $D_Z < \infty$ and

$$\|w^\nu - Z\|_{C([0, T]; E_p)} \leq \nu D_Z. \quad (6)$$

Proof. The smooth Euler vorticity is transported, and (5) bounds its velocity. Its support therefore stays in a fixed ball on $[0, T]$. Global smoothness consequently implies $a_Z, b_Z \in L^1(0, T)$.

Set $d = w^\nu - Z$, $u^\nu = K * w^\nu$, and $v = K * d$. Subtracting the equations, with the viscous velocity as transport field, gives the exact identity

$$\partial_t d + u^\nu \cdot \nabla d - \nu \Delta d = -v \cdot \nabla Z + \nu \Delta Z, \quad d(0) = 0. \quad (7)$$

The L^r transport–diffusion norm inequality, for $r = 1, p$, gives

$$\|d(t)\|_r \leq \int_0^t (\|v(s)\|_\infty \|\nabla Z(s)\|_r + \nu \|\Delta Z(s)\|_r) ds. \quad (8)$$

Here the L^1 assertion is Kato’s inequality; for $r = p$ test with $|d|^{p-2}d$ and use Hölder’s inequality. One may regularize the norm at zero to avoid division by $\|d\|_p^{p-1}$. These are global inequalities on $\mathbb{R}^2$, although w^ν has no compact support at positive times. To justify the spatial integrations, insert cutoffs equal to one on B_R , zero outside B_{2R} , with gradient and Laplacian bounded by C/R and C/R^2 . The transport boundary term is at most $C \|u^\nu\|_\infty \|d\|_r^r / R$, and the diffusion boundary term is at most $C\nu \|d\|_r^r / R^2$ (use $\sqrt{d^2 + \epsilon^2} - \epsilon \leq |d|$ for $r = 1$). The source belongs to $L_t^1 L_x^r$ by (4), the vorticity norm bounds, and the bounds for Z . Thus these errors vanish as $R \rightarrow \infty$, and then the convex regularizations can be removed. This proves (8) without a confinement assertion for the viscous solution.

Adding the two inequalities and using (4) yields, for $F(t) = \|d(t)\|_1 + \|d(t)\|_p$,

$$F(t) \leq \int_0^t a_Z(s) F(s) ds + \nu \int_0^t b_Z(s) ds.$$

Gronwall’s inequality in its inhomogeneous form gives

$$F(t) \leq \nu \int_0^t b_Z(s) \exp\left(\int_s^t a_Z(\tau) d\tau\right) ds \leq \nu D_Z.$$

This proves (6). □

2 Smoothing a fixed bounded Euler trajectory

Lemma 3. *Let $f \in L_c^\infty(\mathbb{R}^2)$ have zero integral, and let Y be its unique bounded Euler solution. Fix a nonnegative radial mollifier $\eta \in C_c^\infty(B_1)$ with $\int \eta = 1$, and put $f_\epsilon = \eta_\epsilon * f$. Let Z_ϵ be the smooth Euler solution from f_ϵ . Then*

$$Z_\epsilon \longrightarrow Y \quad \text{in } C([0, T]; E_p) \quad \text{as } \epsilon \downarrow 0.$$

Moreover $\int f_\epsilon = 0$ and, for $\epsilon \leq 1$, the initial supports lie in the unit enlargement of $\text{supp } f$.

Proof. The convolution has all the asserted smoothness, support, and circulation properties. It converges to f in L^1 , L^2 , and L^p , and

$$\|f_\epsilon\|_1 \leq \|f\|_1, \quad \|f_\epsilon\|_\infty \leq \|f\|_\infty.$$

Theorem A of [1] states that solutions from initial data converging strongly in L^2 to a datum in $L^1 \cap L^\infty$ converge uniformly in time in L^2 to its unique bounded Euler solution. Its approximating solutions must be in $C([0, T]; L^1 \cap L^2)$, as our smooth solutions are. It therefore applies here and gives

$$\sup_{t \in [0, T]} \|Z_\epsilon(t) - Y(t)\|_2 \longrightarrow 0.$$

There is no viscosity in this application of the stability theorem.

By (5) all of these Euler velocities have the common speed bound $C \|f\|_1^{1/2} \|f\|_\infty^{1/2}$. If $\text{supp } f \subset B_R$, transport places their supports in one ball $B_{R+1+CT\|f\|_1^{1/2}\|f\|_\infty^{1/2}}$ for $0 \leq t \leq T$. Denote this ball by B . For $h = Z_\epsilon(t) - Y(t)$ we thus have

$$\|h\|_1 \leq |B|^{1/2} \|h\|_2, \quad \|h\|_p^p \leq (2 \|f\|_\infty)^{p-2} \|h\|_2^2.$$

Both bounds are uniform in time. They prove the claimed E_p convergence. If $f = 0$, the statement is immediate. □

3 The fixed radial datum

Proof of Theorem 1, part 1. Write $\omega_0(x) = F_0(r)$, $r = |x|$. Its velocity is tangential:

$$u_0(x) = A_0(r)e_\theta, \quad A_0(r) = \frac{1}{r} \int_0^r s F_0(s) ds. \quad (9)$$

This follows either by solving the radial streamfunction equation or by rotational invariance, incompressibility, and circulation around circles. If $\text{supp } \omega_0 \subset B_R$, zero circulation gives $A_0(r) = 0$ for $r \geq R$. Hölder's inequality gives $|A_0(r)| \leq C_p \|\omega_0\|_p r^{1-2/p}$. Thus u_0 is bounded and compactly supported, so $u_0 \in L^2$. Also $\omega_0 \in L^2$ by interpolation between L^1 and L^p ; the Fourier div-curl identity yields

$$u_0 \in H^1(\mathbb{R}^2), \quad \|\nabla u_0\|_2 = \|\omega_0\|_2.$$

Define

$$\omega^\nu(t) = e^{\nu t \Delta} \omega_0, \quad u^\nu(t) = e^{\nu t \Delta} u_0 = K * \omega^\nu(t).$$

The last equality follows by commutation of the heat semigroup with the Biot-Savart Fourier multiplier, or by the div-curl characterization. For $t > 0$, ω^ν is smooth and radial, and u^ν is tangential. Therefore $u^\nu \cdot \nabla \omega^\nu = 0$ pointwise, and its heat equation is precisely (2). The initial trace is strong in E_p . This is the unique usual Navier-Stokes solution by [2].

For completeness it is also the unique Leray-Hopf solution. Writing $u^\nu = A^\nu(t, r)e_\theta$, one has

$$(u^\nu \cdot \nabla) u^\nu = -\frac{(A^\nu)^2}{r} e_r, \quad \pi^\nu(t, r) = \int_0^r \frac{(A^\nu(t, s))^2}{s} ds.$$

The pressure cancels this centripetal term, so the heat velocity solves the velocity Navier-Stokes equations. Its heat energy identity puts it in the energy class. If U is another Leray-Hopf solution and $v = U - u^\nu$, the relative-energy inequality and the planar Ladyzhenskaya inequality give

$$\begin{aligned} \frac{1}{2} \|v(t)\|_2^2 + \nu \int_0^t \|\nabla v\|_2^2 ds &\leq \int_0^t \left| \int_{\mathbb{R}^2} (v \cdot \nabla) u^\nu \cdot v dx \right| ds, \\ \left| \int (v \cdot \nabla) u^\nu \cdot v \right| &\leq C \|\nabla u^\nu\|_2 \|v\|_2 \|\nabla v\|_2 \\ &\leq \frac{\nu}{2} \|\nabla v\|_2^2 + \frac{C}{\nu} \|\omega_0\|_2^2 \|v\|_2^2. \end{aligned}$$

Heat contraction was used in the last line. The identities are justified by energy approximation; $u_0 \in H^1$ gives an integrable comparison coefficient even at $t = 0$. Gronwall's inequality gives $v = 0$.

The stationary Euler solution is $\omega^{\text{st}}(t) = \omega_0$: polar integration against any compactly supported smooth test function gives zero angular flux. The same computation applies to every admissible renormalization $\beta(\omega_0)$, since it is radial. Finally, for $r = 1, p$,

$$\sup_{0 \leq t \leq T} \|\omega^\nu(t) - \omega_0\|_r \leq \sup_{0 \leq s \leq \nu T} \|e^{s \Delta} \omega_0 - \omega_0\|_r \longrightarrow 0.$$

Indeed, Gaussian convolution tends strongly to the identity in finite L^r spaces, by continuity of translations and approximation by compactly supported continuous functions. This proves convergence of the whole viscosity family. $\square$

4 Selecting the strongly approximated branch

Proof of Theorem 1, part 2. For each fixed n , apply Lemma 3 to f_n . Choose $0 < \epsilon_n \leq 1$ so that, with

$$g_n = \eta_{\epsilon_n} * f_n, \quad Z_n = \text{the smooth Euler solution from } g_n,$$

one has

$$\|Z_n - \Omega_n\|_{C([0,T];E_p)} \leq 2^{-n}. \quad (10)$$

Every g_n is smooth, compactly supported, and has zero integral. Evaluating (10) and (3) at zero already proves $g_n \rightarrow \omega_0$ in E_p .

Let $D_n = D_{Z_n}$ be the finite constant in Lemma 2. Set $\nu_0 = 1$ and choose recursively

$$\nu_n = \min \left\{ \frac{\nu_{n-1}}{2}, \frac{2^{-n}}{1 + D_n} \right\} > 0. \quad (11)$$

Then $\nu_n \downarrow 0$ and, for $w_n = \text{NS}_{\nu_n}(g_n)$,

$$\|w_n - Z_n\|_{C([0,T];E_p)} \leq \nu_n D_n \leq 2^{-n}.$$

The triangle inequality gives

$$\|w_n - \Omega\|_{C([0,T];E_p)} \leq 2^{1-n} + \|\Omega_n - \Omega\|_{C([0,T];E_p)} \longrightarrow 0. \quad (12)$$

If $\text{supp } f_n \subset B_{R_*}$ for all n , then $\text{supp } g_n \subset B_{R_*+1}$. This completes the proof. $\square$

Corollary 4 (Application to the finite-stack Euler branch). *Suppose a planar Euler construction supplies a radial compactly supported zero-circulation datum $\omega_0 \in E_p$ and a nonstationary branch ω^{ns} satisfying (3) with the finite bounded zero-circulation Euler stacks. Then the fixed-data inviscid limit is uniquely ω^{st} , while a sequence of smooth zero-circulation initial data converging strongly to ω_0 selects ω^{ns} under vanishing viscosity, both in $C([0,T];E_p)$.*

Remark 5. The argument fixes the approximation index before choosing viscosity. The quantities D_n , the finite-stack L^∞ norms, and the smoothing scales need not have uniform bounds. It establishes existence of a selecting sequence rather than a scale law relating ν_n to the original cascade radii. If the perturbed initial data were all radial, their solutions would again be heat evolutions and would converge to the stationary branch. The selection of a nonstationary branch therefore requires nonradial perturbations.

References

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- [2] I. Gallagher and T. Gallay, *Uniqueness for the two-dimensional Navier–Stokes equation with a measure as initial vorticity*, Mathematische Annalen **332** (2005), 287–327. arXiv:math/0406297.