Unforced two-dimensional Euler on the torus
An annular cascade and the nonlinear transfer estimate

Revised research proof for Umberto Pappalettera

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Scope and verification. This is a self-contained revised research argument for the theorem below. Fresh independent reviewers checked the analytic contour construction, full material spectrum, interaction estimates, nonlinear cascade, and compactness. Their checks identified a missing forced endpoint-comparison lemma, an invalid application of the small-noise argument to the deepest initial source, and an implicit support estimate on an outgoing compact source window. Each is repaired here; the two growth phases also have distinct continuation arguments. Exact symbolic identities provide supplementary checks of the linear algebra. These are internal mathematical reviews, not external peer review or a formal proof-assistant certificate. The argument should be assessed from its lemmas and proofs, rather than from the reviewers' failure to find an error.

Proposed theorem 1. *For every $2 < p < \infty$, the square torus admits a mean-zero $\omega_0 \in L^p(\mathbb{T}^2)$, with zero-mean associated velocity, and two different solutions of*

$$\partial_t \omega + u \cdot \nabla \omega = 0, \qquad u = \nabla^\perp \Delta^{-1} \omega, \qquad \int_{\mathbb{T}^2} u = 0,$$

in $C([0,T]; L^p)$ with initial value ω_0. Both solutions are renormalized. The construction gives a stationary solution and a nonstationary one.

The construction uses shielded, linearly unstable annuli and a fixed amplitude ratio. A nonlinear pulse can have oscillatory late forcing. Its *whole* discounted fourth moment is used, rather than a pointwise sign condition. Nonvanishing of that transform is obtained by choosing the amplitude ratio generically. Only a fixed time window is compared with a canonical orbit; the remaining evolution is estimated by transport, support and moment bounds.

1 Parameters and the exact stationary datum

Identify $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and put the origin at the centre of a fundamental square. Throughout, C denotes a finite constant independent of the truncation depth N and of sufficiently small q. It can depend on fixed parameters and reference windows. For brevity, every occurrence of $\mathrm{poly}(L)$ below means at most $C(1+L)^{20}$; it is one fixed power for the entire induction. The estimates displayed explicitly use powers no larger than fourteen; twenty also covers the eighth-moment source bound and one history integration. No exponent is increased at successive stages. Set

$$\rho^2 = \frac{3}{5}, \qquad \sigma = \frac{\rho^5}{2}, \qquad \tau = \frac{3}{10}, \qquad W = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}.$$

Thus $W = -2/5$ on B_ρ, $W = 3/5$ on $B_1 \setminus B_\rho$, and $\int W = 0$. The planar velocity of $AW(|x|/R)$ is tangential and vanishes outside B_R. Its inner solid rotation is $-A/5$. Choose

$$A_j = A_1 \beta^{j-1}, \qquad R_j = R_1 q^{j-1}, \qquad \beta \in [4,5], \quad R_1 = q, \quad L = \log(1/q), \quad \varepsilon = q^6.$$

The choices of β and q will be specified below; $A_1 > 0$ is arbitrary. Require $\beta q^{2/p} < 1$ and $\beta q < 1$. Then

$$\bar\omega = \sum_{j \ge 1} A_j W(|x|/R_j) \in L^p, \qquad \sum_j \|u_j\|_\infty \le C \sum_j A_j R_j < \infty.$$

Choose a compactly supported streamfunction for each zero-circulation radial summand and extend it periodically. Its mean-zero velocity is exactly its planar velocity. Hence $\bar\omega$ is an exact stationary torus solution: the velocity is tangential wherever the vorticity is radial. The same observation applies to every renormalization of $\bar\omega$. On the gaps $R_{j+1} < r < \rho R_j$, $\bar\omega(r) = -(2/5)\sum_{i \le j} A_i$; the datum is unbounded. Write $\bar\omega_N = \sum_{j \le N} A_j W(|x|/R_j)$.

2 Local contours, coordinates and the isolated orbit

2.1 An analytic material equation

Fix $0 < \gamma < \gamma_0 < 1$. Use the little Hölder space

$$E = c^{1,\gamma}(S^1;\mathbb{R}^2)^2$$

of two material embeddings, with fourfold symmetry, near $X_i^0(\theta) = r_i e^{i\theta}$, $r_1 = \rho, r_2 = 1$. The planar velocity field V on these contours is analytic on a fixed chord-arc neighbourhood. Here is an explicit justification of the differentiability needed below.

For one reference disk D, extend boundary perturbations harmonically: $Hh \in C^{1,\gamma}(\overline{D})$. Dacorogna–Moser's divergence right inverse [2, Theorem 2] gives

$$B : C^{\gamma}_{\int=0}(D) \longrightarrow C^{1,\gamma}_{\text{zero trace}}(D;\mathbb{R}^2), \qquad \operatorname{div} Bf = f.$$

Put $J(h) = \operatorname{Area}(X^0 + h)/|D|$. The map

$$(h, f) \longmapsto \det D(\mathrm{Id} + Hh + Bf) - J(h)$$

takes values in mean-zero C^{γ}, by the determinant null-Lagrangian identity and its boundary trace, and its f derivative at zero is the identity. The analytic implicit-function theorem supplies $f(h)$, and an extension $E(h)$ with the prescribed boundary and $\det DE(h) = J(h)$. In the complexification, the area is its oriented boundary polynomial, and the determinant is its usual polynomial in the matrix entries. The mean-zero identity therefore persists holomorphically; it is not a statement about complex-valued Lebesgue area.

Hernandez's complex near-identity bound [1, Definition 1.1, Proposition 1.3], applied to the convex disk D, makes

$$F_D(X)(a) = \int_D K(X(a) - X(b))\, db$$

Banach analytic in $C^{1,\gamma}$. To see the norm statement, take Cauchy coefficients on any complex parameter line. Pointwise coefficients are holomorphic by dominated kernel integration. Difference quotients in the spatial variable converge uniformly, with error $O(|\Delta a|^{\gamma})$, on a smaller complex disc, so their derivatives are holomorphic as well. Cauchy estimates applied also to spatial derivative differences retain the full $C^{1,\gamma}$ norm bound. The resulting coefficient series converges in that norm, proving Fréchet holomorphy. His bound does not impose a Jacobian-one constraint on X. For real embeddings the extension is a diffeomorphism. The real change-of-variables formula gives the *physical* self-interface velocity as

$$J(h)\operatorname{Tr} F_D(E(h)).$$

The factor $J(h)$ is essential. The two self terms are treated on their own disks. Cross-interface contour integrals have separated boundaries and nonsingular logarithmic kernels, and are analytic by composition. The torus regular kernel is analytic as well. This proves, after normalizing $x = Rz,\ t = s/A$,

$$\left\| V(X^0 + h) - V(X^0) - DV(X^0)h \right\|_E \le C\,\|h\|_E^2\,. \tag{1}$$

Here is the little-space completion explicitly. For smooth real embeddings, the physical contour velocity is smooth: split its logarithmic boundary kernel into $\log|2\sin((\theta-\phi)/2)|$ and a smooth kernel; convolution of the first with smooth boundary derivatives remains smooth. Approximate little-space embeddings by smooth ones. Big-space analytic continuity then puts their velocities in the closed little subspace. Finite differences along real little directions put all Taylor coefficients in that subspace; restricting and complexifying the convergent analytic series proves bounded analyticity on E. No smooth-preserving property of the chosen divergence inverse is needed. There is no nonlinear radial-graph ODE in this argument.

Remove the known solid rotation from all larger colours. Then the planar reference inner and outer material rotations are $\Omega_1 = -1/5, \Omega_2 = 0$. Independent constant label shifts commute with the material equation. The stationary label gauge is the semilinear flow

$$\partial_s X_i = V_i(X) - \Omega_i \partial_\theta X_i.$$

Label translation is a strongly continuous isometric group on the little Hölder space. In displacement coordinates this is exactly

$$h_s = \mathcal{A}_0 h + [V(X^0 + h) - V(X^0)], \qquad \mathcal{A}_0 h = -\Omega_i \partial_\theta h_i,$$

since $\mathcal{A}_0 X^0 + V(X^0) = 0$. Mild Grönwall keeps small displacements in a fixed analytic neighbourhood for all $|s| \le P$. Differentiating the mild fixed point gives C^2 fixed-time maps; there is no need to follow a rotating material reference outside that neighbourhood. This gauged flow fixes X^0. At $P = 10\pi$ the label translations are identities, so its time-P map is the material time-P map.

2.2 The full spectral split

For a positive geometric normal Fourier mode m, the linear matrix is

$$L_m = \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

Fourfold symmetry allows $m = 4, 8, \ldots$. At $m = 4$ the exponents are $\pm\sigma + i\tau$, the growing normal vector is

$$v_+ = (1, -i\rho), \qquad \ell = (1/2, i/(2\rho)), \qquad \ell v_+ = 1.$$

For $m \ge 8$ the frequencies are

$$m/5 - 1/2 - \delta_m, \qquad 3/10 + \delta_m, \qquad 0 < \delta_m = O(\rho^{2m}/m).$$

The normal period multipliers are $-e^{\pm iP\delta_m}$, uniformly separated from 1. Their diagonalizers differ from the identity by exponentially summable Fourier coefficients. For $n \ge 1$,

$$\sum_{m=8,12,\ldots} \left| e^{inP\delta_m} - 1 \right| \le C \sum_m \min\{2, n\rho^{2m}/m\} \le C(1 + \log(1+n)).$$

This bounds the centre powers on the Hölder space.

The full material calculation is explicit. Write $h_i = r_i e^{i\theta}(z_i + i\xi_i)$, with real normal and tangential functions. A normal perturbation of the disk with strength c_l and radius r_l gives the m-th streamfunction coefficient

$$-\frac{c_l r_l^2}{2m} z_{l,m} \begin{cases} (r/r_l)^m, & r < r_l, \\ (r_l/r)^m, & r > r_l. \end{cases}$$

Taking its radial and tangential derivatives, together with the reference-field evaluation derivative, gives for $m > 0$

$$z_s = L_m z, \quad \xi_s = T_m \xi + C_m z, \qquad T_m = \begin{pmatrix} im/5 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_m = \frac{1}{2} \begin{pmatrix} 1 & -\rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

Set $H_m = \mathrm{diag}(i, -i)$. Entrywise multiplication gives

$$H_m L_m - T_m H_m = C_m.$$

For negative modes take complex conjugates. Thus H is a componentwise Hilbert-transform multiplier, bounded on E, and $\zeta = \xi - Hz$ satisfies $\zeta_s = T\zeta$. At $P = 10\pi$, $e^{PT} = I$, so the full nonzero-mode period operator is exactly

$$\begin{pmatrix} M & 0 \\ H(M - I) & I \end{pmatrix}.$$

There is no derivative loss in its shear. Tangential relabelings are neutral at period times; continuously they are label translations. The Fourier symbols of the ungauged DV are order zero; the factors m in the displayed matrices belong to the label generator. For the means, $z_{1,s} = z_{2,s} = 0$, $\xi_{1,s} = 0$, and $\xi_{2,s} = (3/5)(z_2 - z_1)$. This is a Jordan block of order two. We retain these ambient area variations, avoiding a nonlinear area chart. Thus, with P_u, P_c the full unstable and complementary projections,

$$\|S_c(s)\| \le C(1+s), \qquad \|S(s)\| \le C_\eta e^{(\sigma+\eta)s}. \tag{2}$$

The continuous bounds follow by bounded interpolation over one period.

The C^2 period map has a two-real-dimensional local strong unstable manifold. Here is the backward construction, including the direction of propagation on the complement. Let its displacement map be $h \mapsto Mh + R(h)$, with $R(0) = DR(0) = 0$. Fix $0 < \kappa < \sigma$, prescribe the unstable component u_0, and on sequences $n \le 0$ use the norm $\sup_{n\le 0} e^{-\kappa n P}\|h_n\|$. The fixed-point equations are

$$u_n = M_u^n u_0 - \sum_{l=n}^{-1} M_u^{n-1-l} P_u R(h_l),$$
$$c_n = \sum_{l=-\infty}^{n-1} M_c^{n-1-l} P_c R(h_l).$$

Only inverse unstable powers and *forward* complementary powers appear. The former decay exponentially, and the latter grow at most linearly. Their weighted convolution bounds are finite; the quadratic remainder makes this map a contraction for a small fixed ball. Its C^2 dependence on u_0 follows by differentiating the contraction with bounded weighted kernels. The result is a graph tangent to the full growing eigenspace. An isolated subperiod flow commutes with the period map and preserves backward exponential decay. Uniqueness of the backward construction therefore makes the graph continuously invariant wherever it stays in the local chart. Each enclosed area is preserved by the real unforced flow and equals its value at the backward limit. Thus this ambient graph automatically has both reference areas.

2.3 Exact geometric moments and the canonical cap

For the two enclosed domains define

$$Q_i(X) = \frac{1}{2\pi r_i^6} \int_{D_i(X)} \bar{z}^4 \, dz, \qquad a(X) = \ell Q(X).$$

Stokes expresses Q_i as a polynomial functional of X_i, X_i'. It is analytic and invariant under independent relabeling. If z_i is the positive fourth Fourier coefficient of relative normal displacement, then $Q_i = z_i + O(\|h\|^2)$. Consequently $Da(X^0)$ is the full unstable left dual. Pair a with the fixed linear complementary coordinate $b = P_c h$. The inverse-function theorem recovers h analytically from (a, b). Write the unstable graph as $b = g(a)$, $g(a) = O(|a|^2)$.

Only the moment coordinate is differentiated continuously:

$$a_s = DQ(X)V(X) \cdot \ell.$$

The label generator contributes zero. This is a bounded analytic finite-dimensional functional, with linear term $(\sigma + i\tau)a$ and quadratic remainder. Centre Duhamel estimates can be carried out for the C^2 time-P maps; no unbounded generator is conjugated by a nonlinear chart. Between periods, isolated invariance of the graph and bounded flow maps give the same error bounds up to a fixed factor.

The action $X_i(\theta) \mapsto e^{i\phi} X_i(\theta - \phi)$ fixes the circles and commutes with the planar gauged flow. It sends $a \mapsto e^{-4i\phi} a$. Hence the graph is equivariant and its $|a| = \delta$ physical cap is one rotation family.

Choose one ancient orbit $w_*(s)$, with cap at $s = 0$. Near the backward limit,

$$B_*(s) := \int \bar{z}^4 w_*(s,z)\,dz = b_* e^{(\sigma+i\tau)s} + O(e^{(\sigma+\kappa)s}), \qquad b_* \neq 0,$$

because

$$B_* = 2\pi[-\rho^6 Q_1 + \rho^2 Q_2] = 2\pi(-\rho^6 - i\rho^3)a + O(|a|^2 + \|b - g(a)\|). \tag{3}$$

2.4 A causal comparison lemma for forced material period maps

All norms in this subsection are norms of the full material space $E = c^{1,\gamma}(S^1;\mathbb{R}^2)^2$, including tangential components. Let

$$h = X - X^0, \qquad A = -\operatorname{diag}(\Omega_i)\partial_\theta, \qquad N(h) = V(X^0 + h) - V(X^0).$$

Since $V(X^0) = \operatorname{diag}(\Omega_i)\partial_\theta X^0$, the gauged equation is

$$h_s = Ah + N(h) + f(s). \tag{4}$$

Here N is analytic on a fixed local contour neighbourhood, and $G(s) = e^{sA}$ is a strongly continuous isometric group on E. The forcing f is only required to be strongly measurable and locally integrable in E. It may be the external velocity evaluated on the actual contour; no time derivative of that evaluation is assumed. All comparisons below are made on a smaller neighbourhood on which the autonomous flow and its coordinate charts exist for every $|s| \le P$. This is achieved by choosing the cap smaller than the fixed neighbourhood used to construct the period map.

Lemma 1 (Forced endpoint and subperiod comparison)**.** *Let h solve (4) on $[t, t+r]$, $0 \le r \le P$, and let u solve the autonomous equation with $u(t) = h(t)$. Suppose both curves lie in the stated neighbourhood and N has Lipschitz constant K_0 there. Then*

$$\|h(t+r) - u(t+r)\|_E \le e^{K_0 P} \int_t^{t+r} \|f(v)\|_E \, dv. \tag{5}$$

More generally, different starting states add $e^{K_0 P}\,\|h(t) - u(t)\|_E$ to the right-hand side. The same assertion holds backwards on an existing solution interval, with the integral over the corresponding past interval.

Let Φ_P be the autonomous time-P map and let χ be either of the locally Lipschitz charts

$$\chi_0(h) = (a(h), P_c h), \qquad \chi_g(h) = (a(h), P_c h - g(a(h))).$$

At every completed period contained in the solution interval,

$$\chi(h(t+P)) = (\chi \circ \Phi_P \circ \chi^{-1})(\chi(h(t))) + \xi_t, \qquad \|\xi_t\| \le C \int_t^{t+P} \|f(v)\|_E \, dv. \tag{6}$$

All constants are uniform on the chosen neighbourhood.

Proof. Subtract the two mild equations. The group is isometric, so

$$\|h(t+r) - u(t+r)\|_E \le K_0 \int_0^r \|h(t+v) - u(t+v)\|_E \, dv + \int_t^{t+r} \|f(v)\|_E \, dv.$$

Integral Grönwall proves (5). For a backwards interval use $G(-s)$, reverse time, and apply the same estimate. Apply the Lipschitz chart only to the two endpoints to obtain (6). In particular the change of $g(a)$ caused by the forcing is already included in ξ_t. No nonlinear chart is applied to the unbounded generator A. □

Moment differentiation is also compatible with this mild setting. Undo the label translations to obtain the material contour solution, differentiate its relabeling-invariant moment there, and translate back. In the stated gauged equation this gives

$$a_s = Da(X)[V(X) + f(s)]$$

at almost every time (and continuously when the evaluated forcing is continuous). This identity does not apply Da to AX as an E-valued vector. Additional spatial rotating-frame terms are bounded moment terms and are computed by rotation covariance.

The same mild comparison, without the coordinate-chart assertion, applies along any fixed compact regular chord-arc trajectory. Cover that trajectory by finitely many neighbourhoods on which the material contour field is locally Lipschitz in E, and subdivide time so each comparison stays in one such neighbourhood. The constants then depend on the fixed trajectory and interval. This uses the usual local contour-flow theorem on regular separated interfaces; the period map and its near-circle spectral split are not asserted outside the near-circle chart.

For every $0 \le r \le P$, the family of autonomous endpoint maps is uniformly C^2. In either chart its derivative at zero is $S(r)$. Thus the raw complementary endpoint has the form $S_c(r)b + O(|a|^2 + |a|\|b\| + \|b\|^2)$. Continuous graph invariance makes the flattened complementary endpoint $S_c(r)y + O(|a|\|y\| + \|y\|^2)$: its remainder is zero at $y = 0$, and its derivative in y at $(0,0)$ is zero. The same forcing error $C\int_t^{t+r} \|f\|_E$ applies in both charts by Lemma 1. These uniform statements include the last incomplete period, with no use of future forcing.

Write $M = D\Phi_P(0)$. In the raw chart, its unstable and complementary blocks are M_u and M_c. Consequently the forced completed-period recurrence is

$$\begin{aligned} a_{n+1} &= M_u a_n + R_u(a_n, b_n) + \xi_{u,n}, \\ b_{n+1} &= M_c b_n + R_c(a_n, b_n) + \xi_{c,n}, \\ \|R_u\| + \|R_c\| &\le C(|a|^2 + |a|\,\|b\| + \|b\|^2), \qquad \|\xi_n\| \le C\int_{nP}^{(n+1)P} \|f\|_E \, ds. \end{aligned} \tag{7}$$

The displayed remainder follows from the C^2 endpoint map and the derivative of χ_0 at zero. Convolution with $\|M_c^n\| \le C(1+n)$ is therefore a discrete Duhamel estimate with a genuine $L^1_s E$ forcing error.

For clarity, the corresponding causal subperiod bound does not have a linear term proportional to the unstable amplitude. If $S(s) = e^{s(A+DN(0))}$, then P_c commutes with $S(s)$: the spectral projection of $S(P)$ commutes with every $S(s)$. Applying P_c to the mild equation linearized at zero gives, for a last incomplete period $[nP, nP + r]$,

$$\|b(nP+r)\| \le C\,\|b_n\| + C\int_{nP}^{nP+r} \left(|a|^2 + |a|\,\|b\| + \|b\|^2 + \|f\|_E \right) ds. \tag{8}$$

Here the analytic inverse raw chart makes $\|h\|$ equivalent to $|a| + \|b\|$, and the Taylor remainder is quadratic. Only times at or before the current time occur in this estimate.

Thus, on a stopped interval of length $O(L)$, with $\|b(0)\| = O(\varepsilon)$, $|a| \le q^3$, and $\|f\|_E \le C\varepsilon L^4 + Cq^8 K\delta$, the bootstrap $\|b\| \le B\varepsilon L^6$ gives

$$\|b\| \le C\varepsilon L^6 + Cq^3 L^2 B\varepsilon L^6 + CB^2\varepsilon^2 L^{14} + Cq^8 K\delta L^2.$$

Completed periods use (7); the last fraction uses (8). Choosing B, then q, closes the stated raw complementary bound at every stopped current time.

In the flattened chart $y = b - g(a)$, invariance of the isolated graph gives instead

$$y_{n+1} = M_c y_n + \widetilde{R}_c(a_n, y_n) + \widetilde{\xi}_{c,n}, \qquad \left\| \widetilde{R}_c(a,y) \right\| \le C(|a| + \|y\|)\,\|y\|\,, \tag{9}$$

with the same endpoint forcing bound. Indeed $\widetilde{R}_c(a,0) = 0$ exactly, and its derivative in y at $(0,0)$ is zero. The fractional-period estimate is obtained by comparing with the autonomous graph trajectory starting at the point having the same a_n:

$$\|y(nP+r)\| \le C\,\|y_n\| + C\int_{nP}^{nP+r} \|f\|_E\, ds. \tag{10}$$

The initial states differ by $O(\|y_n\|)$, and the autonomous reference remains on the graph; Lemma 1 proves the claim.

With $\|M_c^n\| \le C_\eta e^{\eta n P}$, choose the cap so small that the coefficient contributed by $C_\eta C(|a| + \|y\|)$ is at most η. Discrete Grönwall in (9), followed by (10), yields

$$\|y(s)\| \le C\varepsilon(1+L)^6 e^{2\eta s}$$

from $\|y(0)\| \le C\varepsilon L^6$ and $\|f\|_E \le C\varepsilon(1+L)^4$. Once the independent radial growth estimate supplies the post-q^3 clock $s \le 4L/\sigma + C$, this gives

$$\|y_{\mathrm{cap}}\| \le C q^{6-8\eta/\sigma}(1+L)^6.$$

The endpoint and subperiod statements establish no future cap by themselves; they hold on the actual stopped history.

Native and incoming rotating frames. Define the reference-fixing action

$$(\mathcal{R}_\phi X)_i(\theta) = e^{i\phi} X_i(\theta - \phi).$$

It is a strongly continuous isometry of E, fixes X^0, and commutes with the isolated planar gauged flow. The projections and graph may be chosen equivariant, so

$$a(\mathcal{R}_\phi X) = e^{-4i\phi} a(X), \qquad P_c \mathcal{R}_\phi h = \mathcal{R}_\phi P_c h, \qquad g(e^{-4i\phi} a) = \mathcal{R}_\phi g(a).$$

Raw and flattened complementary norms are therefore unchanged. Passing from the receiver's native frame to the incoming frame adds $\mathcal{R}_{\theta/5}$ in receiver time θ. It changes the moment exponent by $-4i/5$, from $\sigma + i\tau$ to $\sigma - i/2$. Since $P/5 = 2\pi$, its additional action over one period is the identity. Hence the same autonomous period map and complementary power bounds apply, up to the isometry specifying the initial frame angle. The transformed forcing has the same E norm. The torus image field is included in that forcing; no continuous rotation symmetry of the square torus is used. Finally $s = \beta\theta$ changes the receiver period to βP and replaces f_θ by $f_s = f_\theta/\beta$, preserving its integrated norm. These observations justify the period-map estimates in either normalization.

Chemin's general tangential-regularity theorem [3, Theorem 1, p. 522] supplies global finite-time continuation of this signed two-interface patch. Indeed choose a divergence-free C^{γ_0} field tangent and nonzero on both interfaces, using local defining functions on disjoint tubes. Its derivative of the signed piecewise constant vorticity vanishes distributionally; the vorticity is smooth off the tubes. The theorem has no positivity assumption. Constructing the local manifold first at exponent γ_0 and using uniqueness in the lower space gives the same orbit in little $c^{1,\gamma}$ on every compact time interval. Conservation of L^1, L^∞ gives

$$\|K w_*\|_\infty \le C, \quad \operatorname{supp} w_*(s) \subset B_{C(1+s)}, \quad |B_*(s)| \le C(1+s)^4 \quad (s \ge 0).$$

3 A nonlinear transfer coefficient that cannot vanish identically

Define the bilateral transform

$$F(z) = \int_{\mathbb{R}} e^{-zs} B_*(s)\, ds, \qquad 0 < \operatorname{Re} z < \sigma.$$

The bounds just proved give holomorphy, including differentiation under the integral on compact substrips. It is not identically zero: on a vertical line it is the Fourier transform of the nonzero L^1 function $e^{-cs}B_*(s)$. The identity theorem therefore lets us fix $\beta \in [4,5]$ such that

$$F(z_\beta) \neq 0, \qquad z_\beta = \frac{\sigma - i/2}{\beta}. \tag{11}$$

If necessary shrink the cap by translating the same ancient orbit. For $B_\delta(s) = B_*(s + t_\delta)$,

$$F_\delta(z) = e^{zt_\delta}F(z), \qquad |F_\delta(z_\beta)| \asymp \delta^{1/\beta}.$$

Thus nonvanishing survives any sufficiently small cap, and $\delta/|F_\delta(z_\beta)| \to 0$. Fix the cap-independent local, weight and packet constants described below, then choose the cap small enough for all local inequalities and retain the notation B_*, F.

For any fourfold zero-circulation source inside both receiving interfaces, write $B_{\mathrm{phys}} = \int \bar{z}^4 \omega_{\mathrm{in}}\, dz$. The logarithmic potential expansion gives the relative normal fourth Fourier forcing at a receiving circle of radius r:

$$f_4(r) = \frac{iB_{\mathrm{phys}}}{4\pi r^6}.$$

These are Fourier coefficients, not real cosine amplitudes. The growing projection is

$$D_4 \frac{B_{\mathrm{phys}}}{R_j^6}, \qquad D_4 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0.$$

Let $k = j + 1$. The larger-colour background is $\Omega_j^{\mathrm{ext}} = -(1/5)\sum_{i<j} A_i$, and $\Omega_k^{\mathrm{ext}} = \Omega_j^{\mathrm{ext}} - A_j/5$. In the incoming rotating frame, the receiver exponent is

$$A_j(\sigma + i\tau - 4i/5) = A_j(\sigma - i/2).$$

In incoming time $s = A_k(t - T_k)$, its leading equation is therefore

$$a_s = z_\beta a + D_4 q^6 B_*(s).$$

The full discounted seed is $q^6 D_4 F(z_\beta)$. Set $c_0 = |D_4 F(z_\beta)| > 0$.

Choose a fixed S large enough that the estimates for both canonical tails, and the actual tails in Section 8, are below $c_0/100$. Choose a fixed cap tube whose finite-window continuity error is below $c_0/100$. The actual reset error will in fact tend to zero; a fixed tube alone would give $O(\mathrm{tube})$, not $o(1)$. The canonical inner boundary has positive distance from the origin on $[-S, S]$. Fourfold symmetry fixes the origin, and the transported domain continues to contain it. This supplies a fixed central hole for the small deeper packet.

4 Exact interaction estimates and torus images

Write each transported colour as

$$w_l = A_l[-\mathbf{1}_{D_{l,1}} + \rho^2 \mathbf{1}_{D_{l,2}}].$$

Each domain has its reference area. Every colour separately preserves zero circulation, fourfold symmetry, $L^1 \leq CA_l R_l^2$ and $L^\infty \leq CA_l$. For a colour in its local contour neighbourhood let $x_l = \|h_l\|_E$. Exterior harmonic expansion, using the vanishing moments of degrees one through three and zero circular exterior field, gives

$$\begin{aligned} l > i: &\quad \|\text{relative field on core } i\| \leq CA_l q^{6(l-i)} x_l, \\ l < i: &\quad \|\text{field after subtracting solid rotation on } i\| \leq CA_l q^{2(i-l)} x_l. \end{aligned} \tag{12}$$

The norms include the full $C^{1,\gamma}$ evaluation on a perturbed receiving contour. To see the powers directly, an exterior degree-four moment is $O(A_l R_l^6 x_l)$, and relative velocity is that moment divided by R_i^6. In an interior constant-vorticity disk the nonradial harmonic velocity starts at cubic order: $O(A_l x_l |x|^3 / R_l^2)$. Derivative bounds have the same powers after normalization; higher harmonics improve them. These bounds are analytic in the contour perturbations. For an arbitrary spent packet the first bound is replaced by its conserved mass times the fourth power of its radius.

Near zero the regular part of the square torus Green function is

$$H(z) = c + c_2 |z|^2 + \operatorname{Re} \sum_{n \geq 1} c_{4n} z^{4n}.$$

Convolve with a *complete constrained colour*. Its circulation and first three holomorphic moments vanish. The quadratic term gives zero velocity; the degree-four term gives only a constant potential. The first possible nonzero image velocity has source degree four and target degree three, from the degree-eight term. Its relative size is

$$C A_l R_l^6 R_i^2 x_l.$$

Relative to (12) this gains R_i^8 for an inner source, R_l^8 for an outer source, and R_i^8 for self interaction. All are at most $R_1^8 = q^8$. This cancellation would be false for unconstrained individual disks. For actual colours it is exact.

5 The active packet needs no shape control

For every transported colour, the torus velocity bound is

$$\|K_{\mathbb{T}} w_l\|_\infty \leq C \sqrt{\|w_l\|_1 \|w_l\|_\infty} + C \|w_l\|_1 \leq C A_l R_l.$$

Consequently the velocity induced by all colours $l \geq k$ is bounded by $C A_k R_k/(1-\beta q)$, independently of N. All initial domains are nested; the common Yudovich homeomorphism preserves every inclusion.

Suppose the larger contours remain inside fixed small caps, and the packet remains in their central plateaux. Their exact constant rotations do not change its radius. Fourfold symmetry eliminates linear harmonic strain, leaving the cubic estimate from (12). If d is the packet radius divided by R_k and $s = A_k(t - T_k)$, the upper Dini derivative obeys

$$d'(s) \leq C + C\beta^{-1} q^2 d(s)^3. \tag{13}$$

For $0 \leq s \leq C_\beta L$, bootstrap $d \leq D_0(1+s)$ and integrate: the cubic contribution divided by $1+s$ is at most $C q^2 (1+L)^3$. Choose D_0 first, then q, to improve the bootstrap. Thus

$$d(s) \leq C(1+s), \qquad |B_{\geq k}(s)| \leq C A_k R_k^6 (1+s)^4.$$

In particular

$$\int_S^{C_\beta L} e^{-\sigma s/\beta} \frac{|B_{\geq k}(s)|}{A_k R_k^6} \, ds \leq C_\beta e^{-\sigma S/\beta} (1+S)^4. \tag{14}$$

At the next receiving cap its radius is $C q L R_{k-1}$. The deeper packet $l > k$, throughout the history before T_k, has radius at most $C q L R_k$. Before the previous cap this follows from the earlier stage; afterwards apply (13) with the bounded summed waiting time. Its moment in incoming units is

$$O(q^6 L^4).$$

On the preceding fixed window $[-S, 0]$ this pre-cap bound puts it inside the source's inner hole. On the outgoing half $[0, S]$, the separate compact-tube cubic-radius bootstrap in Lemma 4 proves the same bound even when the source leaves its near-circle chart. Its full contour forcing is $O(q^6(1+L)^4)$. Later it is treated as part of the aggregate.

6 Inactive contours: a weighted estimate uniform in depth

We always take $L \geq 1$. A receiving stage has current index j, incoming index $k = j + 1$, initial time T_k, and current time $\theta = A_j(t - T_k)$. Its deadline is

$$H = \frac{9}{\sigma} L.$$

For $j = 1$, every inactive maximum and inactive feedback below is defined to be zero. All constants are independent of j, the finite depth N, and sufficiently small q.

Local estimates and induction data

Let $E = c^{1,\gamma}(S^1; \mathbb{R}^2)^2$. In each colour's stationary material gauge and its own frame with the exact larger-colour solid rotation removed, write its perturbation as h_i, and put $x_i = \|h_i\|_E$. Rotations and the label translations used in these gauges are bounded uniformly in this norm. Fix a local neighbourhood of radius $r_* > 0$, sufficiently small that its inner domains contain $B_{r_{\mathrm{pl}}}$ in normalized coordinates for one fixed $r_{\mathrm{pl}} > 0$. Choose this inner neighbourhood small enough that every isolated subperiod orbit starting there stays in a larger fixed comparison neighbourhood for $0 \leq r \leq P$. The constants below are taken on these fixed neighbourhoods, before choosing δ or q.

The isolated linear flow satisfies, for $s \geq 0$,

$$\|S(s)\| \leq M_\eta e^{(\sigma+\eta)s}, \qquad \|S_c(s)\| \leq M(1+s), \qquad \|M_c^n\| \leq M(1+n), \quad M_c = S_c(P). \tag{15}$$

The full local equation has a mild form with this linear flow, and its isolated nonlinear remainder has norm at most Cx_i^2. The full E-norm interaction coefficients, after removal of solid rotation, are bounded by

$$\begin{cases} CA_l q^{6(l-i)} x_l, & l > i, \\ CA_l q^{2(i-l)} x_l, & l < i. \end{cases} \tag{16}$$

Torus image interactions satisfy the corresponding bounds with an additional factor at most q^8; a local self-image term is bounded by $CA_i q^8 x_i$. These are bounds for both material contours, including tangential components and the full $C^{1,\gamma}$ norm.

The current geometric coordinates are (a, b), with inverse chart and unstable graph satisfying

$$x_j \leq C_{\mathrm{ch}}(|a| + \|b\|), \qquad |a| + \|b\| \leq C_{\mathrm{ch}} x_j, \qquad \|g(a)\| \leq C_g |a|^2, \qquad y = b - g(a). \tag{17}$$

The exact continuous moment equation, in any of the rotating frames needed here, is

$$a_\theta = \lambda a + N(a, b) + F_a(\theta), \qquad \Re\lambda = \sigma, \qquad |N(a, b)| \leq C_a(|a|^2 + |a| \, \|b\| + \|b\|^2). \tag{18}$$

The scalar forcing is bounded by the full material forcing norm.

We use the following form of the separately proved forced-endpoint lemma. For every $r \in [0, P]$, its raw and flattened maps satisfy

$$b(\theta + r) = S_c(r) b(\theta) + R_{b,r} + E_{b,r}, \quad \|R_{b,r}\| \leq C_E(|a(\theta)|^2 + |a(\theta)| \, \|b(\theta)\| + \|b(\theta)\|^2), \tag{19}$$

$$y(\theta + r) = S_c(r) y(\theta) + R_{y,r} + E_{y,r}, \quad \|R_{y,r}\| \leq C_E(|a(\theta)| \, \|y(\theta)\| + \|y(\theta)\|^2), \tag{20}$$

$$\|E_{b,r}\| + \|E_{y,r}\| \leq C_E \int_\theta^{\theta+r} \|F_j(v)\|_E \, dv. \tag{21}$$

The constants are uniform in r, and the bounds hold whenever the actual stopped trajectory and the bounded-time isolated comparisons remain in their respective fixed neighbourhoods. The graph is invariant for every isolated subperiod flow. Thus these statements include bounded interpolation and do not differentiate the unbounded label generator through g.

Let $d(s)$ be the support radius of the aggregate packet $l \geq k$, divided by R_k, with $s = A_k(t-T_k) = \beta\theta$. While the packet is in the central plateaux and all larger colours are local, transport gives the upper Dini inequality

$$D^+ d(s) \leq C_v + C_r q^2 d(s)^3. \tag{22}$$

The packet starts with $d(0) \leq d_0$, where d_0 is fixed. If $d(\beta\theta) \leq D_0(1+\beta\theta)$ and central separation holds, its full material forcing on the current receiver is at most $C_p\varepsilon(1+\theta)^4$; here and below C_p may depend on the fixed D_0. Its weighted forcing on the inactive system is at most $C_p\varepsilon(1+\theta)^4/D$. These follow from conserved mass, fourfold cancellation, and separated exterior kernel derivatives.

Fix $D > \beta$, and define

$$w_i = (D\varepsilon)^{j-i}, \qquad I_j(\theta) = \max_{i<j} \frac{x_i(\theta)}{w_i}.$$

The initial induction data, reindexed from the completed cap of k, are

$$I_j(0) \leq DK\delta\varepsilon, \qquad x_j(0) \leq DK\delta\varepsilon, \qquad d(0) \leq d_0. \tag{23}$$

All preceding completed stages have their clock, current-time prefix, support, and source-window conclusions. Lemma 5 supplies these data for the deepest source, uniformly in N.

The signed-transfer estimate proved in Section 8 is used in the following stopped-interval form. There are fixed $\Theta_w > 0$ and $c_s > 0$ such that, if the current receiver remains local with $|a| < q^3$, has the raw-centre and upper-amplitude bounds proved below, and satisfies the prefix and packet barriers, then

$$\left|e^{-\lambda\theta} a(\theta)\right| \geq c_s\varepsilon \quad (\Theta_w \leq \theta \text{ before its first } q^3 \text{ crossing}). \tag{24}$$

Here the source is already capped; its negative history is part of the preceding completed-stage data. In the first pulse it is supplied by the direct base-pulse comparison. No cap of the current recipient is an assumption of (24). The bound is proved by the whole discounted source pulse, including its negative history, rather than by a sign of instantaneous forcing.

The inactive system

Lemma 2 (Uniform weighted Volterra estimate)**.** *Choose $0 < \eta < \sigma/32$. One may choose D, then a fixed $\alpha_* < \sigma/2$, and finally sufficiently small q, so that on every interval with*

$$I_j \leq 2K\delta, \qquad x_i \leq r_* \ (i \leq j), \qquad d(\beta\theta) \leq D_0(1+\beta\theta),$$

and central separation, there is a common exponent $0 < \alpha \leq \alpha_*$ for which

$$I_j(\theta) \leq C_w e^{\alpha\theta} I_j(0) + C_w \int_0^\theta e^{\alpha(\theta-v)} [x_j(v) + \varepsilon(1+v)^4]\, dv. \tag{25}$$

The normalized current feedback and its discounted integral satisfy

$$\|F_{\mathrm{out}}(\theta)\|_E \leq C_f D q^8 I_j(\theta), \tag{26}$$

$$q^8 \int_0^\theta e^{-\sigma v} I_j(v)\, dv \leq Cq^8 I_j(0) + Cq^8\varepsilon + Cq^8 \int_0^\theta e^{-\sigma v} x_j(v)\, dv. \tag{27}$$

The constants do not depend on the number of inactive colours.

Proof. For an inactive receiver $i < j$, an inner inactive source $l = i + d < j$ has weighted coefficient

$$\frac{A_l}{A_j} q^{6d} \frac{w_l}{w_i} = \frac{A_i}{A_j} (\beta/D)^d.$$

An outer source $l = i - d$ has weighted coefficient

$$\frac{A_l}{A_j} q^{2d} \frac{w_l}{w_i} = \frac{A_i}{A_j} (Dq^8/\beta)^d.$$

Their row sums are bounded by

$$\frac{C}{\beta} \left(\frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right).$$

The weighted self-image row is at most Cq^8/β. Since $w_i \le D\varepsilon$, the local quadratic term divided by w_i is at most

$$C \frac{A_i}{A_j} w_i I_j^2 \le \frac{2CDK\delta\varepsilon}{\beta} I_j.$$

Torus cross-image rows are smaller than the already displayed rows. The current source has coefficient $q^{6(j-i)}/w_i = D^{-(j-i)} \le 1/D$. The packet row is bounded as stated in the mass and support bounds in Section 5. Set

$$\nu = \frac{\sigma + \eta}{\beta}, \qquad r(D, q) = \frac{C}{\beta} \left(\frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} + q^8 + 2DK\delta\varepsilon \right).$$

Taking the maximum of the full mild equations, using $A_i/A_j \le 1/\beta$, gives

$$I_j(\theta) \le M_\eta e^{\nu\theta} I_j(0) + M_\eta \int_0^\theta e^{\nu(\theta - v)} \{ r(D, q) I_j(v) + C[x_j(v) + \varepsilon(1 + v)^4]/D \} \, dv.$$

This is a scalar Volterra inequality for a finite maximum; no summation over the number of rows occurs. Multiplying by $e^{-\nu\theta}$ and applying integral Grönwall gives (25) with $\alpha = \nu + M_\eta r(D, q)$ and a fixed C_w. Choose D so that $\nu + M_\eta r(D, 0) < \alpha_* < \sigma/2$; then choose q so that $Dq^8 < \beta/2$ and $\alpha \le \alpha_*$.

For feedback into j, the source $i = j - d$ has coefficient

$$\frac{A_i}{A_j} q^{2d} w_i = (Dq^8/\beta)^d.$$

The geometric sum proves (26). To prove (27), insert (25) and use Tonelli's theorem for its nonnegative majorants. For any v,

$$\int_v^\theta e^{-\sigma u} e^{\alpha(u - v)} \, du \le \frac{e^{-\sigma v}}{\sigma - \alpha_*}.$$

The polynomial source has a finite exponentially discounted integral. This gives the asserted bound with constants uniform in α. □

Lemma 3 (Current-time prefix improvement)**.** *Suppose that, at every current time θ in a stopped stage, $|a(\theta)| \le \delta$, and that for all $v \le \theta$*

$$x_j(v) \le C_x \, |a(\theta)| \, e^{-\sigma_0(\theta - v)} + C_n \varepsilon (1 + L)^{12}, \qquad \sigma_0 = \frac{3\sigma}{4}. \tag{28}$$

Then

$$I_j(\theta) \le C_{\mathrm{pr}} \, |a(\theta)| + C_{\mathrm{rem}} \varepsilon (1 + L)^{12} e^{\alpha\theta}, \qquad 0 \le \theta \le H. \tag{29}$$

Here C_{pr} is independent of B, C_n, K, provided C_x is fixed in advance. The remainder constant may depend on these fixed choices. Choosing $K \ge 4C_{\mathrm{pr}}$ and then sufficiently small q improves the stopped prefix $I_j \le 2K\delta$ to $I_j \le K\delta/2$.

Proof. Insert (28) into (25). The above-noise convolution is at most

$$\frac{C_w C_x}{\sigma_0 - \alpha_*} |a(\theta)| .$$

The other terms, including $I_j(0) \le DK\delta\varepsilon$, are bounded by $C_{\mathrm{rem}}\varepsilon(1+L)^{12}e^{\alpha\theta}$. For the polynomial packet term use $\alpha \ge \nu > 0$, so that $\int_0^\infty e^{-\alpha v}(1+v)^4\,dv$ is uniformly bounded. This proves (29). Since

$$\varepsilon(1+L)^{12}e^{\alpha H} \le q^{6-9\alpha_*/\sigma}(1+L)^{12} \longrightarrow 0,$$

choose q so that the remainder is at most $K\delta/4$. The amplitude term is at most $K\delta/4$. Hence $I_j \le K\delta/2$, a strict half-size improvement even relative to the desired output bound $K\delta$. □

7 Closing the nonlinear estimate before q^3

Start a stage at T_k, and normalize by the receiver amplitude A_j. Its inherited full contour norm is $O(\varepsilon)$. Stop at the first of: $|a| = q^3$, loss of a local contour or packet neighbourhood, $I_j = 2K\delta$, or the deadline

$$H = \frac{9}{\sigma}L.$$

The packet bounds and the stopped prefix bound give actual external forcing $O(\varepsilon L^4)$; $q^8 I_j$ and the self-image term are smaller. All polynomial exponents here are fixed independently of N.

In the raw coordinate b, analytic period-map Taylor bounds have quadratic terms $C(|a|^2 + |a|\,\|b\| + \|b\|^2)$, and complementary powers at most $C(1+n)$. On this stopped interval $|a|^2 \le q^6 = \varepsilon$. Bootstrapping $\|b\| \le B\varepsilon L^6$ gives

$$\|b\| \le C\varepsilon L^6 + Cq^3L^2\,B\varepsilon L^6 + CB^2\varepsilon^2L^{14} + Cq^8K\delta L^2.$$

Choose B first and q afterwards. Every nonconstant bootstrap coefficient tends to zero; the last term is smaller than εL^6. Hence

$$\|b(\theta)\| \le C\varepsilon L^6. \tag{30}$$

Bounded interpolation over one period proves the continuous bound. This proof requires neither a signed seed nor a future cap time.

The moment equation, in any fixed rotating frame, is

$$a_\theta = \lambda a + \varepsilon f(\theta) + N_a + f_{\mathrm{outer}} + f_{\mathrm{image}} + f_{\mathrm{higher}}, \quad \operatorname{Re}\lambda = \sigma,$$

where the leading forcing has polynomial growth and its reference value is $f_*(\theta) = \beta D_4 B_*(\beta\theta)$ in the incoming rotating frame, with $\lambda = \sigma - i/2$. The factor β cancels under $s = \beta\theta$ in the discounted integral. In particular

$$|N_a| \le C(q^3 + \varepsilon L^6)\,|a| + C\varepsilon^2L^{12}.$$

Moment evaluation on a perturbed receiving contour changes the circular-source projection by $C\varepsilon(1+\theta)^4(|a| + \|b\|)$. For an upper bound it fits the same small terms.

Set $U(\theta) = e^{-\sigma\theta}\,|a(\theta)|$. Inserting (25) into the scalar Duhamel inequality and interchanging integrals, using $\sigma - \alpha > 0$, gives

$$q^8\int_0^\theta e^{-\sigma v}I_j(v)\,dv \le Cq^8\varepsilon L^6 + Cq^8\int_0^\theta U(v)\,dv.$$

The polynomial source has a bounded exponentially discounted integral. Therefore

$$U(\theta) \le C\varepsilon + C(q^3 + \varepsilon L^{12} + q^8)\int_0^\theta U(v)\,dv.$$

The coefficient times $H = O(L)$ tends to zero. Grönwall yields

$$|a(\theta)| \le C\varepsilon e^{\sigma\theta}, \qquad I_j(\theta) \le C\varepsilon e^{\sigma\theta} + C\varepsilon L^6 e^{\alpha\theta}. \tag{31}$$

For the signed lower bound keep the complex interaction-picture coefficient. Until the first q^3 crossing, the discounted errors are

$$\begin{aligned}
\int e^{-\sigma\theta}\,|a|^2\,d\theta &\le C\varepsilon q^3 L, \\
\int e^{-\sigma\theta}(|a|\,\|b\| + \|b\|^2)\,d\theta &\le C\varepsilon^2 L^{13}, \\
\int e^{-\sigma\theta}\,|f_{\text{outer}}|\,d\theta &\le C\varepsilon q^8 L^7, \\
\int e^{-\sigma\theta}\,|\text{moving-contour correction}|\,d\theta &\le C\varepsilon q^3\,\text{poly}(L) + C\varepsilon^2\,\text{poly}(L).
\end{aligned} \tag{32}$$

All are $o(\varepsilon)$. Higher harmonics and negative history are handled in Section 8; images give $O(\varepsilon q^8\,\text{poly}(L))$. More explicitly, their sum, excluding the signed leading pulse and the separately treated source history, is bounded by

$$C(q^9 + q^{12} + q^{14})(1+L)^{20} = o(q^6).$$

This is the explicit closure of the unresolved nonlinear estimate. Absolute values have been used for upper bounds, not to infer a sign of the pulse.

8 The full signed pulse and its negative history

Assume the earlier completed stages have supplied their pointwise prefix bounds, cap reset, packet confinement and $9L/\sigma A_i$ clocks. Fix incoming $k = j+1$. The total history, in incoming units, is at most

$$A_k T_k \le \frac{9\beta}{\sigma(\beta-1)} L + C(S_0).$$

The recipient's backward discount thus gains at most

$$Cq^{-9/(\beta-1)} \le Cq^{-3}. \tag{33}$$

Throughout every earlier stage the recipient j was inactive; its pointwise prefix estimate gives $x_j \le C\varepsilon\delta$. Its entire own nonlinear prehistory is consequently

$$O(q^{12} L \cdot q^{-3}) = o(q^6).$$

Its moving-contour forcing error has the same type of bound. Reverse-prefix forcing before T_k is smaller still: in a prior stage $l \ge k$, every outer $i < j$ has $x_i \le K\delta(D\varepsilon)^{l-i}$. The nearest outer core contributes $O(q^2\varepsilon^2)$; summing the geometric rows and discounting gives $O(q^{14}\,\text{poly}(L)q^{-3}) = o(q^6)$. The recipient self-image remainder is also $O(q^{14}\,\text{poly}(L))$ before this discount. At T_k, the whole state is $O(\varepsilon)$; fixed-time contour continuity keeps it so on the incoming window. No early-growth dichotomy or assumption about its future cap is necessary.

For a non-base incoming core $k < N$, introduce a noise level

$$a_* = C_*\varepsilon L^{12}.$$

Before its first q^3 crossing, (30) applies. At $|a| = a_*$, the ratio of actual forcing to $|a|$ is $O(L^{-8})$, and $\|b\| \,/\, |a| = O(L^{-6})$. The exact analytic moment equation therefore gives, for a small fixed cap and small q,

$$\frac{d}{ds}\,|a| \ge \sigma_0\,|a|\,, \qquad \sigma_0 = \frac{3\sigma}{4}, \tag{34}$$

until q^3. Section 9 extends this inequality to the cap. The crossing is one-way. At any current time, its previous above-noise amplitudes are bounded by backward exponential growth; this is not an estimate assumed from an unknown future cap. Relative to the completed incoming cap at $s = 0$,

$$x_k(s) \le C\delta e^{\sigma_0 s} + C\varepsilon\,\mathrm{poly}(L), \qquad s \le 0.$$

Before the receiving stage for k, the pointwise earlier prefix bound is even $O(\varepsilon)$.

The older packet has normalized moment $O(q^6(1+L)^4)$ throughout this negative history. Below-noise core moments are bounded instead by $O(q^6(1+L)^{12})$, from the chosen noise threshold and raw centre. Their history integral is therefore at most $Cq^{6-9/(\beta-1)}(1+L)^{13}$. By (33), their weighted integral is

$$O(q^{6-9/(\beta-1)}\,\mathrm{poly}(L)) = o(1).$$

The above-noise negative tail before $-S$ is bounded by

$$C\delta e^{-(\sigma_0-\sigma/\beta)S},$$

since $\sigma_0 > \sigma/\beta$. The future tail after S is (14).

Lemma 4 (Uniform fixed source window)**.** *For every completed non-base cap T_k, and the adjacent receiving stage stopped as above, the normalized source moment satisfies*

$$\sup_{|s|\le S} |\mathcal{B}_k(s) - e^{i\phi_k} B_*(s)| \le Cq^{6-8\eta/\sigma}(1+L)^{20} = r(q) \longrightarrow 0.$$

The preceding window is part of the completed stage. The following window is available on the stopped receiving interval before q^3. For the base source the corresponding bound is Lemma 5.

Proof. The cap reset in Section 9 gives a full material distance at most $Cq^{6-8\eta/\sigma}(1+L)^6$ from a rotated canonical cap. The upper radial estimate (45) makes the preceding $[-S, 0]$ window lie after the completed source's q^3 crossing. Its full forcing there is $O(q^6(1+L)^4)$. Backwards bounded-time comparison from the known cap therefore controls that negative window. This uses a cap already completed by the descending induction.

For $0 \le s \le S$, the current receiver has $\theta = s/\beta$. Its raw complementary and upper amplitude bounds give $x_j \le Cq^6(1+L)^6$ on that fixed window. Thus its nonradial feedback into k is $O(q^2 x_j) = O(q^8(1+L)^6)$; the larger inactive contours give a smaller geometric row. The deeper packet has radius $CqLR_k$ and full field $O(q^6(1+L)^4)$ on the source interfaces. The constrained image field on a fixed compact source tube is $O(q^8)$, by conserved mass and the degree-eight cancellation. Hence the actual full source forcing on this window tends to zero with the stated powers.

To justify use of that tube, first stop on departure from a fixed chord-arc tube of the canonical $[0, S]$ trajectory or loss of its central hole. The canonical interfaces are regular, separated, and have a positive origin distance on this compact interval. Choose q so the deep packet radius is less than one quarter of this hole and the whole compact source tube lies in the larger receiver's plateau. The smaller packet requires one extra, coupled support bound here, since the source k may leave the near-circle chart. Put $e(s) = \mathrm{rad}(\mathrm{supp}\sum_{l>k} w_l)/R_k$. At the completed cap, $e(0) \le C_d q(1+L)$. On the compact source tube its origin hole has constant vorticity $-2A_k/5$. Subtract its exact solid rotation. Quarter-turn symmetry makes the remaining harmonic field cubic, with coefficient bounded by $C_S A_k/R_k^2$; no small shape norm is required. The local larger colours have the same cubic bound with smaller scale factors. The summed self velocities of $l > k$ are at most $CA_{k+1}R_{k+1}$, so, in source time, the upper radius derivative satisfies

$$D^+ e(s) \le C\beta q + C_S e(s)^3.$$

Include in the compact-tube stop the barrier $e(s) \le 4C_d q(1+L+\beta s)$, enlarging fixed C_d once. On $0 \le s \le S$, integration bounds the cubic increment by $C_S q^3(1+L+\beta S)^3 S = o(q(1+L))$. Thus

this barrier improves to $e(s) \le 2C_d q(1 + L + \beta s)$. Choose q so that this lies inside one quarter of the canonical hole. The deep mass is $O(A_k R_k^2 \beta q^2)$, hence its fourth moment is $O(A_k R_k^6 q^6 (1+L)^4)$, and separated kernel derivatives give the full $O(q^6(1+L)^4)$ forcing on the source contours. This independent radius improvement makes the hole and forcing assumptions causal. The mass and support estimates now give all the preceding forcing bounds on the stopped tube. The local contour vector field is Lipschitz on it: each self term uses the standard chord-arc contour estimate, and the two cross-interface terms are nonsingular. See [4, Appendix, local contour equation]. Its signed strengths only multiply these operators. The label gauge can be removed by translations or kept in a mild equation. Bounded-time Grönwall from the completed cap improves the tube and hole margins strictly, so neither can fail first. No future receiving cap is needed.

The resulting full source distance is bounded by a fixed multiple of

$$q^{6-8\eta/\sigma}(1+L)^6 + q^6(1+L)^4 + q^8(1+L)^6.$$

The moment is Lipschitz on this compact contour tube, giving the claim and its unit rotation phase. The receiver's upper amplitude bound delays its first q^3 crossing by at least $3L/\sigma - C$, exceeding S/β. The two-phase first-exit proof in Section 9 excludes all other exits before that window. □

Accordingly the circular-receiver leading coefficient, integrated from the actual initial time to any time after S before q^3, differs from

$$q^6 e^{i\phi} D_4 F(z_\beta)$$

by at most $c_0 q^6/10 + o(q^6)$, after the fixed tails and tube have been chosen. For the first source $k = N$, replace the preceding noise argument by the fixed-window base-pulse lemma below. Its omitted canonical past before $-S_0$ is placed within the same margin by choosing fixed $S_0 \ge S$. The phase ϕ is arbitrary and harmless.

Higher source harmonics start at q^{10}. A deliberately crude bound over negative history costs $q^{-3}\,\mathrm{poly}(L)$, leaving $O(q^7\,\mathrm{poly}(L)) = o(q^6)$. Positive tails have a fixed exponential discount and polynomial moments. The same calculation applies to image errors, which have an extra q^8. Together with the recipient remainders and its own prehistory, the complete interaction-picture error is bounded by

$$\frac{c_0}{4}\varepsilon + C\varepsilon\, r(q) + C(q^7 + q^9 + q^{12} + q^{14})(1+L)^{20},$$

where $r(q) \to 0$ is the fixed-window cap-comparison error. The reset and the bounded-time comparison give, for example, $r(q) \le C q^{6-8\eta/\sigma}(1+L)^{20}$ after increasing a fixed constant. All constants are independent of the depth and stage. Choose q so the last two terms are below $c_0\varepsilon/4$. This gives

$$\left| e^{-\lambda\theta} a(\theta) \right| \ge \frac{c_0}{2} q^6$$

after the fixed source window, as long as $|a| < q^3$. Thus the first q^3 crossing occurs no later than

$$\theta = \frac{3}{\sigma} L + C, \tag{35}$$

with a harmless fixed adjustment for the source window. If the stopped interval reached that time without crossing, the displayed lower bound would contradict $|a| < q^3$.

9 Reset, prefix improvement and the two-phase stopping proof

The two continuation phases

Proposition 1 (Uniform stopped-stage continuation)**.** *Under the proved local estimates and induction data, one can choose fixed $\delta, D, K, D_0, B, C_*, Y$ and then sufficiently small q, uniformly*

in j, N, such that the current receiver reaches its first cap $|a| = \delta$ before H. Its first q^3 crossing occurs by $3L/\sigma + C$; the cap occurs by $7L/\sigma + C$. Up to the cap,

$$I_j \le K\delta/2, \qquad d(\beta\theta) \le \frac{D_0}{2}(1+\beta\theta),$$

and all local and central-separation bounds improve strictly. At the cap, with elapsed time measured from the first q^3 crossing,

$$\|y_{\mathrm{cap}}\| \le Cq^{6-8\eta/\sigma}(1+L)^6 = o(1). \tag{36}$$

Every earlier above-noise amplitude is bounded in terms of the amplitude at the current time, with rate $3\sigma/4$. The time from the first q^3 crossing to the cap is also bounded below by cL, with a fixed $c > 0$. Consequently every fixed backward cap window lies inside this completed stage for small q. No estimate assumes a future cap of this recipient.

Proof. We give the choices and both first-exit arguments explicitly. Choose $0 < \eta < \sigma/32$, then D and α_* as in Lemma 2. Fix $D_0 \ge 4(d_0 + C_v)$ before K. Its constants come from the fixed local neighbourhood and conserved packet mass, independently of the final cap and source windows. Fix C_x larger than twice the inverse chart constant and choose $K \ge 4C_{\mathrm{pr}}$, also enlarging it if necessary for the base-pulse bound in Lemma 5. This choice of K is independent of the later cap size and fixed comparison windows. Shrink δ so that its graph lies well inside the local neighbourhood, so that the moment remainder on $b = g(a) + y$ has $C\delta < \sigma/8$, and so that the small coefficient in the flattened period-map estimate below is less than $\eta P/2$. Then fix the signed-transfer windows and base ancient time for this cap. Their fixed comparison constants may depend on these windows, but neither C_x nor K does.

On the packet barrier $d(s) \le D_0(1+s)$, (22) implies

$$d(s) \le d_0 + C_v s + \frac{C_r q^2 D_0^3}{4}[(1+s)^4 - 1].$$

For $s \le \beta H$, division by $1+s$ gives

$$\frac{d(s)}{1+s} \le d_0 + C_v + \frac{C_r q^2 D_0^3}{4}(1+\beta H)^3 \le \frac{D_0}{2} \tag{37}$$

for small q. Moreover choose q so that

$$qD_0(1+\beta H) < r_{\mathrm{pl}}/2.$$

Thus the whole packet lies in half the prescribed central plateau radius of the current receiver, and hence in those of every larger local colour. The radius and central-separation barriers improve independently of the signed-transfer calculation.

On all the stopping intervals used below, (26), the packet bound, and the self-image estimate give

$$\|F_j(\theta)\|_E \le C_p\varepsilon(1+\theta)^4 + C_f D q^8 I_j(\theta) + Cq^8 x_j(\theta) \le F_*\varepsilon(1+L)^4. \tag{38}$$

The fixed F_* exists because $I_j \le 2K\delta$, $x_j \le r_*$, and $q^8/\varepsilon = q^2 \to 0$. It is a full material forcing bound.

Phase I: from the inherited state to q^3. Set $a_* = C_*\varepsilon(1+L)^{12}$, with fixed $C_* \ge 1$. Choose B below, and stop at the first of the following events:

$$|a| = q^3, \quad \|b\| = 2B\varepsilon(1+L)^6, \quad I_j = 2K\delta,$$
$$d(\beta\theta) = D_0(1+\beta\theta), \quad \theta = H,$$
loss of central separation or a local neighbourhood,

The initial data are strictly inside all non-amplitude barriers for small q, and $|a(0)| < a_* < q^3$.

Iterate (19) at period times. There are at most CL periods, and $\sum_{m\le CL}(1+m) \le CL^2$. The stopped bounds and (38), including the final partial period using (19), give

$$\|b\| \le C_b\varepsilon(1+L)^6 + C_b q^3 L^2(2B\varepsilon(1+L)^6) + C_b L^2(2B\varepsilon(1+L)^6)^2. \tag{39}$$

Here the a^2 term is at most $q^6 = \varepsilon$, and its convolution is absorbed in $C_b\varepsilon(1+L)^6$. The inherited $O(\varepsilon)$ centre state is absorbed there as well. Choose $B \ge 4C_b$, and then q so that the last two terms are at most $B\varepsilon(1+L)^6/2$. This improves the centre barrier to

$$\|b(\theta)\| \le B\varepsilon(1+L)^6. \tag{40}$$

The choice of B is fixed before q.

The exact moment equation and (40) imply that, whenever $a_* \le |a| \le q^3$,

$$\frac{|N(a,b)|}{|a|} \le C(q^3 + B\varepsilon(1+L)^6 + B^2\varepsilon/C_*), \qquad \frac{|F_a|}{|a|} \le (F_*/C_*)(1+L)^{-8}.$$

For small q, their sum is at most $\sigma/4$. Since $\Re\lambda = \sigma$,

$$\frac{d}{d\theta}\,|a| \ge \sigma_0\,|a|\,, \qquad \sigma_0 = 3\sigma/4, \quad a_* \le |a| \le q^3. \tag{41}$$

This also prevents an above-noise amplitude from returning below a_*. At each current time, previous above-noise amplitudes are at most the current amplitude times $e^{-\sigma_0(\theta-v)}$. Below noise, (17) and (40) give $x_j \le C_n\varepsilon(1+L)^{12}$. Above noise, $\|b\|\,/\,|a| \le (B/C_*)(1+L)^{-6}$, so for small q $x_j \le C_x\,|a|$, with the fixed C_x chosen before B. Consequently (28) holds at every current time in Phase I. Lemma 3 improves the prefix to $I_j \le K\delta/2$.

The current local norm is at most $C_{\mathrm{ch}}(q^3 + B\varepsilon(1+L)^6) < r_*/2$, and every inactive norm is at most $K\delta(D\varepsilon)^{j-i}/2 < r_*/2$. These strict inequalities prevent local exit. Equation (37) prevents packet or central-separation exit. Thus the only remaining Phase I endpoints are its amplitude target or the deadline.

For completeness, the amplitude upper bound also follows before any signed estimate. Set $U(\theta) = e^{-\sigma\theta}\,|a(\theta)|$. By (27) and $x_j \le C_{\mathrm{ch}}(|a| + B\varepsilon(1+L)^6)$,

$$q^8\int_0^\theta e^{-\sigma v} I_j(v)\,dv \le Cq^8\varepsilon(1+L)^6 + Cq^8\int_0^\theta U(v)\,dv.$$

The packet term in (38) has bounded discounted integral; the self-image term has bound $Cq^8\int U + Cq^8\varepsilon(1+L)^6$. Therefore

$$U(\theta) \le C\varepsilon + C(q^3 + B\varepsilon(1+L)^6 + q^8)\int_0^\theta U(v)\,dv.$$

The additive b^2 term is $O(B^2\varepsilon^2(1+L)^{12}) = o(\varepsilon)$. The displayed coefficient times H tends to zero, giving

$$|a(\theta)| \le C_u\varepsilon e^{\sigma\theta}. \tag{42}$$

In particular a q^3 crossing cannot occur before $3L/\sigma - C$. For small q, this is later than the fixed source-window endpoint Θ_w. All other exits have already been excluded, so that source window is available without assuming a future cap.

Apply the signed-transfer bound (24) on this stopped interval. At

$$\theta_3 = \frac{3L + \log(2/c_s)}{\sigma}$$

it would give $|a| \ge 2q^3$ if no earlier crossing occurred. For small q, $\theta_3 < H$. Thus the first crossing τ satisfies $\tau \le 3L/\sigma + C$. At that crossing,

$$\|y(\tau)\| \le B\varepsilon(1+L)^6 + C_g q^6 \le (B + C_g)\varepsilon(1+L)^6.$$

Phase II: from q^3 to the fixed cap. Restart at $\theta = \tau$, and put $s = \theta - \tau$. Choose the fixed Y below. Retain the same prefix, radius, separation, and local barriers, and stop at the first of

$$|a| = \delta, \qquad \|y\| = 2Y\varepsilon(1+L)^6 e^{2\eta s},$$
$$|a| < q^3 e^{\underline{\sigma} s}, \quad \underline{\sigma} = 2\sigma/3, \qquad \theta = H.$$

or a first exit through any retained barrier. The growth condition means a first downward violation for $s > 0$; equality at $s = 0$ does not stop the phase. Its strict improvement below shows that it holds on a nonempty initial interval and prevents a later downward exit.

Choose q so that

$$2Y\varepsilon(1+L)^6 e^{2\eta H} < \delta.$$

This is possible since $6 - 18\eta/\sigma > 0$. On the stopped interval, $|a| \le \delta$ and $\|y\| \le \delta$. At period times counted from τ, (20) therefore has remainder at most $2C_E\delta\,\|y\|$, and forcing error at most $PC_E F_*\varepsilon(1+L)^4$. Using $\|M_c^n\| \le M_\eta e^{\eta n P}$, multiplication by $e^{-\eta n P}$ gives a discrete Volterra inequality

$$e^{-\eta n P}\,\|y_n\| \le C_\eta(\|y_0\| + \varepsilon(1+L)^4) + \kappa_\eta\delta \sum_{r=0}^{n-1} e^{-\eta r P}\,\|y_r\|\,,$$

with fixed κ_η. The earlier choice of δ makes $\kappa_\eta\delta \le \eta P/2$. Discrete Grönwall and the uniform subperiod endpoint estimate yield

$$\|y(s)\| \le C_Y\varepsilon(1+L)^6 e^{3\eta s/2}.$$

Choose $Y \ge C_Y$, including its dependence on the fixed $B + C_g$ and interpolation constants. This improves the centre barrier by a factor of at least two:

$$\|y(s)\| \le Y\varepsilon(1+L)^6 e^{2\eta s}. \tag{43}$$

The initial centre state is strictly inside the enlarged barrier after increasing Y if needed.

On the weaker growth barrier,

$$\frac{\|y\|}{|a|} \le 2Yq^3(1+L)^6 e^{-(\underline{\sigma}-2\eta)s} \le 2Yq^3(1+L)^6, \qquad \frac{|F_a|}{|a|} \le F_* q^3(1+L)^4.$$

Since $b = g(a) + y$, (18) now gives

$$\frac{|N(a,b)|}{|a|} \le C\delta + C_Y q^3(1+L)^6.$$

The fixed small cap and then small q make the sum with the forcing ratio at most $\sigma/4$. Thus

$$\frac{d}{ds}\,|a| \ge \frac{3\sigma}{4}\,|a| > \underline{\sigma}\,|a|\,. \tag{44}$$

There is also an upper bound from the same exact moment equation:

$$\frac{d}{ds}\log|a| \le \sigma + C_{\mathrm{rad}}\delta + \zeta(q), \qquad \zeta(q) \le Cq^3(1+L)^6 \longrightarrow 0. \tag{45}$$

Both estimates use only the stopped bounds at the current time. Equation (44) strictly improves the weaker growth barrier. More explicitly, $e^{-\underline{\sigma}s}\,|a(s)|$ has a strictly positive derivative as long as the stopped bounds hold; it starts at q^3, and cannot have a later downward first exit. This supplies a local growth interval at the restart time as well.

Below noise in Phase I the contour norm is $O(\varepsilon(1+L)^{12})$. Above noise in Phase I it is at most $C_x\,|a|$. In Phase II, (17), $g(a) = O(a^2)$, and the small y/a ratio give the same bound $x_j \le C_x\,|a|$, after the fixed small-cap choice. The rates (41) and (44) concatenate across τ. Consequently (28)

holds at every current time throughout both phases. Lemma 3 again improves the retained prefix to $K\delta/2$. The packet bound improves by (37). Choose δ initially and then q so that

$$C_{\mathrm{ch}}(\delta + C_g\delta^2) + C_{\mathrm{ch}}\, 2Y\varepsilon(1+L)^6 e^{2\eta H} < r_*/2.$$

The current local norm and all inactive local norms are therefore strictly inside their neighbourhoods. Central separation improves as before. Thus none of the non-amplitude barriers can end Phase II.

By (44), the cap must occur within

$$\frac{1}{\sigma_0}\log(\delta/q^3) = \frac{4}{\sigma}L + C$$

of τ. Together with the Phase I clock, this is $7L/\sigma + C < H$, so the deadline cannot be the first endpoint. Conversely, if s_{cap} is the now established cap time, integrating (45) gives

$$s_{\mathrm{cap}} \geq \frac{\log(\delta/q^3)}{\sigma + C_{\mathrm{rad}}\delta + \zeta(q)} = \frac{3L + \log\delta}{\sigma + C_{\mathrm{rad}}\delta + \zeta(q)} \geq cL$$

for small q, with, for example, $c = 2/(\sigma + C_{\mathrm{rad}}\delta + 1) > 0$. Thus a prescribed finite window $[-S, 0]$, relative to this cap in the current colour's normalized time, lies after τ for small q. This conclusion is made only after the cap has been proved; it is not used to obtain that cap. Finally (43) at that cap gives

$$\|y_{\mathrm{cap}}\| \leq CYq^6(1+L)^6 e^{2\eta(4L/\sigma + C)} \leq Cq^{6-8\eta/\sigma}(1+L)^6 = o(1),$$

which proves (36).

All choices precede the final choice of q. The only sums over colours were geometric row sums, and the only time sums had at most CL periods. Hence every conclusion is uniform in j, N. □

Outputs for the next stage

At the completed cap of j, reindexing to the next receiver $j-1$ is exact:

$$I_{j-1}(T_j) \leq D\varepsilon I_j(T_j) \leq DK\delta\varepsilon/2, \qquad x_{j-1}(T_j) \leq DK\delta\varepsilon/2.$$

Thus the next inherited full contour norm is $O(\varepsilon)$, without an inherited power of L. The packet $l \geq j+1$ has radius $O(qLR_j)$. Adding the newly capped colour j, whose local outer radius is uniformly bounded in units of R_j, gives a fixed initial radius bound for the next aggregate packet. The weaker reset (36), together with Lemma 1, gives convergence to a rotated canonical cap on each fixed preceding and following source window. Its use for a following window requires the usual continuation bootstrap on that compact canonical trajectory and the central hole there; it does not assert a long-time shadowing estimate.

The fixed following window is available: for the next recipient the Phase I upper bound forces its first q^3 crossing to occur only after $3L/\sigma - C$, which exceeds every previously fixed window for small q. The backward source window comes from the already completed stage: (45) ensures that it lies after that stage's first q^3 crossing, and the cap reset and fixed-time inverse comparison control it there. For the deepest source, replace this backward argument by direct comparison from its prescribed ancient initial time $-S_0$.

9.1 The deepest ancient-orbit pulse as a separate base case

The deepest colour is not initialized at a circular state. Its fixed ancient-orbit amplitude at $-S_0$ may exceed q^3 when q is small. The raw complementary and noise-threshold arguments are not applied to it.

Choose a fixed small $r_* > 0$, independent of the cap size, such that the isolated graph orbit remains in its local contour chart for $s \le r_*$. Shrink the cap if necessary. The exact moment equation on the graph implies

$$\frac{d}{ds}\left|a_*(s)\right| \ge \sigma_0 \left|a_*(s)\right|, \qquad \sigma_0 = 3\sigma/4,$$

on this local interval. The graph and inverse chart then give

$$\left\|X_*(s) - X^0\right\|_E \le C_g \delta e^{\sigma_0 s} \qquad (s \le r_*). \tag{46}$$

The constant C_g is uniform for sufficiently small caps and all negative starting times. For $0 \le s \le r_*$, it includes the fixed finite forward-growth factor.

For a finite stack, set

$$s = A_N t - S_0, \qquad I_N(s) = \max_{i<N} \frac{x_i(s)}{(D\varepsilon)^{N-i}}; \qquad I_1 = 0.$$

At $s = -S_0$, initialize $X_N = X_*(-S_0)$ and every larger colour circular, so $I_N(-S_0) = 0$. Use each colour's native frame, removing the exact larger-colour solid rotation. Quarter-turn symmetry, fixed enclosed areas and zero circulation hold exactly.

Lemma 5 (Causal deepest-source initialization)**.** *There are fixed K and sufficiently small q, independent of N, for which the first deepest cap occurs at*

$$T_N = \frac{S_0 + \zeta_N}{A_N}, \qquad |\zeta_N| \le C_{S_0,\delta} q^8,$$

and, throughout the initialization up to that cap,

$$I_N \le K\delta, \qquad \|f_N\|_E \le C q^8 \delta, \qquad \|y_N(T_N)\| \le C_{S_0,\delta}\delta q^8.$$

Here f_N is the normalized source forcing after removal of the exact solid rotations. For every fixed S, with $S_0 \ge S + 1$, the actual normalized source moment, recentered at T_N, is

$$\mathcal{B}_N(s) = \frac{e^{4i\Omega_N^{\mathrm{ext}} s/A_N}}{A_N R_N^6} \int_{\mathbb{T}^2} \bar{x}^4 w_N(T_N + s/A_N, x)\, dx.$$

The support lies in a fixed coordinate disk on this interval for small q; the displayed monomial is used only on that disk. The exponential specifies the source's native rotating frame with its spatial axes fixed at the actual cap. This moment satisfies

$$\sup_{|s|\le S} \left|\mathcal{B}_N(s) - e^{i\phi_N} B_*(s)\right| \le C_{S,S_0,\delta} q^8, \tag{47}$$

for a unit-modulus phase $e^{i\phi_N}$. All constants are independent of the depth N.

Proof. Choose the fixed spectral slack and D first. The weighted rows in deepest time have exterior coefficients $(A_i/A_N)(\beta/D)^d$ and interior coefficients $(A_i/A_N)(Dq^8/\beta)^d$. The direct deepest source contributes at most $CD^{-1}x_N$ to the weighted maximum norm. The diagonal exponent is at most $(\sigma+\eta)/\beta$. Thus, on a stopped prefix $I_N \le 2K\delta$, the same row summation and Volterra estimate as for the inactive subsystem yield an exponent $a_b < \sigma/2$ and the causal bound

$$I_N(s) \le \frac{C}{D} \int_{-S_0}^{s} e^{a_b(s-v)} x_N(v)\, dv. \tag{48}$$

Its constants do not depend on S_0 or N. The nonlinear stopped row coefficient is $O(Dq^6 K\delta)$; image rows have the same or better powers. Choose K, before choosing S_0, larger than four times

$$\frac{C(C_g+1)e^{\sigma_0 r_*}}{D(\sigma_0 - \sigma/2)}.$$

It can simultaneously exceed the constants required for subsequent receiving stages.

The feedback into the deepest source is bounded by

$$\sum_{i<N} \frac{A_i}{A_N} q^{2(N-i)} x_i \le I_N \sum_{d\ge 1} (Dq^8/\beta)^d \le CDq^8 I_N.$$

The constrained self-image field is $O(R_N^8 x_N)$, hence $O(q^8 x_N)$. The other image terms gain a further factor at most q^8. Consequently

$$\|f_N(s)\|_E \le Cq^8 (x_N(s) + DI_N(s)). \tag{49}$$

These are full contour-space bounds. The source lies inside the larger central plateaux on the stopped local tube; their radii are separated by q, and the known solid rotations have already been removed.

Stop at the first source-tube departure, $I_N = 2K\delta$, the first cap, or the fixed deadline $s = r_*$. On this interval (49) is $O(q^8\delta)$. Comparing with the autonomous ancient orbit from the identical state at $-S_0$ by Lemma 1, iterated over finitely many periods, gives

$$\sup \|X_N(s) - X_*(s)\|_E \le C_{S_0} \delta q^8. \tag{50}$$

Choose q after S_0 so that this error is smaller than the fixed source-tube margin and than $\delta e^{-\sigma_0 S_0}$. Combining it with (46) gives the causal envelope

$$x_N(s) \le (C_g + 1)\delta e^{\sigma_0 s} \qquad (-S_0 \le s \le r_*).$$

Inserting this into (48) yields

$$I_N(s) \le \frac{C(C_g+1)}{D(\sigma_0 - a_b)} \delta e^{\sigma_0 s} \le K\delta/2 \qquad (s \le r_*),$$

after the fixed choice of K. Thus neither stopped tube nor prefix bound can fail first. The possibly large constant in (50) constrains q, not K.

The cap is transverse. The moment is differentiable along the material equation, with the label generator absent because of relabeling invariance. The exact bounded moment functional and (50) give uniform convergence of a_N and its moment derivative to their canonical counterparts near zero. The canonical derivative of $|a_*|$ is at least $\sigma_0\delta$ at zero. On a fixed smaller interval around zero it remains at least $c\delta$, and the actual radial derivative is at least $c\delta/2$ for small q. Before that interval the ancient monotonicity gives a fixed relative gap below the cap. The intermediate value theorem and this derivative bound therefore give the unique first cap $s = \zeta_N$, with $|\zeta_N| \le C_{S_0,\delta} q^8$. At that actual cap the canonical graph has zero flattened coordinate at the same comparison time, so the Lipschitz chart and (50) give $\|y_N(T_N)\| \le C_{S_0,\delta}\delta q^8$. Equivariance identifies the graph point with this actual unstable phase and $|a| = \delta$ as a rotation of the canonical cap.

It remains to justify the outgoing fixed window without any generic receiving-stage argument. Recenter time at the completed cap and compare with that rotated canonical trajectory. Its compact $[0, S+1]$ trajectory has a fixed chord-arc tube and a fixed support bound $C_S R_N$. On this tube a far-field source bound, using its conserved mass and support radius, gives a weighted source term C_S/D in the outer equations. From the cap prefix, the causal estimate is now

$$I_N(T_N + s/A_N) \le Ce^{a_b s} K\delta + \frac{C_S}{D} \int_0^s e^{a_b(s-v)}\, dv.$$

Let M_S be twice a fixed upper bound for this right-hand side on $0 \le s \le S+1$, using $a_b \le \sigma/2$. Stop the outgoing comparison at $I_N = 2M_S$, outer local-tube departure, or source compact-tube departure. Choose q after M_S, so that $Dq^6 M_S$ is within the outer local chart, the nonlinear row

coefficient is small, and $C_S q \ll \rho$. The last condition keeps the source in each larger plateau. The prefix estimate improves to $I_N \le M_S$, and the source feedback and image estimates become

$$\|f_N\|_E \le C_S q^8 (1 + D M_S).$$

Compact contour-flow comparison from the completed cap then gives an $O(q^8)$ full E source error, improving the compact tube stop. No future receiving cap has been used.

On the negative window, the completed cap is already known to exist, and $S_0 \ge S + 1$, together with $|\zeta_N| = O(q^8)$, makes the whole interval $[-S, 0]$ available. Apply the backwards version of Lemma 1 from the actual cap to the rotated ancient orbit. The cap error and the past forcing are $O(q^8)$, so this gives the same full E error on $[-S, 0]$. The moment functional is Lipschitz on the compact contour tube; its rotation covariance supplies the unit phase in (47). This proves the fixed-window assertion without applying a raw-centre estimate to the initial source. □

The outgoing constant M_S is deliberately distinct from the universal cap prefix constant $K\delta$. There is no assertion that the spent source keeps $I_N \le K\delta$ throughout its outgoing window. Reindexing gives, for $N \ge 2$,

$$I_{N-1} \le D\varepsilon I_N, \qquad x_{N-1} \le D\varepsilon I_N.$$

Thus the new recipient and its reindexed prefix are $O(\varepsilon)$ through any fixed outgoing window, even when M_S is large. At the actual cap they satisfy the stronger inherited bounds with $I_N \le K\delta$. This also proves directly that the new recipient cannot reach a fixed cap within any fixed incoming time window.

For the first discounted pulse, the initial incoming time after recentering is $-S_0 - \zeta_N$. Backwards compact comparison over this fixed history, moment Lipschitz continuity, and the $O(q^8)$ endpoint shift imply

$$\int_{-S_0-\zeta_N}^{S} e^{-z_\beta s} \mathcal{B}_N(s)\, ds = e^{i\phi_N} \int_{-S_0}^{S} e^{-z_\beta s} B_*(s)\, ds + O_{S,S_0,\delta}(q^8).$$

Choose S_0 to make the omitted canonical negative tail smaller than the desired fixed fraction of c_0; choose S for the canonical and subsequent transport-controlled positive tails. The finite-window base error in the receiving seed is then $O(q^6 q^8) = O(q^{14})$. The safe parameter order is

$$(\beta, D, \eta, D_0)\ ;\ K\ ;\ \delta\ ;\ (S, S_0)\ ;\ (M_S, \text{compact-tube constants})\ ;\ q.$$

In particular one must not enlarge K to dominate M_S/δ, which would introduce an avoidable circular dependence of the cap size, tail windows and prefix constant.

10 Finite stacks and passage to a second solution

For each N, initialize the deepest colour on the isolated ancient orbit at $-S_0$, scaled by A_N, R_N; initialize every other colour circular. The domains are nested and fourfold, and each has its fixed area. The finite total vorticity is bounded, so it has a global Yudovich solution on the torus. Transport the colours by its common flow. For each fixed N, the signed finite collection of initially disjoint C^{1,γ_0} interfaces remains regular by the same tangential regularity theorem used above: choose tangent nonvanishing fields in the finitely many disjoint initial tubes. The periodic argument uses the same local singular kernel and a smooth regular remainder, so the tangential commutator estimates are unchanged up to bounded smooth-field terms. Its constants may depend on N; we never use them to estimate a spent packet. The local contour and separation estimates below have constants independent of N.

Lemma 5 supplies the deepest first cap, the universal cap prefix bound, and its fixed outgoing pulse window. Its initial amplitude is fixed independently of q; no raw-centre or noise-level argument is applied to that deepest colour.

Apply Sections 5–9 successively to $j = N-1, \dots, 1$. At each cap the prefix reindexes with a factor Dq^6; the next full recipient norm is $O(q^6)$, with no inherited power of L. All noise and centre exponents are consequently fixed, not enlarged at successive levels. The reset bound (36) is uniform and does not multiply the preceding cap error: its forcing estimate uses only packet mass/support and the prefix, not long-time trajectory shadowing. The packet is in the next plateau, since $CqL \ll \rho$. Every stage therefore reproduces its hypotheses with constants independent of N. The final cap times t_N are bounded above by

$$T \le \frac{S_0 + C}{A_N} + \sum_{j=1}^{N-1} \frac{9L}{\sigma A_j} \le C + \frac{9L}{\sigma A_1} \frac{\beta}{\beta - 1}.$$

At the outer cap $a_1 = \delta e^{i\vartheta_N}$ and $y_1 = o(1)$. By (3),

$$|B_1(t_N)| \ge cA_1 R_1^6 \delta$$

for a small fixed δ and then sufficiently small q. The deeper packet's moment is at most $CA_1 R_1^6 \beta q^6 L^4$, and every circular reference colour has zero fourth moment. Take a smooth cutoff $\chi = 1$ on B_2, supported in B_3, and the periodic compactly supported test

$$\varphi(x) = \chi(x/R_1)(\bar{x}/R_1)^4.$$

It equals the fourth-moment test on the entire outer colour and inner packet, and $\|\varphi\|_{p'} \le CR_1^{2/p'}$. The total pairing has magnitude at least $cA_1 R_1^2 \delta$. Hölder gives the fixed, depth-independent separation

$$\|\omega_N(t_N) - \bar{\omega}_N\|_p \ge cA_1 R_1^{2/p} \delta > 0. \tag{51}$$

No material-coordinate norm is substituted for an L^p norm.

The only initial change is the deepest colour, and

$$\|\omega_N(0) - \bar{\omega}_N\|_p \le CA_N R_N^{2/p} \longrightarrow 0.$$

Together with the geometric tail this gives $\omega_N(0) \to \bar{\omega}$ strongly in L^p.

Lemma 6 (Strong compactness for these exact approximations)**.** *If exact bounded torus Euler solutions have initial data converging strongly in L^p, $p > 2$, a subsequence converges strongly in $C([0,T]; L^p)$ to a renormalized Euler solution.*

Proof. Transport bounds ω_N uniformly in $L_t^\infty L_x^p$. Biot–Savart bounds u_N uniformly in $L_t^\infty W_x^{1,p}$; the velocity equation bounds $\partial_t u_N$ in $L_t^\infty L_x^p$. Interpolation and $p > 2$ yield strong compactness of u_N in $C_t C_x$, and the vorticity equation yields compactness in $C_t L^p_{\text{weak}}$. The flux passes to the limit because the velocity converges uniformly. For $u \in L_t^\infty W_x^{1,p}$, $\omega \in L_t^\infty L_x^p$, the DiPerna–Lions commutator is integrable since $2/p < 1$. Thus the limit transport equation is renormalized and has its initial trace. Apply renormalization to smooth truncated versions of $|r|^p$, integrate over the torus, then increase the truncation; monotone convergence proves conservation of its L^p norm. The finite-p transport trace theorem gives the strongly L^p-continuous representative, including at the initial time.

Every approximating norm equals its initial norm, which converges to that conserved limiting norm. For any $s_N \to s$, weak convergence of $\omega_N(s_N)$ to $\omega(s)$ and convergence of norms imply strong convergence by uniform convexity. A contrary sequence of times would contradict this conclusion, so the convergence is uniform in time. □

Apply this lemma on a common interval containing all cap times. Pass also to $t_N \to t_*$. Strong convergence and (51) imply $t_* > 0$ and $\omega(t_*) \ne \bar{\omega}$. The limiting initial datum is $\bar{\omega}$, so this branch and the stationary branch give the conclusion of Theorem 1.

Parameter choices and scope of the conclusion

Choose a sufficiently small reference planar cap to define the ancient orbit and its nonzero transform. Select a good $\beta \in [4, 5]$. Fix η, the weight constant D, the packet constant D_0, the inverse-chart bound, and K independently of the final cap and source windows. Shrink the cap δ by ancient time translation; nonvanishing persists. Choose the finite tail windows S, S_0, with $S_0 \geq S + 1$, then the compact-tube, packet, centre and outgoing-window constants. Finally choose $q > 0$ so every strict smallness inequality holds, $\beta q^{2/p} < 1$, and $\beta q < 1$. Only finitely many fixed conditions occur, all satisfied as $q \downarrow 0$. In particular $q^a(1 + \log(1/q))^b \to 0$ for each positive exponent used here. All these choices are made once, before the finite depth N. The fixed cap prefix $K\delta$ is never enlarged to absorb a spent source's possibly large fixed-window bound M_S.

The constructed datum is unbounded. The theorem makes no claim in the bounded-vorticity uniqueness class. Both branches solve the unforced equation with the same zero-mean velocity convention, and renormalization and strong L^p continuity are proved in the compactness step, rather than imposed on a contour-norm limit.

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