Lemma 1.1 (Admissibility of a fixed infinite smooth stack)**.** *Let W be a fixed real smooth compactly supported radial density with zero integral, and let Φ_θ be a bounded smooth family with common compact support, zero integral, and $B_4(\Phi_\theta) = 0$. Suppose Φ_θ is fourfold symmetric. Fix $\alpha > 0$, $A_j = A_1\beta^{j-1}$, $R_j = R_1 q^{j-1}$, and $A_1, R_1 > 0$, $\beta > 1$, $0 < q < 1$, and $\beta q^{2/p} < 1$, where $2 < p < \infty$. For any phases θ_j, the single datum*

$$\omega_0(x) = \sum_{j\geq 1} A_j[W(x/R_j) + \alpha\Phi_{\theta_j}(x/R_j)]$$

converges absolutely in $L^1\cap L^p$, has compact support and zero circulation, and satisfies $B_4(\omega_0) = 0$. Changing only phase θ_N changes its $L^1 \cap L^p$ norm by at most $C\alpha A_N(R_N^2 + R_N^{2/p})$, which tends to zero.

Proof. For either summand its L^r norm is at most $CA_jR_j^{2/r}$, uniformly in phase, for $r = 1, p$. The geometric ratio is less than one for both norms. All supports are in a common ball. Integral and phase claims follow by absolute convergence. The absolute fourth moment of each summand is bounded by $CA_jR_j^6$; this series converges since $\beta q^6 < 1$. Radial symmetry gives $B_4(W) = 0$, so termwise integration proves the whole-moment assertion. □

2 An isolated nonlinear active stage for a fixed smooth profile

This section proves the nonlinear activation estimate for one isolated annulus with a fixed smooth radial profile. The smoothing length is independent of viscosity and of the small parameter below.

Theorem 2.1 (Isolated smooth activation)**.** *There exist a real, radial, compactly supported C^∞ vorticity W_ϵ, with zero circulation, a number $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with $\sigma_\epsilon > 0$, and a nonzero complex number b_ϵ, with the following property. For every $2 < p < \infty$ and $K_* > 0$ there is $q_0 > 0$ such that the conclusions below hold for $0 < q < q_0$. Put*

$$L_q = \log(1/q), \quad \alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad \delta_q = L_q^{-4}, \quad H_q = \frac{1}{\sigma_\epsilon}\log\frac{\delta_q}{\alpha_q}.$$

There is a 2π-periodic family of smooth real seeds Φ_ϑ, supported in two fixed annular collars, such that

$$\int \Phi_\vartheta = 0, \qquad B_4(\Phi_\vartheta) = 0, \qquad \Phi_\vartheta(ix) = \Phi_\vartheta(x).$$

Let $\omega^{q,\vartheta}$ be the unforced Navier–Stokes solution with viscosity κ_q and initial vorticity $W_\epsilon + \alpha_q\Phi_\vartheta$. Then, uniformly in $\vartheta \in \mathbb{R}$,

$$\sup_{0\leq t\leq H_q} \|\omega^{q,\vartheta}(t) - e^{\kappa_q t\Delta}W_\epsilon\|_{L^1\cap L^p} \leq C\delta_q, \tag{1}$$

$$\int_0^{H_q} \|K * (\omega^{q,\vartheta}(t) - e^{\kappa_q t\Delta}W_\epsilon)\|_\infty \,\mathrm{d}t \leq C\delta_q, \tag{2}$$

$$B_4(\omega^{q,\vartheta}(H_q)) = \delta_q e^{i(\vartheta+\tau_\epsilon H_q)}b_\epsilon + o(\delta_q). \tag{3}$$

Here $\|f\|_{L^1\cap L^p} = \|f\|_1 + \|f\|_p$; all fourth absolute moments in (3) are finite. In particular, (3) is a statement about the whole moment, rather than only a compactly tested moment. The constants depend on the fixed profile, p and K_*, but not on q or ϑ.

The logarithmic cap satisfies $H_q\sqrt{\delta_q} \to 0$. In the fixed-data construction, q and hence this positive cap are subsequently fixed.

2.1 The fixed profile and its positive angular spectrum

Fix $\rho^2 = 3/5$ and choose a nonnegative, radial, radially decreasing $\chi \in C_c^\infty(B_1)$ with integral one. Define

$$\chi_\epsilon(x) = \epsilon^{-2}\chi(x/\epsilon), \qquad W_\epsilon = \chi_\epsilon * (-\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}).$$

The convolution is smooth, radial, supported in $B_{1+\epsilon}$ and has zero integral. It is constant on $B_{\rho-\epsilon}$. Choose $\epsilon_* < (1-\rho)/4$ and subsequently $0 < \epsilon < \epsilon_*$. The limits below refer first to the mollified family as $\epsilon \downarrow 0$. After the spectral and moment limits here and the transfer-coefficient limits in Lemma 4.2 and the interior-field calculation following it have been checked, choose one sufficiently small ϵ satisfying all their nonvanishing and spectral conclusions, and retain that choice for every later finite expansion and derivative order. Its radial derivative has the form

$$W_\epsilon'(r) = D_{\rho,\epsilon}(r) - \rho^2 D_{1,\epsilon}(r), \qquad D_{a,\epsilon} \geq 0, \quad \operatorname{supp} D_{a,\epsilon} \subset [a-\epsilon, a+\epsilon], \quad \int_0^\infty D_{a,\epsilon}(r)\,\mathrm{d}r = 1. \tag{4}$$

To verify the sign, express the radial decreasing mollifier by its superlevel disks. The area of intersection of two disks is nonincreasing in their center separation: along each line parallel to their displacement, the overlap of the two interval sections is nonincreasing. Layer-cake integration gives radial monotonicity of the disk convolution. Its value is one at the origin and zero at infinity, giving the displayed derivative mass. The support assertion follows because the convolution is exactly constant at points whose distance to the disk boundary exceeds ϵ.

Set

$$\Omega_\epsilon(r) = r^{-2}\int_0^r sW_\epsilon(s)\,\mathrm{d}s.$$

On the two derivative collars, respectively,

$$\Omega_\epsilon(r) = \Omega_0(r_i) + O(\epsilon), \qquad (r_1, r_2) = (\rho, 1), \quad (\Omega_0(r_1), \Omega_0(r_2)) = (-1/5, 0). \tag{5}$$

Indeed disk translation changes its L^1 norm by at most a constant times the translation length. Thus $\|W_\epsilon - W_0\|_1 \leq C\epsilon$, where $W_0 = -\mathbf{1}_{B_\rho} + \rho^2\mathbf{1}_{B_1}$. The radial circulation formula, on radii bounded away from zero, gives $|\Omega_\epsilon - \Omega_0| \leq C\epsilon$. The piecewise smooth Ω_0 is Lipschitz there. Direct integration gives the interface values.

For a positive angular mode $m \geq 1$ use the convention $f(r)e^{im\theta}$. Put

$$h_m(r,s) = \left(\frac{\min(r,s)}{\max(r,s)}\right)^m, \qquad \psi_f(r) = -\frac{1}{2m}\int_0^\infty h_m(r,s)f(s)s\,\mathrm{d}s.$$

The function $\psi_f e^{im\theta}$ solves $\Delta\psi = fe^{im\theta}$, is regular at the origin and decays at infinity. This follows either by solving the radial second-order equation on each side of $r = s$, or by checking the derivative jump $1/s$ of the displayed Green kernel. Since $u_r = -im\psi_f/r$, the frozen Euler linearization at W_ϵ is

$$\mathcal{L}_m f = -im\Omega_\epsilon f + im\psi_f W_\epsilon'/r. \tag{6}$$

For $m = 0$ the linearization is zero: two radial vorticities have tangential velocities and do not advect one another. Negative sectors are the complex conjugates of the positive sectors. From this point onward all angular sectors belong to $4\mathbb{Z}$. No spectral assertion is made for other symmetry classes.

Lemma 2.2 (The fixed smooth unstable pair)**.** *There are $\epsilon_0 > 0$ and $\eta > 0$ such that, for every $0 < \epsilon < \epsilon_0$, $0 < \eta < \sigma_\epsilon/4$ and the following modal conclusions hold. For each $m \in 4\mathbb{N}$ and each*

integer $k \geq 0$, realize (6) as a bounded operator on $H^k(I)$, using the Green potential of the zero extension from a fixed interval $I \Subset (0,\infty)$ containing both derivative collars strictly in its interior. The only spectral value of any of these individual operators with real part greater than η is the algebraically simple $m = 4$ eigenvalue $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$. Its radial eigenfunction f_ϵ is smooth and supported in the two derivative collars. For negative sectors use the conjugate operators; their only spectral value with real part greater than η is $\overline{\lambda_\epsilon}$ in sector $m = -4$. The radial generator is zero. Moreover

$$\lambda_\epsilon \longrightarrow \rho^5/2 + 3i/10 \quad \text{as } \epsilon \downarrow 0.$$

For each fixed finite set $S \subset 4\mathbb{Z}$ and integer $k \geq 0$, let $X_{k,S} = \bigoplus_{m\in S} H^k(I)$ and let $\mathcal{L}_S$ be its bounded direct-sum generator. Its propagator has the decomposition

$$e^{t\mathcal{L}_S} = e^{\lambda_\epsilon t}P_+ + e^{\overline{\lambda_\epsilon} t}P_- + \mathcal{R}_{k,S}(t), \qquad \|\mathcal{R}_{k,S}(t)\|_{X_{k,S}\to X_{k,S}} \leq C_{k,S}e^{2\eta t}, \quad t \geq 0. \tag{7}$$

The growing terms are included only if their sectors are present. The constants may depend on ϵ, k, S. Smooth collar-supported inputs retain their support. No assertion is made about the spectrum or propagator of an infinite angular direct sum.

Proof. Fix $0 < \eta < \rho^5/8$ before the angular cutoff is chosen. First consider an eigenvalue λ in an individual sector with $\Re\lambda > 0$. The eigenvalue equation forces $f = 0$ off the derivative collars and gives

$$f(r)r\,\mathrm{d}r = g(r)W_\epsilon'(r)\,\mathrm{d}r, \qquad g(r) = \frac{im\psi_f(r)}{\lambda + im\Omega_\epsilon(r)}.$$

There is no division by W_ϵ' here. In the Hilbert space $L^2(|W_\epsilon'|\,\mathrm{d}r)$ the equation for g is

$$\lambda g = -im\Omega_\epsilon g - \frac{i}{2}\int h_m(r,s)g(s)W_\epsilon'(s)\,\mathrm{d}s.$$

Each same-collar block is skew-adjoint, since the sign of W_ϵ' is constant there and $h_m(r,s)$ is real and symmetric. Angular multiplication is also skew-adjoint. The two opposite-collar blocks have operator norms at most their Hilbert–Schmidt norms, bounded by

$$Ca_*^m, \qquad a_* = (\rho + \epsilon_*)/(1 - \epsilon_*) < 1,$$

because their two weighted masses are 1 and ρ^2 and their kernels are bounded by a_*^m. Consequently

$$\Re\lambda \leq Ca_*^m. \tag{8}$$

This fixes a finite angular cutoff in any half-plane $\Re\lambda \geq \eta > 0$, independently of ϵ.

For one of the finitely many positive sectors, eliminate f to obtain the equation on the two collars

$$\psi(r) = -\frac{i}{2}\int h_m(r,s)\frac{W_\epsilon'(s)\psi(s)}{\lambda + im\Omega_\epsilon(s)}\,\mathrm{d}s =: \mathcal{T}_\epsilon(\lambda)\psi(r). \tag{9}$$

Work first in the sup-norm space of continuous functions on the two collars. Let P evaluate at r_1, r_2 and let E extend a vector to constants on the two collars, so $PE = I$. The receiving-radius derivative of the kernel is bounded there for fixed m. Hence

$$\|(I - EP)\mathcal{T}_\epsilon(\lambda)\| \leq C\epsilon$$

uniformly on compact subsets of $\Re\lambda \geq \eta$. For $\psi = Ep + w$, $Pw = 0$, solve the w equation by an analytic Neumann series. The remaining two-dimensional equation has a determinant converging uniformly on these compact sets to the determinant of

$$p_i = \frac{i}{2} \sum_j \frac{c_j h_m(r_i, r_j)}{\lambda + im\Omega_0(r_j)} p_j, \qquad (c_1, c_2) = (-1, \rho^2).$$

The sign follows from $W_0' = -\sum c_j \delta_{r_j}$. Using interface variables $f(r) r\,\mathrm{d}r = \sum c_i r_i^2 q_i \delta_{r_i}$, this equation is equivalent to the matrix eigenvalue problem

$$M_m q = \lambda q, \qquad M_m = \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

For $m = 4$ its roots are $\pm\rho^5/2 + 3i/10$, both simple. For $m = 8, 12, \ldots$ they are imaginary, since the real-matrix discriminant is

$$\left(\frac{2(m-4)}{5}\right)^2 - 4\rho^{2m+2} > 0.$$

At $m = 8$ the first term is $64/25$ and the second is $4(3/5)^9 < 1$; thereafter the first increases and the second decreases. Large $|\lambda|$ gives no omitted roots: once $|\lambda| > C_m$, the operator in (9) has norm less than one, uniformly in ϵ, because the denominator then has size comparable with $|\lambda|$. Uniform determinant convergence and Rouché's theorem therefore give one simple root near the growing $m = 4$ root and no other roots in $\Re\lambda \geq \eta$. Combined with (8), this proves the claimed positive spectrum for all sufficiently small ϵ. By the growing-root limit, decrease ϵ_0 so that $4\eta < \sigma_\epsilon$ throughout $0 < \epsilon < \epsilon_0$. The chosen η is unchanged in this step.

We give the operator-domain and multiplicity details needed for the propagator claim. Choose a closed interval $I \Subset (0, \infty)$ containing both collars strictly in its interior. On $H^k(I)$ define ψ_f by the Green integral against the zero extension of f from I. This extension need not be globally H^k: multiplication of the potential by W_ϵ' restricts output strictly inside I. The radial elliptic equation, or direct differentiation of its Green kernel, gives two local derivatives. Thus the coupling in (6) is compact on $H^k(I)$ and the angular multiplier is bounded there. Its spectrum is imaginary: off the multiplier's value range the inverse is multiplication by $(\lambda + im\Omega_\epsilon)^{-1}$. The compact-operator Fredholm alternative consequently reduces all spectrum in $\Re\lambda > 0$ to eigenfunctions. The equation bootstraps such eigenfunctions to smoothness and forces their support into the collars, so the roots already found are exactly the positive spectrum on every $H^k(I)$.

To check algebraic simplicity independently of a sup-norm determinant, start on $X_0 = L^2(I)$. Write $Af = -im\Omega_\epsilon f$, $Cf = \psi_f|_{\text{collars}}$ and $B\psi = imW_\epsilon'\psi/r$. Then $\mathcal{L}_m = A + BC$ and $\mathcal{T}_\epsilon = C(\lambda - A)^{-1}B$. Off the imaginary multiplier range, the resolvent identity is

$$(\lambda - \mathcal{L}_m)^{-1} = (\lambda - A)^{-1} + (\lambda - A)^{-1}B(I - \mathcal{T}_\epsilon)^{-1}C(\lambda - A)^{-1}. \tag{10}$$

Here $C : X_0 \to C(\text{collars})$ is bounded by its elliptic gain, and $B : C(\text{collars}) \to X_0$ is bounded. The invertible complementary Schur factor and the simple two-by-two determinant zero imply a rank-one simple pole in (10). The residue is nonzero, since an eigenfunction exists. On $X_k = H^k(I)$ the resolvent equation and its radial elliptic bootstrap give, locally uniformly for λ off the imaginary multiplier range,

$$\|v\|_{H^k(I)} \leq C_k(\|f\|_{H^k(I)} + \|v\|_{L^2(I)}), \qquad (\lambda - \mathcal{L}_m)v = f.$$

The X_k resolvent is the restriction of the X_0 resolvent; the simple pole has no higher-order singularity by this estimate and has the same nonzero smooth one-dimensional range. This proves algebraic simplicity on every space used below. The $m = -4$ conclusion follows by conjugation; $m = 0$ is the identity propagator.

After removing the growing Riesz projections, the remaining bounded generator has spectral bound at most η. Its exponential is bounded by $Ce^{2\eta t}$. For clarity this last fact does not assume a spectral growth theorem for unbounded generators. For a bounded operator, integrate $e^{tz}(z-\mathcal{L})^{-1}$ on a finite rectangular contour enclosing its remaining spectrum and having right edge $\Re z = 2\eta$. Expansion of the resolvent on a large circle and termwise integration gives the exponential series; contour deformation gives the rectangle formula. The resolvent is bounded on this fixed contour, proving the estimate. The finitely many sectors and derivative orders give (7). Coefficients initially smooth and supported in the collars remain so under this propagator: multiplication preserves that support and the coupling has smooth output flat at the collar endpoints. This proves the lemma. □

2.2 An exactly neutral smooth seed

Normalize the growing radial eigenfunction f_ϵ so that its first collar mass tends to $c_1 r_1^2 = -\rho^2$. The finite-dimensional reduction then implies

$$f_\epsilon(r) r \, \mathrm{d}r \;\rightharpoonup\; -\rho^2 \delta_\rho - i\rho^3 \delta_1.$$

Indeed its interface vector tends to $(1, -i\rho)$, the growing eigenvector of M_4. Define

$$b_\epsilon = 2\pi \int_0^\infty r^5 f_\epsilon(r) \, \mathrm{d}r. \tag{11}$$

The weak convergence on the two compact collars gives

$$b_\epsilon \longrightarrow 2\pi(-\rho^6 - i\rho^3) \neq 0.$$

Consequently $b_\epsilon \neq 0$ for every sufficiently small ϵ. This condition is included in the final fixed choice of the smoothing length. Since $\mathcal{L}_4$ is i times a real operator, $\overline{f_\epsilon}$ is an eigenfunction of $\mathcal{L}_4$ with eigenvalue $-\overline{\lambda_\epsilon} = -\sigma_\epsilon + i\tau_\epsilon$. Define the real seed

$$\Phi_\vartheta(r,\theta) = e^{i\vartheta} \left(f_\epsilon(r) - \frac{b_\epsilon}{\overline{b_\epsilon}} \overline{f_\epsilon(r)} \right) e^{4i\theta} + \text{complex conjugate}. \tag{12}$$

Its circulation is zero and its only initial sectors are ± 4. For the convention $B_4 = \int \bar{z}^4 \omega$, angular integration gives $B_4(f(r)e^{4i\theta}) = 2\pi \int r^5 f(r) \, \mathrm{d}r$. Therefore $B_4(\Phi_\vartheta) = 0$ exactly, while the growing coefficient is $e^{i\vartheta}$. The stable coefficient has modulus one.

2.3 The finite seed–viscosity expansion

For the rest of the proof abbreviate the fixed profile to W, and write $\sigma = \sigma_\epsilon$, $\tau = \tau_\epsilon$, $W_\kappa(t) = e^{\kappa t \Delta} W$ and $U_\kappa = K * W_\kappa$. The latter is an exact radial heat solution. Put $X = L^1 \cap L^p$ with the sum norm. Splitting the Biot–Savart convolution at radius one gives

$$\|K * g\|_\infty \leq C_p \|g\|_X. \tag{13}$$

On the near part use Hölder and $|x|^{-1} \in L^{p'}(B_1)$, which holds because $p' < 2$. On the far part use its bounded kernel and the L^1 norm. This estimate requires no zero-circulation assumption.

The frozen full linear operator is $\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W$. Its sectors are those in (6). Choose integers $M, N \geq 1$, to be fixed below, and construct

$$f_{\mathrm{app}} = \sum_{j=1}^{M} \sum_{\ell=0}^{N} \alpha^j \kappa^\ell f_{j,\ell}, \qquad \omega_{\mathrm{app}} = W_\kappa + f_{\mathrm{app}}. \tag{14}$$

Set $f_{1,0}(0) = \Phi_\vartheta$ and all other initial coefficients zero. With $W_r(t) = t^r \Delta^r W / r!$, solve the triangular recursion

$$\begin{aligned}(\partial_t - \mathcal{L}) f_{j,\ell} = \mathbf{1}_{\ell \geq 1} \Delta f_{j,\ell-1} - \sum_{r=1}^{\ell} \left[(K * W_r) \cdot \nabla f_{j,\ell-r} + (K * f_{j,\ell-r}) \cdot \nabla W_r \right] \\ - \sum_{\substack{j_1+j_2=j,\ j_1,j_2 \geq 1 \\ \ell_1+\ell_2=\ell}} (K * f_{j_1,\ell_1}) \cdot \nabla f_{j_2,\ell_2}.\end{aligned} \tag{15}$$

Every right-hand side uses earlier coefficients in the order $j + \ell$. They are smooth and supported in the fixed collars: Laplacians preserve the support of smooth flat functions; $\Delta^r W$, $r \geq 1$, has that support; and every advective factor differentiates such a supported function. The linear source is supported where W' is supported. Each coefficient has zero circulation, because each forcing is a Laplacian or divergence of an integrable vector field. Its frequencies belong to $4\mathbb{Z}$ and satisfy $|m| \leq 4j$.

Lemma 2.3 (Coefficient bounds and seeded residual)**.** *For every finite derivative order k, the finitely many coefficients in (15) satisfy*

$$\|f_{j,0}(t)\|_{H^k} \leq C_{j,k} e^{j\sigma t}, \tag{16}$$

$$\|f_{j,\ell}(t)\|_{H^k} \leq C_{j,\ell,k} (1+t)^D e^{j\sigma t}, \qquad \ell > 0. \tag{17}$$

There is a finite D, enlarged as needed, such that if $a(t) = \alpha e^{\sigma t} \leq \delta_0 \leq 1$ on $[0, H]$ and $\kappa(1 + H)^D \leq c$, the residual

$$R_{\mathrm{app}} = \partial_t \omega_{\mathrm{app}} + (K * \omega_{\mathrm{app}}) \cdot \nabla \omega_{\mathrm{app}} - \kappa \Delta \omega_{\mathrm{app}}$$

satisfies

$$\|R_{\mathrm{app}}(t)\|_X \leq C \left[a(t)^{M+1} + \kappa^{N+1} (1+t)^D a(t) \right]. \tag{18}$$

All constants are uniform in ϑ.

Proof. The first coefficient is the exact growing-plus-decaying solution from (12). For $j \geq 2$, $\ell = 0$, induction and Sobolev multiplication bound its forcing by $Ce^{j\sigma t}$. Convolution against the growing exponent σ, or the remaining exponent $2\eta < \sigma$, is bounded by a constant times $e^{j\sigma t}$, because $j\sigma - \sigma \geq \sigma$. This proves (16) with no polynomial factor. For positive ℓ the same induction applies, with finite polynomial losses from $W_r(t)$ and possible resonance with the first growing mode. This proves (17). Derivative loss causes no infinite iteration: the coefficient set is finite and the profile is smooth in every order. At each induction step apply the preceding coefficient estimates in a sufficiently higher fixed order and the propagator lemma in the requested order. Compact support and zero circulation also control $K * f$ and its needed derivatives, including low frequencies. More explicitly $|\widehat{f}(\xi)| \leq C|\xi|$ near zero, from its zero mass and finite first absolute moment, while the high-frequency multiplier has order minus one.

Expand the radial heat base only in its couplings to f_{app}. The integral Taylor formula and heat contraction in L^1, L^p give

$$G_N(t) := W_\kappa(t) - \sum_{r=0}^{N} \kappa^r W_r(t), \qquad \|G_N(t)\|_X + \|\nabla G_N(t)\|_X \le C\kappa^{N+1} t^{N+1}.$$

Its residual couplings obey

$$\|(K * G_N) \cdot \nabla f_{\mathrm{app}} + (K * f_{\mathrm{app}}) \cdot \nabla G_N\|_X \le C\kappa^{N+1}(1+t)^D a(t)$$

by (13) and the coefficient bounds. These are the only noncompact factors. Every other coefficient is supported in one fixed bounded set, where its high Sobolev norms control the required L^1, L^p product norms. The recursion cancels exactly all monomials with $1 \le j \le M$, $0 \le \ell \le N$. Terms of seed order $j > M$ are bounded by Ca^{M+1}: the pure-seed bounds contain no time polynomial, and each positive-viscosity polynomial is absorbed after enlarging D and using $\kappa(1+H)^D \le c$. Terms of viscosity order $\ell > N$, including the diffusion of $f_{j,N}$, have the second bound in (18); $a^j \le a$ for $a \le 1$ absorbs their higher seed powers. Crucially the exact radial heat base is part of (14), so no residual term without a seed factor occurs. Finally rotations of (12) change only unit complex phases; all bounds are uniform in ϑ. □

2.4 Nonlinear closure in the actual vorticity norm

Let ω^κ be the NS solution from $W + \alpha\Phi_\vartheta$ and put $\zeta = \omega^\kappa - \omega_{\mathrm{app}}$. Its exact equation, with zero initial data, is

$$\partial_t \zeta + (K * \omega^\kappa) \cdot \nabla \zeta = \kappa\Delta\zeta - (K * \zeta) \cdot \nabla \omega_{\mathrm{app}} - R_{\mathrm{app}}. \tag{19}$$

Kato's inequality gives its L^1 estimate, and multiplication by $|\zeta|^{p-2}\zeta$ gives its L^p estimate; transport disappears by divergence freedom and diffusion is nonpositive in both inequalities. Using (13) and adding gives

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\|\zeta\|_X \le C_p \|\nabla\omega_{\mathrm{app}}\|_X \|\zeta\|_X + \|R_{\mathrm{app}}\|_X. \tag{20}$$

These inequalities follow first with smooth convex regularizations of the norm and compact cutoffs, then by their limits. All solutions here are smooth at positive time, have finite X norm, and are continuous in that norm at time zero, so the integrated inequality starts at zero. No derivative bound on the actual NS vorticity enters its coefficient.

We choose parameters without circular dependence. Fix

$$C_1 = C_p(\|\nabla W\|_X + 1), \qquad r_1 = (C_1/\sigma - 1)_+,$$

before choosing expansion orders. Choose M with $(M+1)\sigma > C_1$ and N with

$$\frac{3}{2}(N+1) > K_* r_1. \tag{21}$$

Construct the finite coefficients, then fix a cap threshold δ_0 using the pure-seed coefficients alone and reduce the positive-viscosity threshold so that $\|\nabla f_{\mathrm{app}}\|_X \le 1$ whenever $a \le \delta_0$ and $\kappa(1+H)^D$ is sufficiently small. This is possible by the finite coefficient bounds: make the pure-seed gradient at most $1/2$ by the cap choice, and the finitely many positive-viscosity contributions at most $1/2$ by the later viscosity threshold. Thus a later increase of N need not reduce a previously fixed pure-seed cap. The exact heat gradient obeys $\|\nabla W_\kappa\|_X \le \|\nabla W\|_X$, so C_1 is the already fixed

exponent in (20). The coefficient constants affect only the later cap threshold; they do not change the previous choice of M, N.

Convolving (18) against $e^{C_1(t-s)}$ gives the all-time estimate

$$\|\zeta(t)\|_X \le Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)_+ t}. \tag{22}$$

The first integral has denominator $(M+1)\sigma - C_1 > 0$. For the second use a bound by $e^{\max(C_1,\sigma)t}$ and an additional polynomial factor if the two exponents coincide. At $H = \sigma^{-1}\log(\delta/\alpha)$ this yields

$$\|\zeta(H)\|_X \le C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \tag{23}$$

For the theorem's parameters, (21) implies that the second bound is $o(\delta_q)$: a strict positive power of q beats every fixed power of $\log(1/q)$. Also $\kappa_q(1+H_q)^D \to 0$, $\delta_q \to 0$, and $\alpha_q/\delta_q \to 0$. The coefficient estimates and (22) therefore prove (1), including uniformity through the entire interval. They also give the actual nonradial density bound

$$\|(I - P_{\mathrm{rad}})\omega^\kappa(t)\|_1 \le Ca(t) + Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)_+ t}. \tag{24}$$

Indeed radial averaging contracts L^1, and the heat base is radial. This estimate concerns the actual density; it is not inferred from a contour coordinate.

Similarly (13), the coefficient bounds and (22) imply

$$\int_0^H \|K * (\omega^\kappa - W_\kappa)\|_\infty \,\mathrm{d}t \le C\delta + C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \tag{25}$$

To integrate the approximate terms use $\int_0^H a(t)^j \,\mathrm{d}t \le \delta^j/(j\sigma)$. The error terms grow at a fixed positive exponential rate; their nondecreasing polynomial factor can be evaluated at H, so there is no additional factor H. This proves (2).

2.5 Whole-moment control and the cap

We verify the tails needed to remove the moment cutoff. Write $u^\kappa = U_\kappa + v^\kappa$, with U_κ tangential, and $B_H = \int_0^H \|v^\kappa\|_\infty \,\mathrm{d}t$. By (25), $B_{H_q} = O(\delta_q)$. Let $d(x) = \sqrt{1+|x|^2}$. Then $|\nabla d| \le 1$, $\Delta d \le 2$, and $U_\kappa \cdot \nabla d = 0$. Kato's inequality with $e^{\lambda d}$ gives, for every $\lambda > 0$,

$$\int e^{\lambda d(x)}|\omega^\kappa(t,x)|\,\mathrm{d}x \le m\exp\{\lambda d_0 + \lambda B_H + 2\kappa\lambda t + \kappa\lambda^2 t\}, \tag{26}$$

where $m = \|\omega^\kappa(0)\|_1$ and $d_0 = \sqrt{1+(1+\epsilon)^2}$. No initial exponential tail is presupposed beyond compact support. To justify the unbounded test use

$$F_L(d) = \frac{e^{\lambda d}}{1 + e^{\lambda d}/L}.$$

It has $F_L' \le \lambda F_L$ and $F_L'' \le \lambda^2 F_L$; therefore $\Delta F_L(d) \le (\lambda^2 + 2\lambda)F_L(d)$. For fixed L add a compact cutoff and let its radius tend to infinity. Finite mass and bounded drift make the cutoff errors vanish. Gronwall then gives (26) uniformly in L, and monotone convergence removes L. The drift is bounded here by (13) and the scalar L^1, L^p contractions.

Choose a fixed $R > 1 + \epsilon$ with $d_R - d_0 > 2c_* > 0$. For small q, $B_{H_q} < c_*/2$ and $\kappa_q H_q \to 0$. Exponential Markov and optimization of (26) give

$$\int_{|x|>s} |\omega^\kappa(t,x)|\,\mathrm{d}x \le m\exp\left\{-\frac{(d_s - d_0 - B_H - 2\kappa t)^2}{4\kappa t}\right\}, \qquad s \ge R.$$

Integrating this mass bound in s gives

$$\sup_{0\le t\le H_q}\int_{|x|>R}|x|^4|\omega^{q,\vartheta}(t,x)|\,\mathrm{d}x\le Ce^{-c/(\kappa_q H_q)}. \tag{27}$$

For example the weighted tail equals R^4 times the mass tail plus $4\int_R^\infty s^3$ times the lower-radius mass tail. The displayed Gaussian estimate bounds both terms after reducing c. This also proves finiteness of the required absolute moments.

Let ξ be a smooth radial cutoff equal to one on B_R and supported in B_{R+1}. Every formal nonradial coefficient is supported in B_R; the exact heat base pairs to zero with both $\bar z^4$ and $\xi\bar z^4$, since it is radial. At the cap the first coefficient has growing moment

$$\alpha e^{\lambda_\epsilon H}e^{i\vartheta}b_\epsilon=\delta e^{i(\vartheta+\tau_\epsilon H)}b_\epsilon$$

and stable moment of size $C\alpha^2/\delta$. The remaining pure-seed coefficients have compactly tested moment $O(\delta^2)$; positive-viscosity coefficients have moment $O(\kappa(1+H)^D\delta)$. The actual error pairs against $\xi\bar z^4$ with bound $\|\xi\bar z^4\|_\infty\|\zeta(H)\|_1$. Using (23) and (27) proves

$$\begin{aligned}|B_4(\omega^\kappa(H))-\delta e^{i(\vartheta+\tau_\epsilon H)}b_\epsilon|\le{}&C\delta^2+C\alpha^2/\delta+C\kappa(1+H)^D\delta+C\delta^{M+1}\\&+C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}+Ce^{-c/(\kappa H)}.\end{aligned}$$

For $\delta=\delta_q$, every bound divided by δ_q tends to zero. This proves (3) and completes the theorem.

Remark 2.4 (Scaling and scope). Let $A,R>0$ and set

$$\omega_{A,R}(t,x)=A\omega^{q,\vartheta}(At,x/R),\qquad W^\nu_{A,R}(t,x)=A(e^{\kappa_q At\Delta}W_\epsilon)(x/R),\qquad \nu=\kappa_q AR^2.$$

Homogeneity of Biot–Savart gives $K*\omega_{A,R}(t,x)=AR(K*\omega^{q,\vartheta})(At,x/R)$, so the physical cap time is H_q/A. For $1\le r<\infty$, change of variables gives

$$\|\omega_{A,R}(t)-W^\nu_{A,R}(t)\|_r=AR^{2/r}\|\omega^{q,\vartheta}(At)-e^{\kappa_q At\Delta}W_\epsilon\|_r.$$

Thus the separate density bounds are $CAR^2\delta_q$ in L^1 and $CAR^{2/p}\delta_q$ in L^p, and their sum is

$$\sup_{0\le t\le H_q/A}\|\omega_{A,R}(t)-W^\nu_{A,R}(t)\|_{L^1\cap L^p}\le CA(R^2+R^{2/p})\delta_q.$$

The cap moment and integrated perturbation velocity satisfy

$$B_4(\omega_{A,R}(H_q/A))=AR^6\delta_q e^{i(\vartheta+\tau_\epsilon H_q)}b_\epsilon+o(AR^6\delta_q),$$

$$\int_0^{H_q/A}\|K*(\omega_{A,R}(t)-W^\nu_{A,R}(t))\|_\infty\,\mathrm{d}t=R\int_0^{H_q}\|K*(\omega^{q,\vartheta}(s)-e^{\kappa_q s\Delta}W_\epsilon)\|_\infty\,\mathrm{d}s\le CR\delta_q.$$

All constants are independent of A,R. These are isolated-colour estimates. Coupled use also requires bounds for the foreign-colour terms in the coefficient and error equations.

3 The canonical ancient profile of the isolated smooth activation

We retain the fixed profile, eigenvalue and seeds constructed in Section 2. Abbreviate them to $W = W_\epsilon$, $\lambda = \sigma + i\tau = \lambda_\epsilon$, $b = b_\epsilon \neq 0$, and Φ_ϑ. In this section the cap $\delta > 0$ is fixed and sufficiently small. It does not decrease with the vanishing parameter. An additional Sobolev estimate permits identification of a unique isolated ancient profile. This identification requires no convergence of an infinite formal series. Here δ specifies the ancient growing amplitude and the linear preparation level. It does not assert that $s = 0$ is the exact first hitting time of $|\mathcal{A}(\Omega_\delta)| = \delta$; at that time $\mathcal{A}(\Omega_\delta(0)) = \delta + O(\delta^2)$.

Theorem 3.1 (The isolated ancient limit)**.** *Fix an integer $k \geq 3$ and $2 < p < \infty$. There exists $\delta_* > 0$ such that, for $0 < \delta < \delta_*$, there is a real quarter-turn invariant Euler solution $\Omega_\delta(s)$ on $-\infty < s < \infty$, with zero circulation, having the following properties. For every fixed $K_* > 0$, put*

$$\alpha_q = q^{K_*}, \qquad \kappa_q = q^{3/2}, \qquad H_q = \sigma^{-1}\log(\delta/\alpha_q), \qquad \vartheta_q = -\tau H_q.$$

Let ω^q be the isolated Navier–Stokes solution with viscosity κ_q and initial vorticity $W + \alpha_q\Phi_{\vartheta_q}$. For every $J < \infty$ and every $0 < \varepsilon < k$,

$$\omega^q(H_q + \cdot) \longrightarrow \Omega_\delta \quad \text{in } C([-J,0];L^1 \cap L^p) \text{ and in } C([-J,0];H^{k-\varepsilon}) \quad \text{as } q \downarrow 0. \tag{28}$$

The parameter δ_* and the limiting profile can be chosen independently of K_*; the threshold for small q may depend on K_*. On $s \leq 0$, Ω_δ is supported in one fixed ball and has a uniform H^k bound. Its whole fourth harmonic moment satisfies

$$B_4(\Omega_\delta(s)) = \delta b e^{\lambda s} + O(\delta^2 e^{2\sigma s}) \qquad (s \leq 0). \tag{29}$$

For $s \geq 0$ it satisfies

$$|B_4(\Omega_\delta(s))| \leq C_\delta(1+s)^4. \tag{30}$$

These assertions concern the isolated annulus.

3.1 A high Sobolev error estimate without derivative loss

Fix once and for all a radius $R > 1 + \epsilon$ with a positive gap in $d(x) = \sqrt{1+|x|^2}$, for example $R = 3+\epsilon$. This radius and the corresponding compact-support norm constants are fixed before the energy exponent or expansion orders. The later cap smallness condition will make the integrated perturbation drift smaller than this fixed gap. For this proof let

$$\|g\|_{Z_k} = \|g\|_1 + \|g\|_{H^k}.$$

The finite expansion from (14) and (15) can be constructed in every finite derivative order. In particular, the residual estimate holds in Z_k. A useful version separates its pure-seed constant from constants depending on the viscosity order:

$$\|R_{\mathrm{app}}(t)\|_{Z_k} \leq [C_M + C_{M,N}\kappa(1+t)^D]a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^D a(t), \qquad a(t) = \alpha e^{\sigma t} \leq 1. \tag{31}$$

Here C_M is independent of N. To verify the additional norm and this separation, the pure-seed products are estimated by their compactly supported coefficient Sobolev bounds. Every other omitted seed-order product contains at least one positive power of κ, and its finite time polynomial is incorporated in D. The noncompact heat remainder products are

$$(K * G_N)\cdot\nabla f_{\mathrm{app}} + (K * f_{\mathrm{app}})\cdot\nabla G_N.$$

The heat Taylor formula gives

$$\|K * G_N\|_{W^{k,\infty}} + \|\nabla G_N\|_1 + \|\nabla G_N\|_{H^k} \le C_{k,N}\kappa^{N+1}t^{N+1}.$$

For the first term use heat Taylor on the smooth initial velocity $K * W$ and contraction of each bounded derivative. For the other terms use heat contraction on $\nabla\Delta^{N+1}W$ in L^1 and H^k. The coefficient bounds give $\|K * f_{\mathrm{app}}\|_{W^{k,\infty}} + \|\nabla f_{\mathrm{app}}\|_1 + \|\nabla f_{\mathrm{app}}\|_{H^k} \le Ca(t)$ after the positive-viscosity threshold is made small. Leibniz's rule proves the claimed seeded remainder in both norms.

Let $\zeta = \omega^\kappa - \omega_{\mathrm{app}}$ and $u_{\mathrm{app}} = K * \omega_{\mathrm{app}}$. Rewriting (19) gives

$$\partial_t\zeta + u_{\mathrm{app}} \cdot \nabla\zeta + (K * \zeta) \cdot \nabla\zeta = \kappa\Delta\zeta - (K * \zeta) \cdot \nabla\omega_{\mathrm{app}} - R_{\mathrm{app}}. \tag{32}$$

We record the estimates that avoid a quasilinear derivative loss. Sobolev embedding and the near–far kernel split give, for $k \ge 3$,

$$\|K * g\|_\infty \le C_k(\|g\|_1 + \|g\|_{H^k}), \tag{33}$$

$$\|\nabla K * g\|_{H^j} \le C_j\|g\|_{H^j}, \qquad j \ge 0. \tag{34}$$

The second is Plancherel's theorem for a bounded order-zero Fourier multiplier. In particular $\|\nabla K * g\|_\infty \le C_k\|g\|_{H^k}$.

For each integer derivative order up to k, the undifferentiated transport term has zero energy pairing. The commutator with $(K * \zeta) \cdot \nabla$ is bounded by

$$C_k\left(\|\nabla K * \zeta\|_\infty\|\zeta\|_{H^k} + \|\nabla\zeta\|_\infty\|\nabla K * \zeta\|_{H^{k-1}}\right) \le C_k\|\zeta\|_{H^k}^2.$$

This commutator inequality follows from Leibniz's rule: the endpoint terms put the first derivative in L^∞ and the last derivative in L^2, and the intermediate terms follow by Sobolev interpolation in dimension two. The embeddings for both displayed endpoint terms hold because $k > 2$. No derivative beyond order k of ζ occurs. Similarly,

$$\|(K * \zeta) \cdot \nabla\omega_{\mathrm{app}}\|_{H^k} \le C\left(\|\nabla\omega_{\mathrm{app}}\|_{H^k} + \sum_{j=1}^{k}\|\nabla^j\omega_{\mathrm{app}}\|_\infty\right)\|\zeta\|_{Z_k}.$$

For the term where no derivative lands on $K*\zeta$, use (33); for every term with a positive number of such derivatives use (34) in L^2 and put the corresponding derivative of the smooth approximation in L^∞. The H^k transport estimate with u_{app} uses only its fixed $W^{k,\infty}$ bounds. Its coefficients are bounded, independently of α, κ, H, by fixed norms of W plus one, once the approximate perturbation is made small in the finitely many required norms.

Together with Kato's L^1 inequality this gives, for $Y = \|\zeta\|_{Z_k}$,

$$\frac{\mathrm{d}^+Y}{\mathrm{d}t} \le C_kY + C_kY^2 + \|R_{\mathrm{app}}\|_{Z_k}, \qquad Y(0) = 0. \tag{35}$$

Diffusion has a nonpositive energy contribution. The constant C_k can be fixed before M, N: the pure-seed approximation is made small by choosing its cap after M, and each positive-viscosity coefficient is made small by choosing its viscosity threshold after N.

Set $E_k = 2C_k$, increasing it if necessary to dominate $C_p(\|\nabla W\|_{L^1\cap L^p} + 1)$ and the compact-support $L^1 \cap L^p$ stability constant used below. Choose one finite M with

$$(M + 1)\sigma > E_k. \tag{36}$$

After its pure-seed coefficients are known, choose $\delta_* > 0$ small enough for all their required norms and their pure-seed remainder. For each subsequently prescribed K_* choose N so that, with $r_k = (E_k/\sigma - 1)_+$,

$$\tfrac{3}{2}(N+1) > K_* r_k. \tag{37}$$

The N-dependent coefficients are controlled by decreasing the threshold for q, not by changing δ_*. In particular arrange $C_{M,N}\kappa(1+H)^D \le 1$ in (31). On a bootstrap interval $Y \le 1$, (35) now gives

$$Y(t) \le C_M a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^{D+1}a(t)e^{(E_k-\sigma)_+ t}. \tag{38}$$

At the cap, the second term is

$$C_{M,N}\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_k} = o(1)$$

in the prescribed polynomial window. The first is smaller than $1/2$ after reducing δ_*. Continuity contradicts a first bootstrap exit, so the estimate holds through the full cap interval. It also yields a uniform H^k bound for the actual vorticity. This bound is obtained from the actual error equation, rather than assumed from the formal approximation.

3.2 Finite homological coefficients and compactness

For a real angle θ define the rotation

$$(\mathcal{T}_\theta f)(r,\varphi) = f(r,\varphi+\theta/4), \qquad \mathcal{L}_{\mathrm{rot}} = \mathcal{L} - (\tau/4)\partial_\varphi.$$

It acts isometrically on every radial or full-plane Sobolev norm, commutes with the radial linearization, and respects Biot–Savart advection. The pure-seed coefficient $f_{j,0}$ with seed phase zero has a limit

$$e^{-j\sigma t}\mathcal{T}_{-\tau t} f_{j,0}(t) \longrightarrow F_j \quad \text{in every finite } H^m \text{ as } t \to \infty. \tag{39}$$

For $j = 1$ the decaying part is $O(e^{-2\sigma t})$ after this normalization, so

$$F_1 = f_\epsilon(r)e^{4i\varphi} + \text{complex conjugate}.$$

For $j \ge 2$, put $g_j = e^{-j\sigma t}\mathcal{T}_{-\tau t} f_{j,0}$. Its equation is

$$\partial_t g_j = (\mathcal{L}_{\mathrm{rot}} - j\sigma)g_j - \sum_{j_1+j_2=j} (K * g_{j_1})\cdot\nabla g_{j_2}.$$

On its finite set of sectors, the homogeneous propagator decays at least as $Ce^{-(j-1)\sigma t}$ by (7). Induction, the coefficient bounds, and dominated convergence in Duhamel's integral prove (39) and the finite homological relations

$$F_j = -(j\sigma - \mathcal{L}_{\mathrm{rot}})^{-1} \sum_{j_1+j_2=j} (K * F_{j_1})\cdot\nabla F_{j_2}, \qquad 2 \le j \le M. \tag{40}$$

The inverse is the integral of the exponentially decaying propagator. All these functions are smooth and supported in the fixed collars. Only the finitely many sectors in $4\mathbb{Z}$ with $|m| \le 4M$ are used. No all-mode Sobolev spectral-growth assertion is needed.

Define the fixed polynomial ancient profile

$$P_M(s) = W + \sum_{j=1}^{M} \delta^j e^{j\sigma s}\mathcal{T}_{\tau s} F_j, \qquad s \le 0. \tag{41}$$

For the phase $\vartheta_q = -\tau H_q$, (39) shows that every pure-seed term of the cap-shifted approximation converges to the corresponding term in (41). The positive-viscosity terms tend to zero because $\kappa_q(1+H_q)^D \to 0$, while $e^{\kappa_q(H_q+s)\Delta}W \to W$. For each fixed $s \le 0$, (38) gives the asymptotic error $C_M(\delta e^{\sigma s})^{M+1}$, since its viscosity remainder tends to zero by (37).

We justify actual compactness separately. The kernel bound and (38), integrated as in (25), give

$$\int_0^{H_q} \|K * (\omega^q(t) - W_{\kappa_q}(t))\|_\infty \, \mathrm{d}t \le C_M\delta + o(1).$$

Choose δ_* so this is less than the positive radial gap fixed before the expansion orders. The exponential Kato proof of (27), with this fixed cap and the already fixed radius R, gives

$$\sup_{0\le t\le H_q} \int_{|x|>R} |x|^4 |\omega^q(t,x)| \, \mathrm{d}x \le Ce^{-c/(\kappa_q H_q)}. \tag{42}$$

Initial velocities have uniformly bounded finite energy, since the initial vorticities are compactly supported and have zero circulation. The NS velocity energy inequality and the uniform vorticity H^k bound therefore bound the velocities in H^{k+1}. The equation bounds $\partial_t\omega^q$ uniformly in H^{k-2} on the cap interval.

On each shifted interval $[-J, 0]$, Rellich compactness and interpolation give local compactness in $C(H^{k-\varepsilon})$. The tail bound and the uniform H^k, hence L^∞, bound give global L^2 tightness. Interpolation applied to a smooth exterior cutoff then gives global $H^{k-\varepsilon}$ tightness; the cutoff is chosen outside the same fixed B_R. Thus compactness holds globally in that norm. The mass tail gives global L^1 tightness, and local convergence plus uniform L^∞ bounds gives compactness in $C(L^1 \cap L^p)$ for every finite p. A diagonal subsequence over J gives an ancient Euler solution: the transport term passes by the stated strong convergence, and $\kappa_q\Delta\omega^q \to 0$ on every compact shifted interval. The solution is supported in B_R on $s \le 0$, since the right side of (42) tends to zero.

By lower semicontinuity of the H^k norm and the strong L^1 convergence, every such limit satisfies, for all $s \le 0$,

$$\|\Omega(s) - P_M(s)\|_{Z_k} \le C_M(\delta e^{\sigma s})^{M+1}. \tag{43}$$

One may first take rational times; continuity in the weaker compactness norm and weak H^k continuity extend the bound to every time.

3.3 Uniqueness of the ancient subsequential limit

We use a norm in which no extra derivative of a comparison orbit is required. If Ω_1, Ω_2 are two ancient subsequential limits, their compact supports and uniform H^k bounds imply a uniform bound for $\|\nabla\Omega_j\|_{L^1\cap L^p}$. Indeed compact support controls the L^1 norm by the L^2 derivative norm, and H^k, $k \ge 3$, controls every finite L^p derivative norm. Their perturbations from W are small in these norms after reducing δ_*, so the stability coefficient is bounded by the fixed constant dominated by E_k above.

For $D = \Omega_1 - \Omega_2$ the exact Euler equation is

$$\partial_s D + (K * \Omega_1)\cdot\nabla D = -(K * D)\cdot\nabla\Omega_2.$$

The same Kato and L^p argument as (20) gives, for $s_0 \le s_1 \le 0$,

$$\|D(s_1)\|_{L^1\cap L^p} \le e^{E_k(s_1-s_0)}\|D(s_0)\|_{L^1\cap L^p}.$$

Both limits have the same polynomial (41). Using (43) at $s_0 = -T$ yields

$$\|D(0)\|_{L^1\cap L^p} \le C_M \delta^{M+1} e^{-[(M+1)\sigma - E_k]T} \longrightarrow 0.$$

The same comparison with any fixed endpoint $s_1 \le 0$ proves equality throughout the ancient interval. The strict inequality (36) uses one finite M only. Thus every subsequence has the same limit, proving the full convergence in (28). Repeating the argument for any different K_* gives the same limit; only its permissible viscosity expansion order and small-q threshold change.

For clarity, a direct H^k difference stability estimate would require an H^{k+1} bound on the comparison vorticity because it differentiates $(K*D)\cdot\nabla\Omega_2$ k times. The $L^1\cap L^p$ comparison above deliberately avoids this extra hypothesis. The high Sobolev estimate is used for compactness and for the available first derivatives.

3.4 Forward extension and the whole-moment asymptotics

The ancient construction initially gives compactly supported H^k vorticity at zero and finite-energy velocity in H^{k+1}, with $k \ge 3$. The finite-Sobolev local two-dimensional Euler construction therefore applies. Its continuation estimates extend that same solution uniquely to every $s \ge 0$, as follows. Its transported vorticity has conserved L^1 and L^∞ norms. The near–far Biot–Savart split gives a time-independent velocity bound. Splitting the gradient kernel at a small radius and using its cancellation gives the logarithmic estimate

$$\|\nabla u\|_\infty \le C_{\|\omega\|_1,\|\omega\|_\infty}[1 + \log(2 + \|u\|_{H^{k+1}})].$$

The near part uses the Hölder norm of ω supplied by H^k; the intermediate annuli give the logarithm, and the far part uses L^1. Choosing the small radius inversely proportional to a fixed positive power of $2 + \|u\|_{H^{k+1}}$ gives the displayed estimate. The velocity Sobolev energy inequality $\frac{\mathrm{d}}{\mathrm{d}s}\|u\|_{H^{k+1}} \le C\|\nabla u\|_\infty \|u\|_{H^{k+1}}$ then prevents finite-time blowup. The local finite-Sobolev solution is obtained by regularization and these same energy bounds; their difference energy bound gives uniqueness. This first extension requires only the finite order k, rather than C^∞ initial vorticity. Finite energy at zero follows from compact support and zero circulation, so these velocity norms are legitimate.

The bounded velocity transports its support into B_{R+Vs}. Consequently

$$\int |x|^4 |\Omega_\delta(s,x)|\,\mathrm{d}x \le \|\Omega_\delta(0)\|_1 (R + Vs)^4 \qquad (s \ge 0),$$

which proves (30). On $s \le 0$, its support is in the fixed B_R. Pairing (43) with $\bar{z}^4$ is therefore bounded by its L^1 error times R^4. The first homological coefficient has moment b, rotation multiplies it by $e^{i\tau s}$, and every higher term is $O(\delta^2 e^{2\sigma s})$. This proves (29) and completes the theorem.

Corollary 3.2 (Every fixed source window, including after the cap)**.** *For the isolated family in Theorem 3.1, every fixed $J < \infty$ satisfies*

$$\omega^q(H_q + \cdot) \longrightarrow \Omega_\delta \quad \text{in } C([-J,J]; L^1 \cap L^p),$$
$$\sup_{|s|\le J} |B_4(\omega^q(H_q + s)) - B_4(\Omega_\delta(s))| \longrightarrow 0.$$

Write $B_q(s) = B_4(\omega^q(H_q + s))$ and $B_\delta(s) = B_4(\Omega_\delta(s))$. For fixed $0 \le S, U < \infty$ and $a \in \mathbb{R}$, its source moment therefore satisfies

$$\int_{-S}^{U} e^{-as} |B_q(s) - B_\delta(s)|\,\mathrm{d}s \longrightarrow 0.$$

Proof. The negative window follows from (28) and (42). On $0 \le s \le J$, let $\zeta_q(s) = \omega^q(H_q + s) - \Omega_\delta(s)$. Its exact equation is

$$\partial_s \zeta_q + (K * \omega^q(H_q + s)) \cdot \nabla \zeta_q = \kappa_q \Delta \zeta_q - (K * \zeta_q) \cdot \nabla \Omega_\delta + \kappa_q \Delta \Omega_\delta.$$

The global Euler extension has bounded H^k norm and compact support on this fixed interval. Since $k \ge 3$, its second derivatives belong to H^1, which embeds in every finite L^p in dimension two; compact support also bounds their L^1 norms. Thus $\|\nabla\Omega_\delta\|_{L^1\cap L^p} + \|\Delta\Omega_\delta\|_{L^1\cap L^p} \le C_J$. Kato's inequality, the L^p inequality and (13) give

$$\sup_{0\le s\le J} \|\zeta_q(s)\|_{L^1\cap L^p} \le e^{C_J J}(\|\zeta_q(0)\|_{L^1\cap L^p} + C_J\kappa_q) \longrightarrow 0.$$

No actual NS derivative bound is used in this comparison.

We remove the moment cutoff separately. Scalar norm contraction gives one uniform post-cap velocity bound V. Choose a fixed radius $R' > R + VJ + d$, also enclosing the Euler support on $[0, J]$, with $d > 0$. The positive post-cap propagator has a forward diffusion with $|X_s - y| \le VJ + \sqrt{2\kappa_q}|B_s|$. For $|y| \le R$, its terminal fourth moment outside $B_{R'}$ is at most $C_J e^{-c_J/\kappa_q}$ by Gaussian endpoint moments. For unrestricted y its full fourth moment is at most $C_J(1 + |y|^4)$. The positive domination of the signed solution by the propagation of $|\omega^q(H_q)|$ therefore gives

$$\sup_{0\le s\le J} \int_{|x|>R'} |x|^4 |\omega^q(H_q + s, x)| \,\mathrm{d}x \le C_J e^{-c_J/\kappa_q} + C_J \int_{|y|>R} (1 + |y|^4)|\omega^q(H_q, y)| \,\mathrm{d}y \longrightarrow 0$$

by (42). A compact radial cutoff of $\bar{z}^4$, the just-proved L^1 convergence, and this tail prove uniform whole-moment convergence. On a fixed weighted time window the weight is bounded, so its final assertion follows by integration. □

Corollary 3.3 (A nonzero bilateral discounted moment)**.** *Define*

$$\mathcal{F}_\delta(z) = \int_{-\infty}^{\infty} e^{-zs} B_4(\Omega_\delta(s)) \,\mathrm{d}s.$$

It is analytic and is not identically zero on $0 < \Re z < \sigma$*. More precisely it extends meromorphically to* $0 < \Re z < 2\sigma$ *as*

$$\mathcal{F}_\delta(z) = \frac{\delta b}{\lambda - z} + \mathcal{G}_\delta(z), \tag{44}$$

where $\mathcal{G}_\delta$ *is analytic on that larger strip. For every fixed real* c*, the exceptional set*

$$\{\beta > 1 : \mathcal{F}_\delta((\sigma - ic)/\beta) = 0\}$$

is discrete in $(1, \infty)$*. In particular one can select arbitrarily large* β *outside it, including for* $c = 1/2$*.*

Proof. The negative-time estimate (29) and the positive-time polynomial bound give absolute, locally uniform convergence on $0 < \Re z < \sigma$, including after any fixed number of z derivatives. Subtract the leading negative-time exponential. Its integral is

$$\int_{-\infty}^{0} \delta b e^{(\lambda - z)s} \,\mathrm{d}s = \frac{\delta b}{\lambda - z}.$$

The remaining negative-time integral is analytic for $\Re z < 2\sigma$, and the positive-time integral is analytic for $\Re z > 0$. This proves (44). Its pole at $z = \lambda$ has nonzero coefficient, so the

transform cannot be identically zero on the smaller strip. Its zeros there are isolated. The curve $\beta \mapsto (\sigma - ic)/\beta$ is nonconstant, lies in that strip for $\beta > 1$, and has no interior accumulation of its zeros. Any terminal interval of large β contains values outside the zero set, proving the last assertion. Accumulation as $\beta \to \infty$ is possible at the boundary $z = 0$ and is not excluded or needed. □

Corollary 3.4 (Changing the canonical cap)**.** *If $0 < \delta' < \delta < \delta_*$ and $s_0 = \sigma^{-1}\log(\delta'/\delta)$, then*

$$\Omega_{\delta'}(s) = \mathcal{T}_{-\tau s_0}\Omega_\delta(s + s_0) \qquad (-\infty < s < \infty), \qquad \mathcal{F}_{\delta'}(z) = e^{(z-i\tau)s_0}\mathcal{F}_\delta(z).$$

Thus the zero set of the transform is independent of the canonical cap. At $z = (\sigma - ic)/\beta$ its modulus scales by $(\delta'/\delta)^{1/\beta}$.

Proof. The time-translated and rotated orbit has the polynomial (41) with δ replaced by δ', since $\delta e^{\sigma s_0} = \delta'$. It has the same order-$M + 1$ remainder. The ancient uniqueness argument therefore identifies it with $\Omega_{\delta'}$ on negative times, and forward Euler uniqueness extends the identity to every time. Rotation $\mathcal{T}_\theta$ multiplies B_4 by $e^{i\theta}$. Substituting $v = s + s_0$ into the absolutely convergent bilateral transform gives the stated exponential factor, which never vanishes. Its modulus at the displayed point is $e^{\sigma s_0/\beta}$. □

Corollary 3.5 (All finite regularity orders and finite jets of one orbit)**.** *The already fixed orbit Ω_δ belongs to every finite spatial Sobolev order at every time. For each integer $\ell \geq k$,*

$$\sup_{s \leq 0} \|\Omega_\delta(s)\|_{L^1 \cap H^\ell} < \infty,$$

and its H^ℓ norm is bounded on each fixed positive-time interval. In particular its vorticity is spatially C^∞. For each fixed $J \geq 1$ and integer $r \geq 0$, the homological coefficients $F_1, \dots, F_J$ satisfy, for some $s_{J,r} < 0$ and $C_{J,r} < \infty$,

$$\left\| \Omega_\delta(s) - W - \sum_{j=1}^{J} \delta^j e^{j\sigma s}\mathcal{T}_{\tau s}F_j \right\|_{L^1 \cap H^r} \leq C_{J,r}(\delta e^{\sigma s})^{J+1}, \qquad s \leq s_{J,r}. \tag{45}$$

The coefficients are given by (40) for each finite target degree, and are independent of the larger cutoff used to construct them. All intersection norms here are sum norms.

Proof. Fix an integer $\ell \geq k$. Repeat the finite-coefficient and actual-error construction above in order ℓ, with amplitude δ_ℓ smaller than δ and both the original and the order-ℓ permissible thresholds. The profile and smoothing length remain fixed. Choose any fixed $K_* > 0$. The exact Navier–Stokes family used in this construction, with cap shift $H_q = \sigma^{-1}\log(\delta_\ell/\alpha_q)$ and seed phase $-\tau H_q$, is precisely the isolated family already identified by Theorem 3.1 at amplitude δ_ℓ. Its full $L^1 \cap L^p$ limit is unique. The order-ℓ construction therefore gives the same Ω_{δ_ℓ} and a uniform H^ℓ bound on negative times. The finite-Sobolev Euler continuation argument just given yields its H^ℓ bounds on every fixed positive-time interval.

Let $s_\ell = \sigma^{-1}\log(\delta_\ell/\delta) < 0$. Corollary 3.4 and forward uniqueness give

$$\Omega_{\delta_\ell}(v) = \mathcal{T}_{-\tau s_\ell}\Omega_\delta(v + s_\ell) \qquad (-\infty < v < \infty).$$

For $s \leq s_\ell$ use the negative-time bound on $\Omega_{\delta_\ell}(s - s_\ell)$. For $s_\ell \leq s \leq 0$ use its positive-time bound on $0 \leq v \leq -s_\ell$. Rotation is an isometry, so these two bounds give the asserted uniform

order-ℓ bound for the same Ω_δ on $s \le 0$. Translating fixed positive intervals gives its remaining bounds. Their constants may depend on ℓ, δ. Since ℓ is arbitrary, Sobolev embedding proves spatial smoothness. No intersection of infinitely many small-amplitude thresholds is used.

For the finite jet, fix J, r and choose $\ell \ge \max\{k, r+1, 3\}$. Choose a finite expansion degree $M \ge J$ large enough that $(M+1)\sigma > E_\ell$, and choose a sufficiently small $\delta_\ell < \delta$ for this finite construction. The same isolated family and uniqueness again identify its limit with Ω_{δ_ℓ}. The pure-seed triangular recursion gives the same $F_1, \dots, F_J$ independently of M; passing its finite expansion and actual-error estimate to the limit gives

$$\left\| \Omega_{\delta_\ell}(v) - W - \sum_{j=1}^{J} \delta_\ell^j e^{j\sigma v} \mathcal{T}_{\tau v} F_j \right\|_{L^1 \cap H^r} \le C_{J,r} (\delta_\ell e^{\sigma v})^{J+1}, \qquad v \le 0.$$

Indeed the error has degree $M+1$, and the finitely many discarded degrees $J+1, \dots, M$ have the displayed degree-$J+1$ bound. Translate by s_ℓ and rotate by τs_ℓ. The amplitude identity $\delta_\ell e^{\sigma(s-s_\ell)} = \delta e^{\sigma s}$ and addition of the rotation phases give (45) for $s \le s_\ell$. The constant and onset time may depend on the finite target orders. This argument proves each finite jet and does not assume convergence of an infinite formal series. □

Remark 3.6 (Scope). The profile and source transform in this section belong to the isolated annulus. Their use in a coupled receiver requires the foreign-colour and weighted source-history estimates in addition to this isolated limit.

4 The smooth receiver's protected coordinate and transfer coefficient

Throughout this section $W = W_\epsilon$ is the one fixed profile of Section 2, $\lambda = \sigma + i\tau$ its growing fourth-sector eigenvalue, and $f_+ = f_\epsilon$ its normalized radial eigenfunction. All angular conventions are unchanged.

Lemma 4.1 (A global growing-coordinate test)**.** *There is a smooth bounded complex test*

$$T(x) = a(r) e^{-4i\theta}$$

whose derivatives of every fixed order are bounded, such that

$$\mathcal{A}(g) = \int_{\mathbb{R}^2} T(x) g(x)\, \mathrm{d}x, \qquad \mathcal{A}(f_+(r) e^{4i\theta}) = 1, \qquad \mathcal{A}(\overline{f_+(r)} e^{4i\theta}) = 0,$$

and $\mathcal{A}(\mathcal{L}g) = \lambda \mathcal{A}(g)$ for smooth integrable perturbations for which the pairings are defined. Near the origin T is a constant multiple of $\bar{z}^4$; outside a fixed ball it is a constant multiple of z^{-4}. In particular, ΔT is compactly supported away from the origin.

Proof. On a fixed radial interval containing the derivative collars, the rank-one Riesz projection in the positive fourth sector has form $P_+ g = f_+ \ell(g)$ with $\ell(f_+) = 1$ and $\ell(\mathcal{L}_4 g) = \lambda \ell(g)$. Represent this continuous L^2 functional by a bilinear density $2\pi a(r) r\, \mathrm{d}r$; complex conjugation is included in the representing density rather than in the pairing. Substitution of (6) gives

$$(\lambda + 4i\Omega_\epsilon(s)) a(s) = -\frac{i}{2} \int_0^\infty a(r) W_\epsilon'(r) h_4(r, s)\, \mathrm{d}r. \tag{46}$$

On the collars, the radial Green equation and this nonvanishing denominator bootstrap a to smoothness. Extend it to every $s > 0$ by the right side of (46) divided by its denominator. This agrees with the original density where W' is supported, so it still satisfies the equation everywhere. Integration of the Green kernel and Fubini then verify the global eigenfunctional identity.

For s smaller than every derivative collar, the numerator is a constant times s^4 and Ω_ϵ is the constant core angular velocity $-1/5$. Thus $a(s) = Cs^4$ there and $T = C\bar{z}^4$. Beyond the profile support the numerator is a constant times s^{-4} and $\Omega_\epsilon = 0$ by zero circulation. Thus $a(s) = C's^{-4}$ and $T = C'z^{-4}$. These two extensions are harmonic; on the remaining fixed annulus the test is smooth. This proves all the regularity and Laplacian assertions. Normalize by $2\pi \int a f_+ r \, \mathrm{d}r = 1$. Since the stable eigenvalue is $-\bar{\lambda} \neq \lambda$, the eigenfunctional identity annihilates its eigenfunction. The angular test kills every sector other than $+4$. □

Lemma 4.2 (Exact inverse-quintic forcing coefficient)**.** *Let a source w be real, integrable, supported in B_{R_s}, quarter-turn symmetric and of zero circulation. For each integer $m \geq 0$ used below, write*

$$B_m(w) = \int_{\mathbb{R}^2} \bar{y}^m w(y) \, \mathrm{d}y, \qquad \int_{\mathbb{R}^2} |y|^m |w(y)| \, \mathrm{d}y < \infty.$$

These absolute moments are finite for this compactly supported source. At points in the exterior region $|z| > R_s$ its complex velocity is

$$U(z) = \frac{i}{2\pi} \sum_{\ell=1}^{\infty} \frac{B_{4\ell}(w)}{\bar{z}^{4\ell+1}}. \tag{47}$$

For every finite $L \geq 1$, its remainder after degree $4L$ satisfies

$$\left| U(z) - \frac{i}{2\pi} \sum_{\ell=1}^{L} \frac{B_{4\ell}(w)}{\bar{z}^{4\ell+1}} \right| \leq \frac{1}{2\pi |z|^{4L+5} (1 - (R_s/|z|)^4)} \int |y|^{4L+4} |w(y)| \, \mathrm{d}y. \tag{48}$$

The first term forces the receiver's positive fourth vorticity sector by $B_4 F_\epsilon(r) e^{4i\theta}$, where

$$F_\epsilon(r) = -\frac{iW_\epsilon'(r)}{4\pi r^5}.$$

Its growing projection is $\gamma_\epsilon B_4$, with $\gamma_\epsilon = \mathcal{A}(F_\epsilon e^{4i\theta})$. For the mollified family before the smoothing length is fixed,

$$\gamma_\epsilon \longrightarrow \gamma_0 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0 \quad (\epsilon \downarrow 0). \tag{49}$$

Proof. In complex notation Biot–Savart is $U(z) = \frac{i}{2\pi} \int (\bar{z} - \bar{y})^{-1} w(y) \, \mathrm{d}y$. For $|z| > R_s$, the geometric series converges uniformly in y on the source support and absolutely after integration. Quarter-turn symmetry gives $B_m = (-i)^m B_m$, eliminating every degree not divisible by four; zero circulation eliminates B_0. This proves (47). Summing the absolute values of the remaining degrees and using $|y| \leq R_s$ proves (48). The exterior expansion is not used in an inner hole of an annular support. For its first term,

$$u_r = \Re(U e^{-i\theta}) = \frac{iB_4}{4\pi r^5} e^{4i\theta} + \text{complex conjugate}.$$

The linearized forcing is $-u_r W'$, giving F_ϵ.

We check nonvanishing without assuming an adjoint convergence theorem. The inhomogeneous resolvent Schur reduction in the proof of Lemma 2.2 applies to this forcing. Its natural radial measure has limit

$$F_\epsilon(r) r \, \mathrm{d}r \rightharpoonup \sum_{i=1}^{2} \frac{i c_i}{4\pi r_i^4} \delta_{r_i}.$$

Thus in interface variables its forcing vector tends to $h = (i/(4\pi\rho^6), i/(4\pi))$. The growing dual of M_4, normalized on $(1, -i\rho)$, is $\ell_0(q) = (q_1 + iq_2/\rho)/2$. Its pairing with h is γ_0 in (49). More explicitly, on a small fixed contour around the growing root the inhomogeneous Schur equations converge uniformly to $(z - M_4)q = h$. Integrating their resolvent in z gives convergence of the rank-one projected forcing coefficient. The eigenfunction normalization agrees with that in Section 2. This proves the limit and nonvanishing for all sufficiently small ϵ. This condition is included in the final fixed choice of the smoothing length. □

The direction of this interaction matters. A smaller source acts on the larger receiver by the inverse-quintic exterior field above. A larger annulus acts in its interior through a cubic harmonic field. They are different coefficients and cannot be interchanged.

For comparison, an exterior annulus's interior harmonic field $U(z) = c_3 \bar{z}^3$ has positive fourth radial coefficient $\bar{c}_3 r^3/2$. Its frozen-base forcing is $-\bar{c}_3 r^3 W'/2$, with growing projection $\gamma_\epsilon^{\mathrm{out}} \bar{c}_3$. The same resolvent limit gives

$$\gamma_\epsilon^{\mathrm{out}} \longrightarrow (\rho^2 + i/\rho)/4 \neq 0.$$

Indeed its interface forcing vector is $(\rho^2/2, 1/2)$, paired against the same growing dual. Thus older cubic fields can genuinely seed the growing coordinate; their size must be bounded, rather than discarded by symmetry. All the required spectral and nonvanishing conditions now hold for every sufficiently small smoothing length. Fix once one ϵ for which Lemma 2.2 holds and $b_\epsilon, \gamma_\epsilon, \gamma_\epsilon^{\mathrm{out}}$ are nonzero, and retain it in every later finite expansion and Sobolev order.

Lemma 4.3 (Duhamel coefficient for a child source)**.** *Let $A_c = \beta A_r$, $R_c = qR_r$, and use receiver time $s = A_r t$. Suppose, in a frame removing all common solid rotation, that the source's normalized fourth moment is $B(u)$, with child time $u = \beta s - s_*$. The receiver core has angular velocity $-A_r/5$; this rotates the child's moment by $e^{4is/5}$ in the receiver frame. The ideal linear growing receiver coefficient obeys*

$$a'(s) = \lambda a(s) + \beta q^6 \gamma_\epsilon e^{4is/5} B(\beta s - s_*).$$

For the response with no independent growing coefficient at $s = -\infty$,

$$e^{-\lambda s} a(s) = q^6 \gamma_\epsilon e^{-\mu_\beta s_*} \int_{-\infty}^{\beta s - s_*} e^{-\mu_\beta u} B(u) \, \mathrm{d}u, \qquad \mu_\beta = \frac{\sigma + i(\tau - 4/5)}{\beta}. \tag{50}$$

This integral exists whenever $B(u) = O(e^{\sigma u})$ at $-\infty$ and has polynomial growth at $+\infty$, provided $\beta > 1$.

Proof. The dimensional fourth moment is $A_c R_c^6 B$. Dividing the child velocity by $A_r R_r$ leaves the factor βq^6, so Lemma 4.2 gives the forcing magnitude. Rotation through angle Ωt multiplies B_4 by $e^{-4i\Omega t}$. The parent constant core has vorticity $-2A_r/5$, hence angular velocity $-A_r/5$, which gives the factor $e^{4is/5}$. Multiplication by the integrating factor $e^{-\lambda s}$ and the substitution $u = \beta s - s_*$ give (50). Its negative tail has exponential exponent $\sigma - \sigma/\beta > 0$; its positive tail has discount $\sigma/\beta > 0$. □

Lemma 4.4 (A nonzero bilateral source transform from an ancient orbit)**.** *Suppose a global Euler source trajectory has a fourth moment $B(u)$ with*

$$B(u) = cb_\epsilon e^{\lambda u} + O(e^{2\sigma u}) \quad (u \to -\infty), \qquad c \neq 0,$$

and $|B(u)| \le C(1+u)^4$ for $u \ge 0$. Define

$$\mathcal{F}(z) = \int_{\mathbb{R}} e^{-zu} B(u)\,\mathrm{d}u, \qquad 0 < \Re z < \sigma.$$

Then $\mathcal{F}$ is analytic and not identically zero. For every finite interval $[B_*, B_*+1] \subset (1,\infty)$ there is a choice of β in that interval such that $\mathcal{F}(\mu_\beta) \neq 0$.

Proof. The two explicit bounds give absolute locally uniform convergence of the integral and of all complex derivatives, proving analyticity. Split at zero and subtract the leading negative-time asymptotic. It gives

$$\mathcal{F}(z) = \frac{cb_\epsilon}{\lambda - z} + \mathcal{G}(z), \qquad 0 < \Re z < \sigma,$$

where $\mathcal{G}$ is analytic in $0 < \Re z < 2\sigma$: the remainder on the negative half-line has rate 2σ, and the positive half-line has polynomial growth. Thus its meromorphic extension has a nonremovable simple pole at $z = \lambda$. In particular it cannot vanish identically on the original strip. The analytic nonconstant curve $\beta \mapsto \mu_\beta$ lies inside that strip for real $\beta > 1$. If the transform vanished along any real interval, its zeros would have an interior accumulation point and the identity theorem would force identical vanishing. Its zeros along the displayed compact interval are therefore discrete, and a nonzero choice exists. □

The canonical ancient Euler orbit of Theorem 3.1 has precisely this negative-time estimate. It comes from the high-Sobolev construction and not merely from the isolated X-norm cap theorem. Its positive-time moment bound is elementary once a compactly supported bounded-vorticity orbit exists: L^1, L^p conservation bounds its velocity by (13), its support radius grows at most linearly, and its fourth absolute moment is therefore polynomial of degree at most four. Lemma 4.4 isolates the precise ancient-orbit input needed for the noncancellation assertion.

Lemma 4.5 (Quantitative window and source errors)**.** *Let $\beta > 1$ be fixed, $\mu = \mu_\beta$, and $S, U \ge 0$. Suppose $\mathcal{F}(\mu) \neq 0$ and the source satisfies the bounds of Lemma 4.4. Define the truncation budget*

$$E_{\mathrm{tr}} = \left| \mathcal{F}(\mu) - \int_{-S}^{U} e^{-\mu u} B(u)\,\mathrm{d}u \right|.$$

It satisfies

$$E_{\mathrm{tr}} \le Ce^{-(\sigma - \Re\mu)S} + C_\beta (1+U)^4 e^{-\Re\mu U}.$$

Replacing the moment on that window by $\widetilde{B}$ has budget

$$E_{\mathrm{rep}} = \int_{-S}^{U} e^{-\Re\mu u} |\widetilde{B}(u) - B(u)|\,\mathrm{d}u.$$

Set $Q = q^6 |\gamma_\epsilon| e^{-\sigma s_*/\beta}$. If the three budgets satisfy

$$E_{\mathrm{tr}} < \tfrac{1}{4}|\mathcal{F}(\mu)|, \qquad E_{\mathrm{rep}} < \tfrac{1}{4}|\mathcal{F}(\mu)|, \qquad |a_0| < \tfrac{1}{4}Q|\mathcal{F}(\mu)|,$$

then adding the independent old growing coefficient a_0 to the modified source response leaves a growing coefficient of modulus greater than $Q|\mathcal{F}(\mu)|/4$. If $s_* = \sigma^{-1}\log(\delta/\alpha)$ and both old and child seed amplitudes have size $\alpha = q^{K_*}$, the old-seed condition is algebraically feasible when

$$K_*(1 - 1/\beta) > 6.$$

Here $0 < \delta \le 1$ is fixed or decreases logarithmically in $1/q$; the remaining source and transfer constants are fixed.

Proof. Put $a = \Re\mu = \sigma/\beta \in (0, \sigma)$. The negative tail is bounded by $Ce^{-(\sigma-a)S}/(\sigma - a)$. For the positive tail, substitute $u = U + v$ and use $(1 + U + v)^4 \le (1 + U)^4(1 + v)^4$ for $U, v \ge 0$. The integral of $e^{-av}(1 + v)^4$ is finite, giving the asserted positive-tail bound. The negative and positive tails together constitute the single budget E_{tr}. By the triangle inequality,

$$\left|\int_{-S}^{U} e^{-\mu u}\widetilde{B}(u)\,\mathrm{d}u\right| \ge |\mathcal{F}(\mu)| - E_{\mathrm{tr}} - E_{\mathrm{rep}} > \tfrac{1}{2}|\mathcal{F}(\mu)|.$$

Equation (50) multiplies this transform by a factor of modulus Q. A further triangle inequality subtracting $|a_0|$ leaves a modulus greater than $Q|\mathcal{F}(\mu)|/4$. Finally $e^{-\sigma s_*/\beta} = (\alpha/\delta)^{1/\beta}$, so

$$\frac{\alpha}{q^6(\alpha/\delta)^{1/\beta}} = q^{K_*(1-1/\beta)-6}\delta^{1/\beta} \longrightarrow 0$$

under the stated power inequality. Fixed logarithmic cap factors do not alter this strict inequality. □

4.1 Exact heat shielding and the remaining coupled terms

The neutral seed (12) has only sectors ± 4 and $B_4 = 0$. Its velocity is zero outside $B_{1+\epsilon}$: in the exterior angular Green formula its coefficient is exactly its fourth moment. Thus its velocity $V_\vartheta = K * \Phi_\vartheta$ is smooth and compactly supported. Heat commutes with Biot–Savart, so a passive physical seed of amplitude $A\alpha$ and radius r has exterior heat velocity bounded, for $|x| \ge R > 2(1 + \epsilon)r$, by

$$|e^{\nu t\Delta}[A\alpha r V_\vartheta(\cdot/r)](x)| \le CA\alpha r \exp[-cR^2/(\nu t)]. \tag{51}$$

Indeed convolution integrates a bounded function supported in that ball, and the Gaussian mass at distance at least $R/2$ has the displayed bound. No damping estimate of a conserved harmonic moment is invoked. The bound concerns a passive heat evolution; its nonlinear moment generation is an additional term.

For precision, the actual foreign-colour equation yields an exact protected-coordinate identity. Suppose a receiver colour w solves

$$w_t + (K * w + b)\cdot\nabla w = \kappa\Delta w, \qquad \operatorname{div} b = 0,$$

and put $f = w - W_\kappa$. Remove any common rigid rotation first. Testing with Lemma 4.1 gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(f) = \lambda\mathcal{A}(f) - \int Tb\cdot\nabla W\,\mathrm{d}x + \mathcal{E}(t), \tag{52}$$

where the exact error is the sum of

$$\kappa\int \Delta T f\,\mathrm{d}x, \qquad -\int T[(K*(W_\kappa-W))\cdot\nabla f+(K*f)\cdot\nabla(W_\kappa-W)]\,\mathrm{d}x,$$
$$\int f(K*f)\cdot\nabla T\,\mathrm{d}x, \qquad \int fb\cdot\nabla T\,\mathrm{d}x, \qquad -\int Tb\cdot\nabla(W_\kappa-W)\,\mathrm{d}x.$$

There is no derivative of the actual f after integrating its transport terms by parts. Smooth heat Taylor and (13) give

$$|\mathcal{E}(t)|\le C[\kappa(1+t)\|f\|_X+\|f\|_X^2]+\int|f||b||\nabla T|\,\mathrm{d}x+\left|\int Tb\cdot\nabla(W_\kappa-W)\,\mathrm{d}x\right|. \tag{53}$$

For the term with ∇f, integration by parts bounds it by $\|\nabla T\|_\infty\|K*(W_\kappa-W)\|_\infty\|f\|_1$; the other heat term uses $\|\nabla(W_\kappa-W)\|_1=O(\kappa t)$. The quadratic self term uses (13). If $b=\Omega(t)Jx$ is a rigid rotation, its base forcing vanishes and its perturbation contribution is exactly $-4i\Omega(t)\mathcal{A}(f)$; it can be removed by its integrating phase.

Equation (52) identifies the estimates required for coupled use. On the receiver collars one must approximate b by the inverse-quintic child field of Lemma 4.2; its projected error is bounded by the collar sup error times $\|T\nabla W\|_1$. One must also control the two weighted foreign terms in (53) and prove a nonlinear density bound for this actual colour. The canonical source transform must be approximated in the weighted norm of Lemma 4.5, including its post-cap history. The actual foreign-colour estimates and weighted source-history control are additional inputs to a coupled receiver argument.

4.2 A nonlinear receiver theorem for the concrete canonical input

The following theorem uses the moment of the canonical isolated Euler orbit and realizes its exterior harmonic field through a prescribed smooth divergence-free extension. Its equation has this auxiliary drift; the source and receiver are not asserted here to solve a jointly unforced system.

Fix an ancient orbit Ω_{δ_c} from Theorem 3.1, with fixed small $\delta_c>0$, and put $B(u)=B_4(\Omega_{\delta_c}(u))$. Choose $\beta>1$ for which

$$C_*:=\gamma_\epsilon\mathcal{F}(\mu_\beta)\neq 0.$$

Such a choice is possible in every sufficiently large fixed interval by Lemma 4.4. Take a smooth radial cutoff ζ_0 equal to zero near the origin and equal to one on and outside the two receiver derivative collars. For $c\in\mathbb{C}$ set

$$\Psi_c(r,\theta)=-\frac{\zeta_0(r)}{16\pi r^4}(ce^{4i\theta}+\bar{c}e^{-4i\theta}), \qquad \mathcal{V}_c=\nabla^\perp\Psi_c.$$

This is a smooth real divergence-free field with every fixed spatial derivative bounded by a constant times $|c|$. On the receiver collars it agrees exactly with the inverse-quintic field of a source moment c.

Theorem 4.6 (Transmission for a prescribed canonical harmonic input)**.** *Fix $K_*>0$ such that $K_*(1-1/\beta)>6$. For small q put*

$$\alpha=q^{K_*},\quad \varepsilon_q=q^6,\quad \delta_q=[\log(1/q)]^{-4},\quad t_0=\frac{1}{\beta\sigma}\log\frac{\delta_c}{\alpha},\quad S_q=\frac{1}{\sigma}\log\frac{\delta_q}{\varepsilon_q|C_*|},\quad H_q=t_0+S_q.$$

For $0 < \kappa \le q^{3/2}$ define the prescribed foreign drift

$$b_q(t,x) = \beta\varepsilon_q \mathcal{V}_{e^{4i(t-t_0)/5}B(\beta(t-t_0))}(x), \qquad 0 \le t \le H_q.$$

Let w^q solve the coloured equation

$$\partial_t w^q + (K * w^q + b_q)\cdot\nabla w^q = \kappa\Delta w^q, \qquad w^q(0) = W + \alpha\Phi_0.$$

Then uniformly in $0 < \kappa \le q^{3/2}$,

$$\sup_{0\le t\le H_q} \|w^q(t) - e^{\kappa t\Delta}W\|_X \le C\delta_q,$$
$$\mathcal{A}(w^q(H_q)) = \delta_q e^{i(\tau S_q + \arg C_*)} + o(\delta_q),$$
$$B_4(w^q(H_q)) = b_\epsilon \delta_q e^{i(\tau S_q + \arg C_*)} + o(\delta_q).$$

The whole fourth moment is finite. Multiplying the source moment in b_q by a fixed unit complex phase multiplies the leading cap phase by the same phase; all bounds are uniform in that phase.

Proof. For completeness, the positive-viscosity solution with this prescribed drift exists on the whole displayed window. On a short interval its mild equation is a contraction in $C_t(L^1 \cap L^p)$: the heat-gradient estimate is $\|\nabla e^{\kappa t\Delta}\|_{L^r\to L^r} \le C(\kappa t)^{-1/2}$, the Biot–Savart map sends X to L^∞, and b_q is bounded there. Scalar convex-energy estimates give the L^1, L^p contractions and hence a time-uniform velocity bound. Iterating the local construction prevents finite-time loss of the mild solution. Smoothing follows by the same mild equation from the smooth spatial coefficients. This argument uses only bounded drift on the finite window, not the conclusion of the theorem. Use $s = t - t_0$ and write $b_q = \varepsilon_q b_1(s)$. The ancient and future moment bounds give

$$\|b_1(s)\|_{C^j} \le \begin{cases} C_j e^{\beta\sigma s}, & s \le 0, \\ C_j(1+s)^4, & s \ge 0. \end{cases}$$

In particular this is bounded by $C_j e^{\sigma s}$ for all s, and its growing-mode weighted time integral is finite, uniformly in the initial time $-t_0$. The first seed-order coefficient solves

$$(\partial_t - \mathcal{L})g_{1,0} = -b_1(t-t_0)\cdot\nabla W, \qquad g_{1,0}(0) = (\alpha/\varepsilon_q)\Phi_0.$$

Its growing coordinate is

$$e^{-\lambda s}\mathcal{A}(g_{1,0}(t)) = \frac{\alpha}{\varepsilon_q}e^{\lambda t_0} + \gamma_\epsilon \int_{-\beta t_0}^{\beta s} e^{-\mu_\beta u} B(u)\,\mathrm{d}u.$$

The first term tends to zero in modulus because $K_*(1 - 1/\beta) > 6$. The integral tends to $\mathcal{F}(\mu_\beta)$ as $t_0, s \to \infty$. Thus its growing coefficient tends to C_*. The complementary propagator has exponent $2\eta < \sigma$; integrating its ancient exponential forcing and its future polynomial forcing gives a complementary bound $Ce^{2\eta s}$ for $s \ge 0$. For $s \le 0$ the whole first coefficient is bounded by $Ce^{\sigma s}$. Indeed its forcing has ancient rate $\beta\sigma > \sigma$ and $(\alpha/\varepsilon_q)e^{\sigma t_0}$ is bounded. All bounds are uniform in q.

Expand in ε_q and κ exactly as in (14), replacing α there by ε_q. In the recursion add $-b_1\cdot\nabla g_{j-1,\ell}$ for $j \ge 2$ and add $-b_1\cdot\nabla W_\ell$ for $j = 1$. The initial first coefficient is the one above and the other initial coefficients vanish. The support and finite angular-sector properties are unchanged. Induction gives pure-seed bounds $Ce^{j\sigma s}$ and positive-viscosity bounds $C(1+t)^D e^{j\sigma s}$, uniformly

in t_0: the first coefficient has just been estimated, and for $j \geq 2$ the external forcing is at most $Ce^{\sigma s}e^{(j-1)\sigma s}$, leaving the same nonresonant gap $j\sigma - \sigma$. Positive-viscosity resonances produce only the finite absolute-time polynomial. Consequently the residual estimate (18) holds with

$$a(t) = \varepsilon_q e^{\sigma(t-t_0)}, \qquad a(0) = q^{6+K_*/\beta}\delta_c^{-1/\beta}, \qquad a(H_q) = \delta_q/|C_*|.$$

The residual contains no forcing-free term: the exact heat base is retained, and $b_1 \cdot \nabla$ of its heat Taylor remainder has the seed factor ε_q. Its bound is $C\kappa^{N+1}(1+t)^D a(t)$ by the displayed input estimate.

The prescribed drift is identical in the actual and approximate equations, so it disappears from the transport part of their Kato comparison. Choose M using the fixed exponent C_1 as before, and choose N with

$$\frac{3}{2}(N+1) > (6 + K_*/\beta)r_1.$$

The exact same X error calculation yields Ca^{M+1} plus a viscosity remainder $o(\delta_q)$ at the cap. It is uniform for $\kappa \leq q^{3/2}$. The higher pure-seed terms contribute $O(\delta_q^2)$ and the positive-viscosity correctors contribute $O(\kappa(1+H_q)^D\delta_q)$. The complementary first-coefficient response at S_q is $\varepsilon_q e^{2\eta S_q} = o(\delta_q)$. Its growing response gives exactly the asserted protected-coordinate cap. These estimates also give the density bound.

For the whole moment, the integrated self-perturbation velocity is $O(\delta_q)$ as in (25). The additional prescribed drift has integral at most $Cq^6(1+H_q)^5 = o(\delta_q)$, by its ancient and future bounds. The radial exponential Kato tail argument therefore applies unchanged. All formal coefficients are supported in the fixed collars, and the leading growing coefficient has moment b_ϵ. The complementary, nonlinear and viscosity errors estimated above, followed by the Gaussian tail bound, prove the whole-moment cap. This completes the proof. □

For coupled use of Theorem 4.6, the actual Biot–Savart field must satisfy bounds for the harmonic remainder, core leakage and feedback. Over a logarithmic stage, a finite harmonic expansion must be carried to an order whose omitted remainder beats the comparison exponent. The exterior remainder in (48) records the spatial estimate; the time-dependent coupled estimates are additional requirements.