

The full material spectrum and the isolated strong unstable orbit

Independent first-principles reconstruction

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This module concerns the planar, isolated two-interface problem. Set

$$\rho^2 = \frac{3}{5}, \quad r_1 = \rho, \quad r_2 = 1, \quad c_1 = -1, \quad c_2 = \rho^2, \quad W = c_1 \mathbf{1}_{B_{r_1}} + c_2 \mathbf{1}_{B_{r_2}}.$$

Our convention is $\Delta\psi = W$, $u = \nabla^\perp \psi$, with $\nabla^\perp = (-\partial_y, \partial_x)$. Thus $u_r = -r^{-1}\partial_\theta\psi$ and $u_\theta = \partial_r\psi$. Let $0 < \gamma < 1$. The real Banach space is the closed subspace

$$E = \{h \in C^{1,\gamma}(S^1; \mathbb{R}^2)^2 : h_i(\theta + \pi/2) = ih_i(\theta)\}$$

of little Hölder perturbations of $X_i^0(\theta) = r_i e^{i\theta}$. Here and below real vectors in the plane are written as complex numbers, without complexifying the real phase space. The little space is the closure of smooth functions in the usual $C^{1,\gamma}$ norm.

The contour input, and the precise extent of this module. We assume the physical contour velocity V is a bounded C^2 map on an open chord-arc neighborhood of X^0 in E , preserves smooth approximations and the enclosed domains, and gives the planar Euler velocity there. Its first two derivatives are bounded on a smaller fixed neighborhood. This is the differentiability assertion proved by the separate contour module; no unstable-manifold theorem, spectral theorem for multipliers, or unproved shape differentiability statement is being used here. All remaining conclusions are proved below. An analytic contour input makes the moment chart analytic; C^2 is sufficient for the strong unstable graph.

1 The material gauge and the layer calculation

The angular velocity of the reference field is

$$\Omega(r) = \begin{cases} -1/5, & 0 < r < \rho, \\ 3/10 - \rho^2/(2r^2), & \rho < r < 1, \\ 0, & r > 1. \end{cases}$$

Consequently $\Omega_1 = -1/5$, $\Omega_2 = 0$. Independent label translations commute with the material equation. The stationary gauge is

$$X_{i,s} = V_i(X) - \Omega_i \partial_\theta X_i.$$

Writing $X = X^0 + h$, this is the semilinear equation

$$h_s = \mathcal{A}_0 h + \mathcal{B}(h), \quad \mathcal{A}_0 h_i = -\Omega_i \partial_\theta h_i, \quad \mathcal{B}(h) = V(X^0 + h) - V(X^0).$$

Label translations form a strongly continuous group of isometries on the little space: continuity holds on smooth functions and then on their norm closure. The mild equation and its first two differentiated integral equations therefore give a local C^2 flow Φ_s . For clarity, on any fixed interval $[-P, P]$, contraction on a short time interval follows from the local Lipschitz bound for $\mathcal{B}$; Gronwall extends the flow on a sufficiently small fixed ball. Its first derivative solves a bounded Volterra equation; the second has the same linear equation with source $D^2\mathcal{B}(h)[D\Phi_s, D\Phi_s]$. These bounds are uniform for $|s| \leq P$. Negative times use the same group and integral equation. No differentiation in time into E is asserted for the unbounded label generator.

Untwist the perturbation by writing

$$h_i(\theta) = r_i e^{i\theta} \{z_i(\theta) + i\xi_i(\theta)\}, \quad z_i, \xi_i \text{ real and } \pi/2\text{-periodic.}$$

This is a bounded real isomorphism of the phase space with four scalar little Hölder spaces. Fourier coefficients use $f(\theta) = \sum_m f_m e^{im\theta}$, so $f_{-m} = \overline{f_m}$. The allowed nonzero geometric indices are $m \in 4\mathbb{Z}$.

A relative normal perturbation z_l of disk l produces the distributional first variation $c_l r_l z_l(\theta) \delta(r - r_l)$ of vorticity. For $m > 0$, its streamfunction coefficient is the unique continuous solution, regular at zero and decaying at infinity, of the radial Poisson equation. Matching the derivative jump at r_l gives exactly

$$\psi_{l,m}(r) = -\frac{c_l r_l^2}{2m} z_{l,m} \begin{cases} (r/r_l)^m, & r < r_l, \\ (r_l/r)^m, & r > r_l. \end{cases} \quad (1)$$

Indeed its outward-minus-inward radial derivative is $c_l r_l z_{l,m}$, as required. Its radial velocity, divided by r_i , at interface i is

$$\frac{ic_l r_l^2}{2r_i^2} \left(\frac{\min(r_i, r_l)}{\max(r_i, r_l)} \right)^m z_{l,m}.$$

Its tangential velocity, divided by r_i , is the same coefficient without i , with a minus sign for $r_i < r_l$ and a plus sign for $r_i > r_l$.

The reference-field evaluation derivative must be included. Away from an interface the ordinary polar calculation gives

$$z_{i,s} = -\Omega_i \partial_\theta z_i + \text{delta}\tau_{r,i}/r_i, \quad (2)$$

$$\xi_{i,s} = -\Omega_i \partial_\theta \xi_i + r_i \Omega'(r_i) z_i + \delta u_{\theta,i}/r_i. \quad (3)$$

For example, the variation of the reference vector under tangential displacement is $-\Omega_i r_i \xi_i e_r$. This cancels the term $+\Omega_i \xi_i$ coming from differentiating $e^{i\theta}$ in the label generator; the normal evaluation derivative leaves $r_i \Omega' z_i$ in the second equation.

At a self interface neither Ω' nor the radial derivative in (1) has a single trace. There is no averaging assumption: their sums have identical one-sided limits. At $r_1 = \rho$ the two values of $r\Omega'$ are 0, 1, and the self-layer tangential values divided by r_1 are $+z_1/2, -z_1/2$. Their sum is $z_1/2$ from either side. At $r_2 = 1$ the corresponding values are $\rho^2, 0$ and $-\rho^2 z_2/2, +\rho^2 z_2/2$, giving $\rho^2 z_2/2$. Equivalently, one may evaluate the moving boundary from either side and differentiate the continuous physical velocity. This cancellation resolves the apparent trace ambiguity.

Combining these terms proves, for every positive allowed m ,

$$\begin{aligned} z_s &= L_m z, & \xi_s &= T_m \xi + C_m z, \\ L_m &= \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}, & T_m &= \begin{pmatrix} im/5 & 0 \\ 0 & 0 \end{pmatrix}, \\ C_m &= \frac{1}{2} \begin{pmatrix} 1 & -\rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}. \end{aligned} \quad (4)$$

Negative indices have the conjugate matrices.

For the mean layer, Gauss' law gives zero velocity inside disk l and tangential velocity $c_l r_l^2 z_{l,0}/r$ outside it. Using the same one-sided cancellation gives the entire mean block

$$z_{1,0,s} = z_{2,0,s} = 0, \quad \xi_{1,0,s} = 0, \quad \xi_{2,0,s} = \rho^2(z_{2,0} - z_{1,0}). \quad (5)$$

It is a nilpotent block of order two. In particular, deleting mean normal modes before constructing the manifold would delete legitimate ambient area variations and is unnecessary.

2 Exact bounded shear and the complete spectral split

On positive indices set $H_m = \text{diag}(i, -i)$, on negative indices set $H_{-m} = \overline{H_m}$, and set $H_0 = 0$. Direct entrywise multiplication in (4) yields

$$H_m L_m - T_m H_m = C_m. \quad (6)$$

Thus H is the negative periodic Hilbert transform on the first component and the positive periodic Hilbert transform on the second, when the standard transform has symbol $-i \text{sgn } m$.

Here are the elementary operator bounds needed in this proof. The periodic Hilbert transform is bounded on C^γ , $0 < \gamma < 1$. Its principal-value kernel is a constant multiple of $\cot(t/2)$. Subtracting $f(\theta)$ uses its odd cancellation and bounds the value by $C[f]_\gamma \int_0^\pi t^{\gamma-1} dt$. For a difference at separation d , split the integrals into the two arcs within $2d$ of the evaluation points and their complement. The first part is bounded by $C[f]_\gamma d^\gamma$ by the same subtraction. In the second part, subtract the value at one evaluation point and use $|K(t+d) - K(t)| \leq Cd/|t|^2$; the bound is $C[f]_\gamma d \int_d^\pi t^{\gamma-2} dt \leq C_\gamma [f]_\gamma d^\gamma$. The constant term created by changing the subtracted value is bounded by $C[f]_\gamma d^\gamma$, using the integral cancellation of the kernel outside the removed arcs. This proves the seminorm estimate. The transform commutes with the derivative on smooth functions, so it is bounded on $C^{1,\gamma}$. It preserves smooth functions and hence their closed little subspace. Secondly, an absolutely summable matrix Fourier symbol (A_m) defines convolution with a uniformly convergent kernel, and has operator norm at most $C \sum_m \|A_m\|$ on each of these Hölder spaces. Such a multiplier is compact, since its finite Fourier truncations converge in operator norm. Both facts remain valid on the symmetry subspace.

Consequently the change of variables $\zeta = \xi - Hz$ is a bounded invertible real linear map, and (6) gives

$$z_s = Lz, \quad \zeta_s = T\zeta$$

on every nonzero mode. The identities first hold on trigonometric polynomials; boundedness and density give the exact mild-flow identities on E . The derivative terms in L and T belong to the label-translation generator. The ungauged derivative DV is order zero: in untwisted variables its normal component is $-\Omega_i \xi_i + \delta u_{r,i}/r_i$ and its tangential component is $u'_\theta(r_i)z_i + \delta u_{\theta,i}/r_i$, with the self combination understood as above. Thus the shear introduces no unbounded operator and loses no derivative.

Set

$$\sigma = \frac{\rho^5}{2}, \quad \tau = \frac{3}{10}, \quad v_+ = (1, -i\rho)^t, \quad v_- = (1, i\rho)^t, \quad \ell_+ = (1/2, i/(2\rho)), \quad \ell_- = (1/2, -i/(2\rho)).$$

At $m = 4$, $L_4 v_\pm = (\pm\sigma + i\tau)v_\pm$ and $\ell_\alpha v_\beta = \delta_{\alpha\beta}$. On the full material space the unstable and stable eigenvectors are respectively $(v_+, H_4 v_+)$ and $(v_-, H_4 v_-)$, with conjugate negative Fourier coefficients to impose reality. In particular, $H_4 v_+ = (i, -\rho)^t$.

For $m \geq 8$, write

$$a_m = m/5 - 1/2, \quad d = 3/10, \quad g_m = a_m - d = (m-4)/5, \quad b_m = \rho^m/2, \quad c_m = -\rho^{m+2}/2.$$

The exact real eigenfrequencies of L_m are

$$\mu_{m,+} = a_m - \delta_m, \quad \mu_{m,-} = d + \delta_m, \quad (7)$$

$$\delta_m = \frac{g_m - \sqrt{g_m^2 - \rho^{2m+2}}}{2} = \frac{\rho^{2m+2}}{2(g_m + \sqrt{g_m^2 - \rho^{2m+2}})}. \quad (8)$$

The square root is real: $g_m \geq 4/5$ and $\rho^{2m+2} \leq (3/5)^9 < 1/64$. Thus $0 < \delta_m \leq C\rho^{2m}/m$. A diagonalizer is

$$D_m = \begin{pmatrix} 1 & -b_m/(g_m - \delta_m) \\ c_m/(g_m - \delta_m) & 1 \end{pmatrix}, \quad D_m^{-1} L_m D_m = i \operatorname{diag}(a_m - \delta_m, d + \delta_m).$$

Its determinant is $1 + b_m c_m / (g_m - \delta_m)^2$, uniformly positive. Both $D_m - I$ and $D_m^{-1} - I$ have absolutely summable symbols. Their conjugate negative-index symbols define bounded real diagonalizers on the little Hölder space. This justifies operator diagonalization, beyond the formal matrix eigenvalue calculation.

Take $P = 10\pi$. The nonzero tangential period operator is the identity. The normal high-mode period multipliers are exactly $-e^{-iP\delta_m}$, $-e^{iP\delta_m}$. They are uniformly separated from 1: for example $P\delta_m \leq (25\pi/4)(3/5)^9 < \pi/2$, and hence $|1 + e^{\pm iP\delta_m}| \geq \sqrt{2}$. If $M = e^{PL}$, the full nonzero-mode material period operator in the original (z, ξ) variables is exactly

$$(z, \xi) \mapsto (Mz, \xi + H(M - I)z). \quad (9)$$

The mean period block is $I + PN_0$, where N_0 is given by (5) and $N_0^2 = 0$.

For every integer $n \geq 1$,

$$\sum_{m=8,12,\dots} |e^{\pm inP\delta_m} - 1| \leq C \sum_m \min(2, n\rho^{2m}/m) \leq C(1 + \log(1 + n)). \quad (10)$$

For the last inequality split at $m = C_\rho \log(2 + n)$, chosen large enough that $n\rho^{2m} \leq 1$ past the splitting point. There are $O(1 + \log(1 + n))$ terms before it and the remaining geometric tail is bounded. The base multiplier is $(-1)^n I$. The summable-multiplier bound and bounded diagonalizers therefore bound all normal high-mode period powers by $C(1 + \log(1 + n))$. This proof does not assume that uniformly bounded scalar multipliers are automatically bounded on Hölder space.

Define $J : \mathbb{C} \rightarrow E$, as a real linear map, by assigning positive fourth coefficients $z_4 = v_+ \alpha$, $\xi_4 = H_4 v_+ \alpha$, their conjugate negative coefficients, and no other coefficients. Define $d_u(h) = \ell_+ z_4$. Then

$$P_u = Jd_u, \quad P_c = I - P_u, \quad d_u J = I_{\mathbb{C}}. \quad (11)$$

These are bounded real complementary projections commuting with the linearized flow. The complement contains the stable fourth modes, all high normal modes, every tangential relabeling, and the whole mean block. In particular $d_u P_c = 0$ despite the material shear.

Let $S(s) = D\Phi_s(0)$, $s \geq 0$, and let $S_c = S|_{P_c E}$. The stable fourth modes decay exponentially, the high normal modes obey (10), the tangential modes are translations, and the means grow at most linearly. Bounded evolution on $[0, P]$ then proves

$$\|S_c(s)\| \leq C(1 + s), \quad \|S(s)\| \leq C e^{\sigma s}, \quad \|S(-s)|_{P_u E}\| \leq C e^{-\sigma s}. \quad (12)$$

The second bound follows since $1 + s \leq C_\sigma e^{\sigma s}$. One can also directly bound continuous high-mode powers: their base inner factor is a translation composed with the bounded sign-phase multiplier $e^{-is \operatorname{sgn}(m)/2}$, and their outer factor is $e^{ids \operatorname{sgn}(m)}$; each sign-phase multiplier is a linear combination of the identity and Hilbert transform. The δ_m corrections satisfy the same summation bound with nP replaced by $|s|$.

For completeness the complexified period spectrum is precisely

$$\{-e^{\sigma P}, -e^{-\sigma P}, 1, -1\} \cup \{-e^{\pm i P \delta_m} : m = 8, 12, \dots\}.$$

The point 1 includes the tangential identity and the mean Jordan block. The high normal operator is a bounded diagonal conjugate of $-I$ plus a compact summable multiplier, and each Fourier subspace supplies its displayed eigenvalues. Conversely, outside the closure of these values, the inverses of the diagonal symbols are bounded constants plus absolutely summable corrections, so give a bounded inverse. This proves equality of the spectrum and the stated set. For the continuous generator the unstable and stable exponents are $\pm\sigma \pm i\tau$; the remaining spectrum consists of 0, the tangent frequencies $\pm im/5$, the normal frequencies $\pm i\mu_{m,\pm}$, and the accumulation points $\pm id$, where $m = 4, 8, \dots$ in the tangent list and $m \geq 8$ in the normal list. To check the converse assertion for this unbounded generator, the pure-translation resolvent follows by integrating the translation group over its period when the spectral parameter is not a translation eigenvalue. Positive and negative frequency projections are bounded combinations of identity, mean projection, and Hilbert transform. On each sign sector the inner normal resolvent differs from such a translated and shifted resolvent by an absolutely summable symbol, while the outer normal resolvent differs from its constant limiting resolvent by an absolutely summable symbol. Parameters off the displayed closed set therefore have bounded resolvents. All calculations initially on polynomials extend to the generator domain and then by the resolvent identity.

3 A direct C^2 strong unstable construction

Write the local period map as

$$\Phi_P(h) = Mh + R(h), \quad M = S(P), \quad R(0) = DR(0) = 0.$$

On a fixed ball, $\|R(h)\| \leq C \|h\|^2$, $\|DR(h)\| \leq C \|h\|$, and $\|D^2R(h)\| \leq C$. Set $M_u = M|_{P_u E}$, $M_c = M|_{P_c E}$. Choose any $0 < \kappa < \sigma$ and let

$$\mathcal{X}_\kappa = \{(h_n)_{n \leq 0} : \|h\|_{\mathcal{X}_\kappa} := \sup_{n \leq 0} e^{-\kappa n P} \|h_n\| < \infty\}.$$

Prescribe $u_0 \in P_u E$. The backward-sequence equations are

$$u_n = M_u^n u_0 - \sum_{l=n}^{-1} M_u^{n-1-l} P_u R(h_l), \quad (13)$$

$$c_n = \sum_{l=-\infty}^{n-1} M_c^{n-1-l} P_c R(h_l), \quad h_n = u_n + c_n. \quad (14)$$

The unstable exponents in the first sum are strictly negative. The complementary exponents in the second sum are nonnegative. Their weighted convolution bounds are respectively bounded by constants times

$$\sum_{j \geq 0} e^{-(\sigma - \kappa)jP}, \quad \sum_{j \geq 0} (1 + j) e^{-\kappa jP}.$$

Thus all series converge in E and define a map on $\mathcal{X}_\kappa$. On a sequence ball of radius r , its nonlinear part has norm at most Cr^2 and Lipschitz constant at most Cr ; the free term has norm at most $C\|u_0\|$. Choosing r fixed and small, and then u_0 small, gives a unique fixed point in this ball. This requires no smooth cutoff or smooth norm on the ambient Banach space.

The fixed-point map is C^2 jointly in the sequence and u_0 . For example, in the second derivative the pointwise source obeys

$$\|D^2\mathbf{R}(h_n)[k_n, l_n]\| \leq Ce^{2\kappa nP} \|k\|_{\mathcal{X}_\kappa} \|l\|_{\mathcal{X}_\kappa},$$

which is stronger than the weight required by either convolution. Continuity of the second derivative is uniform on the shrinking sequence tail because $h_n \rightarrow 0$; finitely many remaining indices use ordinary continuity of $D^2\mathbf{R}$. The derivative of the fixed-point equation with respect to the sequence has inverse given by a convergent Neumann series, since its nonlinear operator norm is less than one. Differentiating it once and twice proves C^2 dependence on u_0 . At zero the solution is $h_n = M_u^n u_0$, and hence $c_0 = G(u_0)$ satisfies

$$G(0) = DG(0) = 0, \quad \|G(u_0)\| \leq C\|u_0\|^2.$$

This proves a two-real-dimensional local graph $\mathcal{W}^u = \{u + G(u)\}$ tangent to the full material unstable eigenspace.

The fixed-point equations imply $h_{n+1} = \Phi_P(h_n)$. Conversely any sufficiently small backward period orbit in $\mathcal{X}_\kappa$ satisfies (13): solve its unstable recursion toward time zero, and solve the complementary recursion from time N to time n . The leftover complementary term is bounded by $C(1+n-N)e^{\kappa NP}\|h\|_{\mathcal{X}_\kappa}$, which tends to zero as $N \rightarrow -\infty$. This proves the uniqueness characterization needed below.

Subperiod flow preserves the graph locally. Indeed, for $|t| \leq P$, the sequence $k_n = \Phi_t(h_n)$ is a backward period orbit because the local autonomous flow commutes with its period map. Its norm is at most $C_P\|h\|_{\mathcal{X}_\kappa}$, since $\Phi_t(0) = 0$ and its first derivative is uniformly bounded. After shrinking the parameter ball this sequence lies in the uniqueness neighborhood of the same construction. It is therefore the fixed-point sequence prescribed by $P_u\Phi_t(h_0)$. This proves continuous local invariance whenever the segment stays in the local neighborhood. Gluing the subperiod segments of the backward sequence constructs an actual ancient isolated trajectory with $\|h(s)\| \leq Ce^{\kappa s}$, $s \leq 0$.

Both enclosed areas are conserved along the physical material flow: their derivative is $\int_{\partial D_i} u \cdot n dS = 0$. Relabeling does not change an enclosed area, so they are also conserved in the stationary gauge. Since every graph trajectory tends backward to X^0 , its two areas are precisely $\pi\rho^2$ and π . Thus the graph lies in the correct area level set automatically, although the construction was in the full ambient space. In particular its total circulation is zero. This argument does not differentiate an unbounded generator through a nonlinear area chart.

4 Exact moment coordinates, symmetry, and the cap family

For each enclosed domain define

$$Q_i(X) = \frac{1}{2\pi r_i^6} \int_{D_i(X)} \bar{z}^4 dA(z), \quad a(X) = \ell_+ Q(X).$$

The boundary expression is the real polynomial functional

$$Q_i(X) = \frac{1}{20\pi i r_i^6} \int_0^{2\pi} \overline{X_i(\theta)}^5 X_i'(\theta) d\theta. \quad (15)$$

Indeed $d(\bar{z}^5 dz) = 10i\bar{z}^4 dA$. This proves analyticity as a map on E , as well as invariance under orientation-preserving independent relabelings. The first shape derivative is

$$DQ_i(X^0)h = \frac{1}{2\pi} \int_0^{2\pi} z_i(\theta) e^{-4i\theta} d\theta = z_{i,4};$$

it has no tangential contribution. Consequently

$$Da(X^0)h = d_u(h), \quad Da(X^0)P_c = 0, \quad Da(X^0)J = I_{\mathbb{C}}. \quad (16)$$

In particular this is a full material left dual, a complex notation for exactly two real coordinates. It is not a complex one-dimensional restriction of the real phase space.

The map

$$h \mapsto (a(X^0 + h), b = P_ch) \in \mathbb{C} \times P_c E$$

has derivative (d_u, P_c) , whose bounded inverse is $(\alpha, b) \mapsto J\alpha + b$. The Banach inverse function argument (equivalently solve the quadratic remainder by contraction) gives a real analytic local coordinate chart. In these coordinates the unstable graph is $b = g(a)$, with $g \in C^2$, $g(0) = Dg(0) = 0$, and $\|g(a)\| \leq C|a|^2$.

Only the geometric coordinate requires a continuous differential equation. Its label-generator term vanishes by relabeling invariance, and thus

$$a_s = \ell_+ DQ(X)V(X) =: F(a, b), \quad F(a, b) = (\sigma + i\tau)a + O((|a| + \|b\|)^2).$$

This is a bounded C^2 (analytic for analytic V) finite-dimensional functional. The linear term follows from $\ell_+ L_4 = (\sigma + i\tau)\ell_+$ and (16). The computation can first be done along smooth material solutions; boundedness and continuity extend its integral form to the little-space flow. There is no conjugation of the unbounded label generator by the nonlinear coordinate chart.

The action

$$(\mathcal{R}_\phi X)_i(\theta) = e^{i\phi} X_i(\theta - \phi)$$

fixes X^0 , preserves fourfold equivariance, and commutes with the planar gauged flow. It is a bounded action on E , with norms uniform in ϕ ; using the natural rotation-invariant Hölder norms it is isometric. It sends a to $e^{-4i\phi}a$ and commutes with P_u, P_c . The backward-sequence uniqueness therefore makes the graph equivariant.

On the graph, $a_s = (\sigma + i\tau)a + O(|a|^2)$. Hence

$$\frac{d}{ds}|a| = \sigma|a| + O(|a|^2) > 0$$

for $0 < |a| \leq \delta$, with a fixed sufficiently small δ . The cap $\{|a| = \delta, b = g(a)\}$ is transverse and all its points are a single rotation family. There are no additional material relabeling parameters on this graph; those neutral modes were in the complementary space and are uniquely slaved by the backward condition. Fourfold rotations are the expected isotropy. Any chosen cap point has one ancient backward orbit. Rotating that orbit gives every phase of the cap and its ancient history.

For a nonzero such orbit, $a(s) \neq 0$ at every finite backward time, by uniqueness of the reduced two-dimensional equation. Since $|a(s)| \leq Ce^{\kappa s}$, the function $(F(a(s), g(a(s))) - (\sigma + i\tau)a(s))/a(s)$ is integrable on $(-\infty, 0]$. Integrating the scalar complex logarithmic derivative yields a nonzero number a_* and

$$a(s) = a_* e^{(\sigma + i\tau)s} + O(e^{(\sigma + \kappa)s}), \quad s \rightarrow -\infty.$$

Nonvanishing follows directly from the exponential representation with a finite integral, rather than from merely taking a possibly zero limit.

The whole fourth moment of the physical signed annulus is exactly

$$B(X) = \int \bar{z}^4 \{c_1 \mathbf{1}_{D_1(X)} + c_2 \mathbf{1}_{D_2(X)}\} dA(z) \quad (17)$$

$$= 2\pi[-\rho^6 Q_1 + \rho^2 Q_2] = 2\pi(-\rho^6 - i\rho^3)a + O(|a|^2 + \|b - g(a)\|). \quad (18)$$

The coefficient $2\pi(-\rho^6 - i\rho^3)$ is nonzero. On the chosen ancient orbit this proves

$$B_*(s) = b_* e^{(\sigma+i\tau)s} + O(e^{(\sigma+\kappa)s}), \quad b_* = 2\pi(-\rho^6 - i\rho^3)a_* \neq 0.$$

The whole moment is essential: no positivity statement about a single interface, a radial strip, or a selected part of the pulse has been used.

5 Normalization and uniformity

For an amplitude $A > 0$ and radius $R > 0$, put $x = Rz$, $s = At$ and divide the material embedding by R . The planar velocity scales as $u(x) = ARV(z)$. Thus all Banach norms, fixed chart radii, multiplier bounds, contraction constants, cap radii, and subperiod bounds in this module depend only on ρ, γ and the fixed unit-scale contour neighborhood. They are independent of A, R . The physical growth rate is $A\sigma$, physical period is P/A , and the whole physical fourth moment is AR^6B . A known ambient solid rotation can be removed by a rotation of the spatial frame; rotational covariance and uniform rotation bounds preserve all these norm constants. The isolated module contains no truncation-depth parameter, and its constants can therefore be reused at every level of every finite stack. Uniform estimates for the genuinely nonconstant fields of other packets require the separate interaction and forced-comparison modules; they do not follow from this isolated statement.

Verification status. The matrices, the one-sided self cancellation, the mean Jordan block, the Sylvester identity, both left/right fourth-mode duals, the full projections, the high-mode diagonalizers and frequencies, the exact moment normalization, and the nonlinear backward-sequence construction have all been derived here. No prior audit was read or used. The sole external mathematical input still requiring a separate proof is the stated C^2 physical contour map on the fixed little Hölder neighborhood.