

1 Actual viscous outer cores and the local harmonic field

This section proves a conditional estimate for the actual coloured Navier–Stokes evolution. Its nonradial-density hypothesis is explicit; it is not deduced from material contours or from named unstable coordinates. We identify $x = (x_1, x_2)$ with $z = x_1 + ix_2$ and identify a velocity with $u_1 + iu_2$. Thus

$$K(z) = \frac{i}{2\pi\bar{z}}, \quad Jz = iz.$$

Let P denote circular averaging. For $f \in L^1 \cap L^\infty$, splitting the Biot–Savart integral at distance d and optimizing d gives

$$\|K * f\|_\infty \leq C\sqrt{\|f\|_1 \|f\|_\infty}. \quad (1)$$

Indeed the near integral is at most $Cd\|f\|_\infty$ and the far integral is at most $C\|f\|_1/d$.

Lemma 1.1 (Fourfold growth at the origin). *If $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is invariant under $z \mapsto iz$, then*

$$|K * f(z)| \leq C\|f\|_\infty |z|.$$

Neither zero circulation nor regularity of f is required.

Proof. For $r = |z| > 0$, the integral over $|y| \leq 2r$ is at most $C\|f\|_\infty r$, by integrating $|z - y|^{-1}$ over $B_{3r}(z)$. For $|y| > 2r$ use the absolutely convergent expansion

$$\frac{1}{\bar{z} - \bar{y}} = -\sum_{n \geq 0} \frac{\bar{z}^n}{\bar{y}^{n+1}}.$$

The inverse moment vanishes unless $n + 1$ is a multiple of four, by changing variables under a quarter turn. The surviving degrees are 3, 7, 11, ... Their absolute sum is at most

$$Cr^3 \int_{|y| > 2r} \frac{|f(y)|}{|y|^4} dy \leq C\|f\|_\infty r^3 \int_{2r}^\infty s^{-3} ds \leq C\|f\|_\infty r.$$

Termwise integration is justified by the integrable majorant $C|f(y)|/|y|$ on the far region. At $z = 0$ the velocity vanishes by rotational covariance and continuity. $\square$

Lemma 1.2 (Exit controlled only by radial drift). *Let a diffusion satisfy*

$$dX_s = v(s, X_s) ds + \sqrt{2\nu} dB_s, \quad X_0 = x,$$

and let σ be its first exit from B_b , where $|x| < b$. Suppose, before exit,

$$|v(s, y) \cdot y/|y|| \leq V(s) \quad (0 < |y| < b), \quad V \geq 0, \quad D := \int_0^T V(s) ds < b - |x|.$$

Then

$$\mathbb{P}(\sigma \leq T) \leq 2 \exp\left(-\frac{(b - |x| - D)^2}{8\nu T}\right). \quad (2)$$

There is no bound on the tangential component of v .

Proof. For $\lambda > 0$ set $\phi(y) = \cosh(\lambda|y|)$. Its even power series shows that it is smooth at zero. In dimension two,

$$\Delta\phi = \lambda^2 \cosh(\lambda r) + \frac{\lambda}{r} \sinh(\lambda r) \leq 2\lambda^2 \phi, \quad v \cdot \nabla \phi \leq \lambda V(s)\phi,$$

using $\sinh a \leq a \cosh a$ for $a \geq 0$. Therefore Itô's formula shows that

$$e^{-2\nu\lambda^2(s\wedge\sigma)-\lambda\int_0^{s\wedge\sigma} V(\tau) d\tau} \phi(X_{s\wedge\sigma})$$

is a nonnegative supermartingale. Stopping at T gives

$$\mathbb{P}(\sigma \leq T) \leq \frac{\cosh(\lambda|x|)}{\cosh(\lambda b)} e^{2\nu\lambda^2 T + \lambda D} \leq 2e^{-\lambda(b-|x|-D)+2\nu\lambda^2 T}.$$

Choose $\lambda = (b - |x| - D)/(4\nu T)$. The stopped calculation remains valid for bounded drifts smooth for positive times: stop first before their time endpoint and then pass to that endpoint. No approximation of a radial drift bound is used. $\square$

Consider real, fourfold symmetric colours solving

$$\partial_t w_l + u^\nu \cdot \nabla w_l = \nu \Delta w_l, \quad u^\nu = K * \sum_l w_l, \quad (3)$$

with geometric scales

$$A_l = A_1 \beta^{l-1}, \quad R_l = R_1 q^{l-1}, \quad 1 < \beta_{\min} \leq \beta \leq \beta_{\max}, \quad q \leq \frac{1}{4}, \quad \beta q \leq \frac{1}{2}.$$

Assume

$$\|w_l(0)\|_1 \leq C_0 A_l R_l^2, \quad \|w_l(0)\|_\infty \leq C_0 A_l. \quad (4)$$

In the countable case assume that the initial series converges absolutely in $L^1 \cap L^p$, $p > 2$, and use its mild Navier–Stokes solution and the associated linear coloured evolutions. The common velocity is bounded and smooth for positive times. Norm contraction identifies its vorticity with the sum of the colours, and (1) implies uniform convergence of the velocity series because $\sum_l A_l R_l < \infty$.

Fix an index i , a window $T = H/A_i$, and

$$\kappa_i = \frac{\nu}{A_i R_i^2}.$$

Suppose that for some fixed $a_0 > 0$ every $l \leq i$ has an initial constant core,

$$w_l(0) = c_l \quad \text{on } B_{a_0 R_l}, \quad |c_l| \leq C_0 A_l. \quad (5)$$

Write $h_l = (I - P)w_l$ and impose the actual density hypothesis

$$\sup_{0 \leq t \leq T} \|h_l(t)\|_1 \leq \delta A_l R_l^2 \quad (l \leq i), \quad 0 < \delta \leq 1. \quad (6)$$

No nonradial hypothesis is imposed on $l > i$.

The scalar stochastic representation used below is

$$w_l(t, x) = \mathbb{E}[w_l(0, X_t)], \quad dX_s = -u^\nu(t-s, X_s) ds + \sqrt{2\nu} dB_s, \quad X_0 = x. \quad (7)$$

Its sign is verified by applying Itô's formula to $w_l(t-s, X_s)$: the drift is $-\partial_t w_l - u^\nu \cdot \nabla w_l + \nu \Delta w_l = 0$. For completeness, first stop at $t - \varepsilon$, where the coefficients are smooth. Boundedness of the drift gives a continuous limiting process at t . Its transition densities away from starting time are bounded by $C/[\nu s]$. This bound follows by L^1 contraction, the Nash inequality $\|f\|_2^4 \leq C \|f\|_1^2 \|\nabla f\|_2^2$, the L^2 energy identity, and the corresponding adjoint estimate, composed at the midpoint. Thus $w_l(\varepsilon) \rightarrow w_l(0)$ in L^1 suffices to pass to the endpoint in expectation. Approximation of the bounded initial colour in L^1 by continuous functions also justifies evaluation at the limiting endpoint. This verifies (7) for the actual colours without smoothing an asserted radial bound.

Proposition 1.3 (Uniform persistence of actual constant cores). *There is $C \geq 1$, depending only on $C_0, a_0, \beta_{\min}, \beta_{\max}$, with the following property. Set*

$$\mathcal{D} = \sqrt{\delta} + \frac{\delta q^2}{\beta - q^2} + \frac{\beta q}{1 - \beta q}, \quad c_* := \frac{9}{512}, \quad \varepsilon_* := \frac{a_0(\beta - 1)}{16CH}.$$

If

$$CH\mathcal{D} \leq \frac{a_0}{16}, \quad 4C_0 e^{-c_* a_0^2 / (\kappa_i H)} \leq \frac{\varepsilon_*}{2}, \quad (8)$$

then, with $a = a_0/2$, for $l \leq i$ and $0 < t \leq T$,

$$\|w_l(t) - c_l\|_{L^\infty(B_{aR_l})} \leq 4C_0 A_l e^{-c_* a_0^2 R_l^2 / (\nu t)}. \quad (9)$$

All constants are independent of i , viscosity, and the number of colours.

Proof. The maximum principle gives $\|h_l\|_\infty \leq 2C_0 A_l$. The radial average induces a purely tangential velocity, so (1) and (6) imply

$$|(K * w_l) \cdot e_r| \leq \|K * h_l\|_\infty \leq C\sqrt{\delta} A_l R_l \quad (l \leq i).$$

Every unrestricted inner colour has $\|K * w_k\|_\infty \leq C A_k R_k$, whence

$$\sum_{k>l} \|K * w_k\|_\infty \leq C A_l R_l \frac{\beta q}{1 - \beta q}. \quad (10)$$

Temporarily assume that all probe-core defects satisfy

$$\|w_k - c_k\|_{L^\infty(B_{aR_k})} \leq \varepsilon A_k.$$

For $k < l$ and $|z| \leq a_0 R_l$, one has $|z| \leq a R_k/2$. Decompose the older colour into

$$w_k = c_k \mathbf{1}_{B_{aR_k}} + (w_k - c_k) \mathbf{1}_{B_{aR_k}} + w_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}}.$$

The first term gives exactly $(c_k/2)Jz$. The second gives a velocity bounded by $C\varepsilon A_k |z|$ by Lemma 1.1. The radial part of the third has zero velocity inside the disk. Its remaining density is $h_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}}$, whose L^1 norm is at most $\delta A_k R_k^2$. Its exterior series starts at degree three, so

$$|K * (w_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}})(z)| \leq C\delta A_k |z|^3 / R_k^2. \quad (11)$$

Here and below the density is evaluated at its actual viscous time. Summing the self, inner and older bounds gives

$$|u^\nu(t, z) \cdot e_r| \leq CA_l R_l \left(\mathcal{D} + \frac{\varepsilon}{\beta - 1} \right) \quad (|z| \leq a_0 R_l, l \leq i). \quad (12)$$

We used the geometric sums

$$\sum_{k < l} \frac{A_k}{R_k^2} \leq \frac{A_l}{R_l^2} \frac{q^2}{\beta - q^2}, \quad \sum_{k < l} A_k \leq \frac{A_l}{\beta - 1}.$$

Fix C large enough for (12).

On a bootstrap interval with $\varepsilon \leq \varepsilon_*$, the integrated radial drift in the backward representation is at most $a_0 R_l / 8$, because $A_l T \leq H$ and (8) controls both terms. Start at $|x| \leq a_0 R_l / 2$ and stop on exiting $B_{a_0 R_l}$. If the backward path does not exit, its endpoint belongs to the initial constant core. The possible discrepancy outside it is at most $2C_0 A_l$. Lemma 1.2 therefore gives

$$|w_l(t, x) - c_l| \leq 4C_0 A_l \exp\left(-\frac{(3a_0 R_l / 8)^2}{8\nu t}\right),$$

which is (9). Subtracting c_l is legitimate: a constant is itself a bounded scalar solution, and bounded stochastic representation does not require its integrability.

The maximum normalized probe defect over the finite set $l \leq i$ is continuous for positive time. It tends to zero at time zero, since the probe has a fixed margin inside the initial constant disk and the common drift is bounded. If it first attained ε_* , the preceding estimate, $R_l \geq R_i$, $t \leq H/A_i$, and (8) would instead bound it by $\varepsilon_*/2$. This closes the bootstrap. The finite maximum adds no dependence on i ; every infinite inner sum was already controlled by (10). $\square$

Proposition 1.4 (Harmonic exterior field and actual density leak). *Under Proposition 1.3, set*

$$b_i = \sum_{l \leq i} K * w_l, \quad \Omega_i = \frac{1}{2} \sum_{l \leq i} c_l, \quad H_i = \sum_{l \leq i} K * (w_l \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_l}}), \quad L_i = \sum_{l \leq i} K * ((w_l - c_l) \mathbf{1}_{B_{aR_l}}).$$

For $|z| \leq aR_i/2$ one has the exact decomposition

$$b_i(t, z) = \Omega_i Jz + H_i(t, z) + L_i(t, z). \quad (13)$$

The field H_i is divergence free and curl free on B_{aR_i} , and

$$|H_i(t, z)| \leq C\delta A_i |z|^3 / R_i^2, \quad |DH_i(t, z)| \leq C\delta A_i |z|^2 / R_i^2, \quad (14)$$

$$|L_i(t, z)| \leq CA_i e^{-c_* a_0^2 / (\kappa_i H)} |z|. \quad (15)$$

Its cubic coefficient is exactly

$$c_{3,i}(t) = -\frac{i}{2\pi} \sum_{l \leq i} \int_{|y| \geq aR_l} \frac{w_l(t, y)}{\bar{y}^4} dy. \quad (16)$$

With $\eta_l(t) = \|h_l(t)\|_1 / (A_l R_l^2)$,

$$|c_{3,i}(t)| \leq C \frac{A_i}{R_i^2} \sum_{m=0}^{i-1} (q^2/\beta)^m \eta_{i-m}(t) \leq C\delta A_i / R_i^2. \quad (17)$$

Furthermore,

$$|H_i(t, z) - c_{3,i}(t)\bar{z}^3| \leq C\delta A_i |z|^7/R_i^6, \quad (18)$$

$$|D(H_i(t, z) - c_{3,i}(t)\bar{z}^3)| \leq C\delta A_i |z|^6/R_i^6. \quad (19)$$

The derivative denotes the real spatial Jacobian. No derivative bound on L_i is asserted.

Proof. The constant disks induce $(c_l/2)Jz$ in the specified region, proving (13). Exterior densities induce curl free velocities away from their supports, and all Biot-Savart velocities are divergence free. Radial averaging commutes with radial truncation, and an exterior radial density induces zero velocity inside its truncation radius. Consequently, on this region,

$$H_i(t, z) = -\frac{i}{2\pi} \sum_{l \leq i} \sum_{m \geq 0} \bar{z}^{4m+3} \int_{|y| \geq aR_l} \frac{h_l(t, y)}{\bar{y}^{4m+4}} dy.$$

The inverse radial moment is zero, so this gives exactly (16). The coefficient integral is at most

$$\frac{\|h_l\|_1}{(aR_l)^4}.$$

This proves (17).

The series and its first spatial derivatives converge uniformly on $|z| \leq aR_i/2$. Indeed each series ratio is at most $1/16$, and the derivative majorant adds only the factor $4m+3$. Thus, for each colour, its harmonic field and derivative are bounded by $C\delta A_l |z|^3/R_l^2$ and $C\delta A_l |z|^2/R_l^2$, respectively. Removing $m=0$ gives $C\delta A_l |z|^7/R_l^6$ and $C\delta A_l |z|^6/R_l^6$. Summing uses

$$\sum_{l \leq i} \frac{A_l}{R_l^d} \leq \frac{A_i/R_i^d}{1 - q^d/\beta} \quad (d = 2, 6).$$

These prove (14), (18), and (19). Finally Lemma 1.1, (9), and $\sum_{l \leq i} A_l \leq A_i \beta / (\beta - 1)$ prove (15). $\square$

Scope for the localized tail problem. If $R_* = \kappa_i^{1/4} R_i \leq aR_i/2$, then throughout B_{R_*} ,

$$|b_i - \Omega_i Jz| \leq C\delta A_i |z|^3/R_i^2 + CA_i e^{-c_* a_0^2/(\kappa_i H)} |z|.$$

The integrated linear-leak coefficient is at most $CH e^{-c_* a_0^2/(\kappa_i H)}$. The actual density leak is therefore quantified; no material hole has been retained by assumption. The harmonic derivative estimate (14) is available for a coupled transfer or energy calculation, while the core leak has only the growth estimate (15). The parameter margins are compatible with $H \asymp \log(1/q)$ and $\kappa_i = q^{3/2}$ if $\sqrt{\delta}H$ is sufficiently small. This compatibility does not prove (6); obtaining that hypothesis and controlling the coupled inner aggregate over its whole receiver window remain separate nonlinear tasks. In particular no fixed-data nonunique vanishing-viscosity theorem is concluded here.