

A multiplicative linear comparison for the fourth angular mode

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Fix

$$\rho^2 = \frac{3}{5}, \quad r_1 = \rho, \quad r_2 = 1, \quad c_1 = -1, \quad c_2 = \rho^2, \quad W = \sum_{i=1}^2 c_i \mathbf{1}_{B_{r_i}}.$$

For $\kappa > 0$ put $W_\kappa(t) = e^{\kappa t \Delta} W$ and

$$\Omega_\kappa(t, r) = r^{-2} \int_0^r s W_\kappa(t, s) ds.$$

This is the angular velocity of the radial heat solution. Its inviscid values at the reference interfaces are

$$\Omega_0(r_1) = -\frac{1}{5}, \quad \Omega_0(r_2) = 0.$$

All constants in this note depend only on the displayed fixed profile and the fixed collars chosen below. They do not depend on κ or H .

We use one *complex positive Fourier coefficient*, with angular index $m = 4$. A real perturbation is $f(t, r)e^{4i\theta} + \overline{f(t, r)}e^{-4i\theta}$; its negative coefficient is recovered by conjugation and imposes no additional restriction on the positive coefficient. Write

$$\mu_t(dr) = f(t, r)r dr, \quad h_m(r, s) = \left(\frac{\min(r, s)}{\max(r, s)} \right)^m, \quad \psi_\mu(r) = -\frac{1}{2m} \int_0^\infty h_m(r, s)\mu(ds).$$

The radial coefficient of the streamfunction is ψ_μ . The equation linearized at the radial heat solution is

$$\partial_t f + im\Omega_\kappa(t, r)f - im\frac{\psi_\mu(r)}{r}\partial_r W_\kappa(t, r) = \kappa \left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2} \right) f. \quad (1)$$

Indeed $u_r = -r^{-1}\partial_\theta \psi$; hence the linearized radial advection is $-im\psi_\mu W'_\kappa/r$. The streamfunction formula follows by solving the radial Poisson equation, regular at zero and decaying at infinity. In particular every normalization in (1) refers to the same positive Fourier coefficient.

Choose $d_0 > 0$ so small that the intervals $I_i = (r_i - 2d_0, r_i + 2d_0)$ are disjoint and their closures are contained in $(0, \infty)$. Fix $\chi_i \in C_c^\infty(I_i)$, with $0 \leq \chi_i \leq 1$, equal to one when $|r - r_i| \leq d_0$. Set

$$d(r) = \min\{\text{dist}(r, \{r_1, r_2\}), d_0\}, \quad q_i(t) = \frac{1}{c_i r_i^2} \int \chi_i(r)\mu_t(dr),$$

$$F(t) = \|\mu_t\|_{\text{TV}}, \quad T(t) = \int d(r)|\mu_t|(dr).$$

The Euler interface matrix is

$$(M_m z)_i = -im\Omega_0(r_i)z_i + \frac{i}{2r_i^2} \sum_j c_j r_j^2 h_m(r_i, r_j)z_j. \quad (2)$$

For $m = 4$, this gives exactly

$$M_4 = \begin{pmatrix} 3i/10 & i\rho^4/2 \\ -i\rho^6/2 & 3i/10 \end{pmatrix}, \quad \lambda_\pm = \pm\sigma + i\tau, \quad \sigma = \rho^5/2 > 0, \quad \tau = 3/10,$$

with eigenvectors $v_+ = (1, -i\rho)$ and $v_- = (1, i\rho)$.

Proposition 1. *There are $a_* > 0$ and $C < \infty$ with the following property. Let $H \geq 1$ and $0 < \kappa H \leq a_*$. Set $\varepsilon = \sqrt{\kappa H}$ and prescribe*

$$\mu_0 = \sum_{i=1}^2 c_i r_i^2 z_i \delta_{r_i}, \quad F_0 = \|\mu_0\|_{\text{TV}}.$$

Equation (1) has a unique bounded measure-valued mild solution on $[0, H]$, weakly continuous at zero with initial measure μ_0 . It is smooth in the radial variable for positive times. For $0 \leq t \leq H$ it satisfies

$$F(t) \leq C F_0 e^{(\sigma + C\varepsilon)t}, \quad (3)$$

$$T(t) \leq C\varepsilon F_0 e^{(\sigma + C\varepsilon)t}, \quad (4)$$

$$|q(t) - e^{tM_4} z| \leq C\varepsilon t e^{(\sigma + C\varepsilon)t} F_0. \quad (5)$$

The norms on the last line can be any fixed norm on $\mathbb{C}^2$.

Lemma 1 (Radial heat layers). *Let $D_i(t, r) = -\partial_r e^{\kappa t \Delta} \mathbf{1}_{B_{r_i}}$. For $0 < t \leq H$ with $\kappa H \leq a_*$ these functions are nonnegative and*

$$\int_0^\infty D_i(t, r) dr = 1 - e^{-r_i^2/(4\kappa t)}, \quad (6)$$

$$\int_0^\infty |r - r_i| D_i(t, r) dr \leq C\sqrt{\kappa t}, \quad (7)$$

$$\int_{|r - r_i| \geq d_0} D_i(t, r) dr \leq C e^{-c/(\kappa t)}. \quad (8)$$

Moreover

$$|\Omega_\kappa(t, r) - \Omega_0(r_i)| \leq C(|r - r_i| + \sqrt{\kappa t}) \quad (r \in I_i). \quad (9)$$

Proof. Convolution of a disk indicator with a radial decreasing positive kernel is radial decreasing, which proves $D_i \geq 0$. To verify this directly and obtain quantitative bounds, differentiate the disk convolution as a boundary integral. With $a = \kappa t$ and $R = r_i$ this gives

$$D_i(t, r) = \frac{R}{2a} e^{-(r^2 + R^2)/(4a)} I_1\left(\frac{rR}{2a}\right). \quad (10)$$

Here $I_1(s)$ is the modified Bessel function, represented by $(2\pi)^{-1} \int_0^{2\pi} e^{s \cos \theta} \cos \theta d\theta$. For $s \leq 1$ its power series gives $I_1(s) \leq Cs$. For $s \geq 1$, integration by parts gives

$$I_1(s) = \frac{s}{2\pi} \int_0^{2\pi} e^{s \cos \theta} \sin^2 \theta d\theta \leq C s e^s (1 + s)^{-3/2}.$$

The last bound follows by $1 - \cos \theta \geq c\theta^2$ near zero and the corresponding bound near 2π , with exponentially small contribution elsewhere. Thus for all $s \geq 0$, $I_1(s) \leq C s e^s (1 + s)^{-3/2}$. On $R/2 \leq r \leq 2R$, formula (10) is bounded by $C a^{-1/2} e^{-(r-R)^2/(4a)}$. On the remaining ranges the same formula, and the preceding global bound on I_1 , give an exponentially small integral, including after multiplication by $|r - R|$. Near zero the additional factor s in the Bessel bound removes any radial singularity. These elementary Gaussian integrals prove (7) and (8), with slightly reduced exponential constant in the tails. The mass identity is also exact: the heat-convolved disk tends to zero at infinity and has value $1 - e^{-R^2/(4a)}$ at the origin.

For (9), the disk translation estimate $\|\mathbf{1}_{B_R}(\cdot - y) - \mathbf{1}_{B_R}\|_1 \leq C_R |y|$ and Gaussian convolution give $\|W_\kappa(t) - W\|_1 \leq C\sqrt{\kappa t}$. For $r \in I_i$, which is bounded away from zero, the displayed formula for Ω_κ bounds $|\Omega_\kappa - \Omega_0|$ by this L^1 error. The function Ω_0 is Lipschitz on the collars, including across its interface values, so the remaining difference is at most $C|r - r_i|$. $\square$

We next justify the measure equation and the estimates used in its proof. Let $P(t, s)$ be the homogeneous propagator obtained by omitting the nonlocal source in (1). In the natural measure $f(r)r dr$ it satisfies

$$\|P(t, s)\eta\|_{\text{TV}} \leq \|\eta\|_{\text{TV}}, \quad (11)$$

$$\int d(r)|P(t, s)\eta|(dr) \leq \int d(r)|\eta|(dr) + C\sqrt{\kappa(t-s)}\|\eta\|_{\text{TV}}. \quad (12)$$

Indeed complex Kato gives

$$\partial_t |f| \leq \kappa \left(\partial_{rr} + r^{-1} \partial_r - m^2 r^{-2} \right) |f|.$$

The angular multiplication is imaginary and disappears. There is also a direct kernel argument avoiding any boundary issue at $r = 0$. The angular- m heat propagator has a positive radial kernel involving I_m , and $0 \leq I_m(s) \leq I_0(s)$ implies domination by ordinary radial heat. Positivity follows from its power series and the upper bound from the circle-integral representation. Interleave this heat propagator with the unit-modulus multiplications $e^{-im\Omega_\kappa(t,r)\Delta t}$. All these products are dominated by radial heat; their bounded-perturbation limit constructs the homogeneous propagator and preserves domination. The bounded coefficient is continuous in time in L^∞ . At time zero, for $r \leq \rho/2$ the heat error in the central constant-vorticity region is $O(e^{-c/(\kappa t)})$, so the integral defining Ω_κ has the same bound there. For $r \geq \rho/2$, the disk L^1 heat error and the factor r^{-2} give $O(\sqrt{\kappa t})$. At positive times L^∞ heat continuity suffices. This proves the claimed continuity. This time-dependent product construction is therefore legitimate and proves (11). The radial function $d(|x|)$ is globally 1-Lipschitz on $\mathbb{R}^2$, so ordinary heat changes its average from a point x by at most $\mathbb{E}|\sqrt{2\kappa}B_{t-s}| \leq C\sqrt{\kappa(t-s)}$, proving (12). For clarity, these facts hold initially for smooth radial coefficients; extension to finite measures follows by approximating measures and using the radial heat kernel. One can also construct P directly by the bounded imaginary perturbation of the angular- m heat propagator. The multiplier Ω_κ is uniformly bounded because $|\Omega_\kappa| \leq \|W\|_\infty/2$.

The nonlocal source as a natural measure is

$$\mathcal{B}_t \mu(dr) = im\psi_\mu(r)W'_\kappa(t, r) dr. \quad (13)$$

It is a bounded map on finite measures, uniformly in $t > 0$: $|\psi_\mu| \leq \|\mu\|_{\text{TV}}/(2m)$ and $\int |W'_\kappa| dr \leq \sum_i |c_i|$. Thus the Volterra equation

$$\mu_t = P(t, 0)\mu_0 + \int_0^t P(t, s)\mathcal{B}_s \mu_s ds$$

has a unique bounded weakly continuous solution by Picard iteration in finite measures on every finite interval. At positive times each source term has an integrable radial density; its integral is therefore a usual strong measure integral. No strong continuity in total variation at zero is asserted for the singular ring measure. The usual iteration estimate is the exponential series with terms bounded by $(Ct)^n/n!$. The singular initial ring measure is therefore harmless. The kernels are smooth for positive times; on any interval separated from zero, this bounded-source construction and the smooth coefficients give the stated radial regularity. All computations below may alternatively be performed after smoothing the ring initial data and passed to the measure limit with the displayed uniform bounds.

Proof of Proposition 1. We first reconstruct the streamfunction from two collar masses. Uniformly in $r > 0$,

$$\left| \psi_\mu(r) + \frac{1}{2m} \sum_i c_i r_i^2 h_m(r, r_i) q_i \right| \leq CT, \quad |\psi_\mu(r)| \leq C(|q| + T). \quad (14)$$

To prove this, insert $\chi_1 + \chi_2$ in the integral defining ψ_μ . Outside the regions where these cutoffs equal one, $d(r)$ is bounded below by a fixed positive constant, so the remaining variation is bounded by CT . On the i th collar, $|\partial_s h_m(r, s)| \leq m/s$ uniformly in the receiving radius r . The difference from $h_m(r, r_i)$ is consequently bounded by $C|s - r_i| \leq Cd(s)$ there, after adjusting the fixed constant where $|s - r_i| \geq d_0$. This proves (14) with no confinement assumption on μ .

Since $W'_\kappa = -\sum_i c_i D_i$, Lemma 1 and (14) give

$$\|\mathcal{B}_t \mu\|_{\text{TV}} \leq C(|q| + T), \quad \int d(r) |\mathcal{B}_t \mu|(dr) \leq C\varepsilon(|q| + T).$$

Apply (11)–(12) to the mild equation. The initial measure has distance moment zero, so

$$F(t) \leq F_0 + C \int_0^t (|q(s)| + T(s)) ds, \quad T(t) \leq C\varepsilon \left[F_0 + \int_0^t (|q(s)| + T(s)) ds \right]. \quad (15)$$

In the source contribution to the second inequality, the additional heat displacement is at most $C\sqrt{\kappa(t-s)} \leq C\varepsilon$ times the source variation.

Testing (1) against χ_i in the natural measure yields

$$\begin{aligned} q'_i &= -\frac{im}{c_i r_i^2} \int \chi_i \Omega_\kappa \mu(dr) \\ &\quad + \frac{\kappa}{c_i r_i^2} \int (\chi_i'' + r^{-1} \chi_i' - m^2 r^{-2} \chi_i) \mu(dr) \\ &\quad + \frac{im}{c_i r_i^2} \int \chi_i \psi_\mu W'_\kappa dr. \end{aligned}$$

The first line equals $-im\Omega_0(r_i)q_i$ up to $C(T + \varepsilon F)$, by (9) and the fixed collar geometry. The second is bounded by $C\kappa F \leq C\varepsilon F$, since the test function is supported away from zero and $\kappa H \leq a_* \leq 1$, $H \geq 1$.

For the third line, $|\partial_r h_m(r, s)| \leq m/r$ on each receiving collar gives $|\partial_r \psi_\mu| \leq CF$. The near-unit mass of D_i , its first distance moment, and the exponentially small D_j mass in the foreign collar imply

$$\int \chi_i \psi_\mu W'_\kappa dr = -c_i \psi_\mu(r_i) + O(\varepsilon F).$$

There is no singular remainder at zero: these tests are supported on fixed positive-radius collars, and Lemma 1 already controls the omitted radial gradient mass. Applying (14) at r_i identifies the main term with the coupling in (2). Consequently

$$q' = M_4 q + e, \quad |e| \leq C(T + \varepsilon F). \quad (16)$$

This identity is valid almost everywhere and in the integrated form from time zero; all right-hand sides are locally integrable by the mild bounds. The initial identity $q(0) = z$ follows from weak convergence of the homogeneous heat evolution against χ_i and the $O(t)$ variation of the source integral.

To exploit the exact spectral growth, write $q = \alpha v_+ + \beta v_-$ and put $y = |\alpha| + |\beta|$. This is an equivalent norm on $\mathbb{C}^2$; the upper Dini derivative in (16) satisfies

$$D^+ y \leq \sigma y + C(T + \varepsilon F).$$

Define

$$Z(t) = F_0 + \int_0^t (|q(s)| + T(s)) ds.$$

Equation (15) gives $F \leq CZ$, $T \leq C\varepsilon Z$, and therefore, for fixed positive constants C_1, C_2, C_3 ,

$$D^+ y \leq \sigma y + C_1 \varepsilon Z, \quad Z' \leq C_2 y + C_3 \varepsilon Z.$$

Choose $a \geq C_1/\sigma$ and $C_4 \geq \max\{aC_2, C_3\}$. For $Y = y + a\varepsilon Z$, direct substitution gives

$$D^+Y \leq (\sigma + C_4\varepsilon)Y.$$

Initially $q = z$ and $y(0) \leq CF_0$, so $Y(0) \leq CF_0$. Gronwall gives $y(t) \leq CF_0 e^{(\sigma+C_4\varepsilon)t}$. Integrating the inequality for Z then yields

$$Z(t) \leq e^{C_3\varepsilon t} \left[F_0 + CF_0 \int_0^t e^{(\sigma+(C_4-C_3)\varepsilon)s} ds \right] \leq CF_0 e^{(\sigma+C_4\varepsilon)t},$$

where the fixed positive σ bounds the denominator. This proves (3) and (4). Finally the diagonalization gives $\|e^{tM_4}\| \leq Ce^{\sigma t}$. Variation of constants in (16), followed by the Z bound, proves

$$|q(t) - e^{tM_4}z| \leq C\varepsilon F_0 \int_0^t e^{\sigma(t-s)} e^{(\sigma+C_4\varepsilon)s} ds \leq C\varepsilon t e^{(\sigma+C_4\varepsilon)t} F_0,$$

as claimed. $\square$

Remark 1 (Unstable coordinate and scale compatibility). *The left unstable coordinate is $a(q) = q_1/2 + iq_2/(2\rho)$. Since $a(M_4q) = (\sigma + i\tau)a(q)$, (5) gives*

$$|a(q(t)) - e^{(\sigma+i\tau)t}a(z)| \leq C\varepsilon t e^{(\sigma+C\varepsilon)t} F_0.$$

If $F_0 \leq C_0|a(z)|$, the relative comparison error on $[0, H]$ is at most $CC_0\sqrt{\kappa}H^{3/2}e^{C\sqrt{\kappa}H^{3/2}}$. This includes the zero-initial-fourth-moment stable/unstable combination, whose stable and unstable coefficients have equal modulus.

For geometric cascade scales, the first receiver has $\kappa_{N-1} = \beta q^2 \kappa_N$. On $H = O(\log(1/q))$, this linear relative error tends to zero even when $\kappa_N \gg \beta q^2$, the condition needed to place the next scale in the low-Re regime.

Remark 2 (Scope of the proposition). *This is a linear statement about a radial heat background. It does not prove the finite-amplitude source-window comparison, the nonlinear stable-complement reset, the control of actual foreign diffusive fields, or a nonunique inviscid limit for one fixed datum. Those statements require additional estimates. In particular, this proposition supplies a multiplicative linear estimate rather than a replacement for the nonlinear contour cascade.*