

1 Fixed seeds: separate initial and forced prefix contributions

The result in this section is a majorant lemma. Its coupling inequality is an explicit hypothesis; no assertion that diffusive colours already satisfy that inequality is made.

Lemma 1.1 (Two prefix norms). *Let $\beta \geq 4$, $A_i = \beta^{i-1}$, and fix a receiving index $j \geq 2$. Suppose nonnegative continuous functions x_i , $1 \leq i < j$, satisfy, on $0 \leq \theta \leq S$,*

$$x_i(\theta) \leq Mbe^{\lambda_0\theta} + M \int_0^\theta e^{\lambda_0(\theta-s)} \left(\sum_{l < j} K_{il} x_l(s) + f_i(s) \right) ds, \quad \lambda_0 = \frac{\sigma + \eta}{\beta}, \quad (1)$$

where $M, C, \sigma > 0$, $\eta, r_* > 0$, $f_i \geq 0$ are integrable, and the nonnegative matrix K obeys

$$K_{ii} \leq Cr_*/\beta, \quad K_{il} \leq \begin{cases} C\beta^{l-j}q^{2(i-l)}, & l < i, \\ C\beta^{l-j}q^{6(l-i)}, & i < l < j. \end{cases}$$

Assume $\beta q^6 < 1$. Define

$$r_B = \frac{C}{\beta} \left(r_* + \frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right), \quad \lambda_B = \lambda_0 + Mr_B.$$

Then $x_i \leq h_i + p_i$, where the initial contribution satisfies

$$\max_{i < j} h_i(\theta) \leq Mbe^{\lambda_B\theta}. \quad (2)$$

For $D > \beta$ with $Dq^8 < \beta$, put $w_i = (Dq^6)^{j-i}$ and

$$r_P = \frac{C}{\beta} \left(r_* + \frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right), \quad \lambda_P = \lambda_0 + Mr_P.$$

The forced contribution satisfies

$$P(\theta) := \max_{i < j} \frac{p_i(\theta)}{w_i} \leq M \int_0^\theta e^{\lambda_P(\theta-s)} \max_{i < j} \frac{f_i(s)}{w_i} ds. \quad (3)$$

All constants are independent of j . One can choose η, r_* small, then D large and q small, so both λ_B and λ_P are strictly smaller than $\sigma/2$.

Proof. The row sums of K are bounded by r_B : for $l = i + d$ the inner coefficient is at most $C\beta^{i-j}(\beta q^6)^d$, and for $l = i - d$ the outer coefficient is at most $C\beta^{i-j}(q^2/\beta)^d$. Use $\beta^{i-j} \leq 1/\beta$ and sum the geometric series.

Solve the positive comparison system

$$Y' = (\lambda_0 I + MK)Y + Mf, \quad Y(0) = Mb\mathbf{1}.$$

Iteration of (1), or its integral comparison principle, gives $x \leq Y$. The variation-of-constants formula splits $Y = h + p$, with

$$h = e^{(\lambda_0 I + MK)\theta} Mb\mathbf{1}, \quad p = M \int_0^\theta e^{(\lambda_0 I + MK)(\theta-s)} f(s) ds.$$

The unweighted maximum norm gives (2). For the weighted maximum norm, the inner row acquires $(\beta/D)^d$ and the outer row acquires $(Dq^8/\beta)^d$. Their geometric sums give r_P and hence (3). Since λ_0 is close to $\sigma/\beta \leq \sigma/4$, first make the diagonal term small. Increasing D makes the remaining q -independent term in r_P small; subsequently decreasing q makes all other terms small. $\square$

Corollary 1.2 (The fixed-seed baseline is small in its actual field). *Assume the hypotheses of Lemma 1.1, with $\lambda_B \leq \lambda_* < \sigma/2$, in the clock $\theta = A_j t$. Suppose the completed cap clocks satisfy*

$$T_{j,N} \leq S/A_N + H \sum_{m=j}^{N-1} A_m^{-1}, \quad N \geq j, \quad S \geq H\beta/(\beta-1).$$

Let $b = \delta e^{-\sigma S}$. Then through $T_{j,N}$ the initial contribution is bounded uniformly by

$$B = M\delta e^{-(\sigma-\lambda_*)S}.$$

Its normalized inward feedback on the receiver j is at most

$$C \sum_{i < j} \beta^{i-j} q^{2(j-i)} h_i \leq CB \frac{q^2/\beta}{1 - q^2/\beta}. \quad (4)$$

The forced contribution has feedback at most

$$CP \frac{Dq^8/\beta}{1 - Dq^8/\beta}.$$

In particular, with $H = h_0 \log(1/q)$ one can take $S = C_0 \log(1/q)$ so that $B = O(q^6)$ and its feedback is $O(q^8)$, with constants independent of depth.

Proof. Writing $a = \beta^{j-N} \in (0, 1]$ gives

$$A_j T_{j,N} \leq Sa + \frac{H\beta}{\beta-1}(1-a) \leq S.$$

Apply (2). The actual inward row in (4) is a geometric series with ratio q^2/β . For $p_i \leq w_i P$ its ratio is Dq^8/β . Choose $C_0 \geq h_0\beta/(\beta-1)$ and $C_0(\sigma - \lambda_*) > 6$. No weighted norm of the initial seeds is used. If $Dq^6 < 1$, the local chart bound follows from $x_i \leq B + w_i P \leq B + Dq^6 P$ whenever the latter quantity is below r_* . $\square$

The corollary repairs a specific scalar obstruction. To use it for Navier–Stokes one must prove (1), including its spatial q^2 and q^6 rows for diffusive colours, and bound the forced term P . Those remain coupled PDE hypotheses.

2 A fixed passive colour is determined by the total Euler path

The next result allows signed smooth colours as well as indicator colours. It therefore does not require a patch reference profile.

Proposition 2.1 (Recovery of fixed passive colours). *Let u be bounded and divergence free on $[0, T] \times \mathbb{R}^2$, with $u \in L^1([0, T]; W_{\text{loc}}^{1,p})$ for some $p > 1$. For each compactly supported $f_0 \in L^1 \cap L^\infty$ there is at most one solution of*

$$\partial_t f + u \cdot \nabla f = 0, \quad f(0) = f_0,$$

in $L^\infty([0, T]; L^1 \cap L^\infty)$ with its weak initial trace. It is renormalized and belongs to $C([0, T]; L^2)$.

Suppose $u_n \rightarrow u$ uniformly, $\sup_n \|u_n\|_\infty \leq V$, $\nu_n \rightarrow 0$, and $0 \leq \nu_n \leq 1$. Let f_n solve

$$\partial_t f_n + u_n \cdot \nabla f_n = \nu_n \Delta f_n, \quad f_n(0) = f_0,$$

with the L^1 , L^2 , and L^∞ contraction bounds and the weak-form equation. Then their transport limit f is unique and

$$f_n \rightarrow f \quad \text{in } C([0, T]; L^1 \cap L^2).$$

The same conclusion holds for transported colours with $\nu_n = 0$.

Proof. We give the renormalization argument needed for uniqueness. Let $f_\varepsilon = f * \eta_\varepsilon$ be a spatial mollification. It satisfies

$$\partial_t f_\varepsilon + u \cdot \nabla f_\varepsilon = r_\varepsilon, \quad r_\varepsilon(x) = \int \nabla \eta_\varepsilon(y) \cdot [u(x) - u(x - y)] f(x - y) dy.$$

On a compact spatial set the difference-quotient estimate bounds its L^1 norm by $C\|\nabla u\|_p\|f\|_{p'}$, on a slightly larger set. This follows by writing the velocity difference as the integral of its gradient along the segment and then applying Hölder's inequality. Approximate u in $L_t^1 W_{\text{loc}}^{1,p}$ by divergence-free spatial convolutions. For a smooth approximating velocity, divergence freedom allows subtraction of $f(x)$ in the commutator integral. Translation continuity of f then makes the commutator tend to zero in $L_{t,x}^1$. The uniform difference-quotient bound handles the velocity approximation. Thus $r_\varepsilon \rightarrow 0$ locally in L^1 .

Multiplication by a bounded derivative of a C^1 function and passage to the limit shows that bounded renormalizations satisfy the transport equation, including their initial trace. For the difference of two bounded solutions, use a renormalization equal to s^2 on their bounded value range. Integration against expanding cutoffs has a flux bounded by $C\|u\|_\infty R^{-1} \int |f|^2$. Since bounded integrable functions are square integrable, that flux tends to zero. The initial square norm is zero, which proves uniqueness. The same argument preserves the square norm of one solution. Weak time continuity and preservation of that norm give strong L^2 time continuity.

For convergence, the weak-form identity gives, for compact smooth φ ,

$$\left| \frac{d}{dt} \int \varphi f_n \right| \leq \|f_0\|_1 (V \|\nabla \varphi\|_\infty + \|\Delta \varphi\|_\infty).$$

The contraction bounds and this equicontinuity give weak L^2 convergence uniformly in time along subsequences. The Kato inequality and a radial cutoff also give uniform tightness:

$$\sup_{t \leq T} \int_{|x| > 2R} |f_n(t, x)| dx \leq \int_{|x| > R} |f_0(x)| dx + CT \|f_0\|_1 (V/R + 1/R^2).$$

Passing to the distributional equation identifies every limit as a bounded integrable transported solution with the given initial trace. Uniqueness identifies all subsequences.

To prove uniform strong L^2 convergence, suppose otherwise and choose bad times $t_n \rightarrow t$. Weak time equicontinuity gives $f_n(t_n) \rightharpoonup f(t)$ in L^2 , while contraction and renormalization give

$$\limsup_n \|f_n(t_n)\|_2 \leq \|f_0\|_2 = \|f(t)\|_2.$$

Hilbert-space uniform convexity gives strong convergence, a contradiction. Local L^1 convergence follows from L^2 , and tightness gives global uniform L^1 convergence. $\square$

For $p > 2$ the kernel split gives the explicit useful bound

$$\|K * h\|_\infty \leq C_p(\|h\|_1 + \|h\|_p) : \quad (5)$$

the near kernel belongs to $L^{p'}(B_1)$ because $p' < 2$, and the far kernel is bounded. Thus strong convergence of total vorticity paths in $C_t(L^1 \cap L^p)$ implies uniform convergence of their velocities. Whenever the limiting velocities have the Sobolev regularity in Proposition 2.1, the proposition applies to every fixed bounded initial colour. In particular, if two total Euler paths coincide, their recovered fixed colours coincide. No uniqueness of the total nonlinear Euler equation is used.

Let $f_{1,0}, \dots, f_{k,0}$ be finitely many fixed bounded compact colours and choose complex constants c_l . A suitable observable is

$$a_u(t) = \sum_{l=1}^k c_l \int \Phi_l(x) f_l[u](t, x) dx, \quad \Phi_l(x) = \chi_l(|x|) \bar{z}^4, \quad (6)$$

where each χ_l is a fixed smooth radial compact cutoff. The weak-form estimate makes a_u uniformly Lipschitz for a bounded family of drifts and $\nu \leq 1$, and the proposition makes it continuous under the strong total-path convergence just specified. These tests cover one smooth outer colour and also a moment coordinate made from several patch colours. If uncut moments are wanted in an Euler limit, the cutoffs can equal one on a common support ball: bounded velocity gives finite propagation. Indeed test $|f|$ against a nondecreasing smooth function of $|x| - R_0 - Vt$; its material derivative is nonpositive and its initial pairing is zero.

3 Protected first caps and phase shooting

A phase evaluated at an arbitrary cap time need not distinguish trajectories. The following first-cap functional does distinguish them and requires no common cap clock.

Lemma 3.1 (Continuity of a protected first-cap phase). *Fix $\delta, c, e, h, L > 0$, with $2e < T$. Let $a : [0, T] \rightarrow \mathbb{C}$ be L -Lipschitz, let*

$$\tau(a) = \inf\{t \geq 0 : |a(t)| \geq \delta\}, \quad \mathcal{P}(a) = a(\tau(a))/\delta,$$

and suppose

$$\begin{aligned} |a(0)| &\leq \delta - h, & e &\leq \tau(a) \leq T - e, \\ |a(t)| &\leq \delta - h & (0 \leq t \leq \tau(a) - e), \\ |a(t)| - |a(s)| &\geq c(t - s) & (\tau(a) - e \leq s < t \leq \tau(a) + e). \end{aligned}$$

For any continuous b with $\varepsilon = \|a - b\|_\infty < \min(h/2, ce/4)$, its first cap exists,

$$|\tau(a) - \tau(b)| \leq 2\varepsilon/c, \quad |\mathcal{P}(a) - \mathcal{P}(b)| \leq \delta^{-1}(1 + 2L/c)\varepsilon.$$

Families satisfying the displayed conditions with common constants retain a protected first cap under uniform convergence. Hence $\mathcal{P}$ is a continuous functional on that protected class.

Proof. Before $\tau(a) - \varepsilon$ the strict margin excludes any cap of b . On $[\tau(a) - \varepsilon, \tau(a)]$ the radial lower-slope inequality gives $|a(t)| \leq \delta - c(\tau(a) - t)$. Thus b cannot cap before $\tau(a) - 2\varepsilon/c$. At $\tau(a) + 2\varepsilon/c$, its modulus is greater than or equal to $\delta + \varepsilon$, so it has capped by that time. This proves the time estimate. Since both first-cap values have modulus δ , the phase estimate follows from

$$|a(\tau(a)) - b(\tau(b))| \leq L|\tau(a) - \tau(b)| + \|a - b\|_\infty.$$

For a uniformly convergent protected sequence, take a convergent subsequence of its cap times. The common Lipschitz bound, the precap margin, and the integrated radial lower-slope inequality pass to the limit. They identify the limiting time as the unique first cap. This argument applies to every subsequence and proves the final assertion. $\square$

Corollary 3.2 (Separation of total inviscid paths). *Consider one fixed total initial datum and fixed bounded initial colours. Suppose two Navier–Stokes subsequences converge strongly in $C_t(L^1 \cap L^p)$, their limits satisfy the velocity regularity of Proposition 2.1, and their observables (6) have uniformly protected first caps. If their limiting first-cap phases are different, the two total Euler paths are different. The cap times may differ.*

Proof. Proposition 2.1 and Lemma 3.1 pass the colours, observables, and phases to the limit. Equal total paths would have equal velocities and recovered colours, hence equal first-cap phases, a contradiction. $\square$

Proposition 3.3 (Conditional degree shooting). *Fix $\nu > 0$ and an active index $N \geq 2$. Suppose there is a continuous initial-data family indexed by $(r, \theta) \in [0, 1] \times \mathbb{S}^1$: the active seed at N has phase θ , and all other seed amplitudes are multiplied by r . Assume the total and passive-colour solution paths at this fixed viscosity depend continuously on these parameters. Assume their outer observables have first caps protected by the common constants of Lemma 3.1.*

If at $r = 0$ all other colours are radial and the active seed family is obtained by spatial rotation through angle $-\theta/4$, then the output phase map at $r = 1$ is surjective onto $\mathbb{S}^1$.

Proof. The first-cap map is continuous by Lemma 3.1. At $r = 0$, rotational covariance and the radial cutoffs in (6) give

$$\mathcal{P}(a_{0,\theta}) = e^{i\theta} \mathcal{P}(a_{0,0}).$$

Its degree is one. The degree is constant along the homotopy in r : a continuous argument lifted on the rectangle $[0, 1] \times [0, 2\pi]$ has endpoint difference $2\pi d(r)$, and the continuous integer $d(r)$ is constant. A circle map omitting one point admits a periodic continuous argument and has degree zero. The map at $r = 1$ therefore cannot omit a point. $\square$

The continuity hypothesis at fixed positive viscosity is routine and does not hide a uniform-in-viscosity estimate. For completeness, in $X = L^1 \cap L^p$, (5) makes $(K * w)w$ locally Lipschitz on bounded X balls. The mild vorticity equation gives

$$d(t) \leq d(0) + \frac{C_p M}{\sqrt{\nu}} \int_0^t (t-s)^{-1/2} d(s) ds.$$

Short intervals on which the integral coefficient is at most $1/2$, followed by iteration, prove continuity for each fixed ν . The passive mild equation has the same kernel and an additional source controlled by the uniform drift difference. It gives continuity of colours, including when their initial bounded domains vary continuously. Uniform cap protection across the homotopy, by contrast, is a substantive coupled PDE hypothesis.

Proposition 3.4 (Conditional sparse-phase construction). *Fix $p > 2$ and $\beta q^{2/p} < 1$. Let a family of infinitely seeded, zero-circulation compact initial data be indexed by a phase at every depth. Assume its series converges absolutely in $X = L^1 \cap L^p$, that changing only the phase at depth N changes the datum by at most $\eta_N \rightarrow 0$, uniformly in all phases, and that depths $N \geq 2$ leave the fixed outer initial colours unchanged.*

Suppose $\nu_N \downarrow 0$, and for every depth and every choice of all other phases, Proposition 3.3 applies with common cap-protection constants. Finally suppose that for every datum in this family its Navier–Stokes family is relatively compact in $C_t X$, and every limit satisfies the velocity hypothesis of Proposition 2.1. Then some single datum in this family has two distinct vanishing-viscosity accumulation paths.

Proof. Start with arbitrary default phases at all depths. At step k , choose a new depth $N_k > N_{k-1}$ with $\nu_{N_k} < \nu_{N_{k-1}}/2$ and

$$\eta_{N_k} < 2^{-k} \min(1, r_1, \dots, r_{k-1}).$$

Shoot its phase to give first-cap output $(-1)^k$ at ν_{N_k} , using Proposition 3.3, and call the resulting datum g_k . Fixed-positive-viscosity continuity gives $r_k > 0$ such that an admissible datum within r_k of g_k changes that output by less than $1/4$. All admissible future phase choices already have protected caps by hypothesis; openness of a derivative lower bound in the uniform topology is not asserted.

The summable phase changes give one limiting datum g_∞ ; each coordinate is changed at most once and the absolutely convergent series identifies the limit with that phase sequence. For each k ,

$$\|g_\infty - g_k\|_X \leq \sum_{l>k} 2^{-l} r_k < r_k.$$

Hence its selected-viscosity output lies within $1/4$ of $(-1)^k$. Relative compactness supplies limiting paths along each parity. Their first-cap phases lie in the disjoint closed $1/4$ -neighbourhoods of $+1$ and -1 . Corollary 3.2 proves that the total paths are different. Zero circulation and compact support pass by strong L^1 convergence. $\square$

The proposition is an implication with explicit uniform cap and coupling hypotheses. It is not an unconditional Navier–Stokes counterexample. Every intermediate datum retains the full unbounded infinite background; the proof does not assert arbitrary-small macroscopic controllability around a bounded finite prefix.