

1 Actual viscous outer cores and the local harmonic field

This section proves a conditional estimate for the actual coloured Navier–Stokes evolution. Its nonradial-density hypothesis is explicit; it is not deduced from material contours or from named unstable coordinates. We identify $x = (x_1, x_2)$ with $z = x_1 + ix_2$ and identify a velocity with $u_1 + iu_2$. Thus

$$K(z) = \frac{i}{2\pi\bar{z}}, \quad Jz = iz.$$

Let P denote circular averaging. For $f \in L^1 \cap L^\infty$, splitting the Biot–Savart integral at distance d and optimizing d gives

$$\|K * f\|_\infty \leq C\sqrt{\|f\|_1 \|f\|_\infty}. \quad (1)$$

Indeed the near integral is at most $Cd\|f\|_\infty$ and the far integral is at most $C\|f\|_1/d$.

Lemma 1.1 (Fourfold growth at the origin). *If $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is invariant under $z \mapsto iz$, then*

$$|K * f(z)| \leq C\|f\|_\infty |z|.$$

Neither zero circulation nor regularity of f is required.

Proof. For $r = |z| > 0$, the integral over $|y| \leq 2r$ is at most $C\|f\|_\infty r$, by integrating $|z - y|^{-1}$ over $B_{3r}(z)$. For $|y| > 2r$ use the absolutely convergent expansion

$$\frac{1}{\bar{z} - \bar{y}} = -\sum_{n \geq 0} \frac{\bar{z}^n}{\bar{y}^{n+1}}.$$

The inverse moment vanishes unless $n + 1$ is a multiple of four, by changing variables under a quarter turn. The surviving degrees are 3, 7, 11, ... Their absolute sum is at most

$$Cr^3 \int_{|y| > 2r} \frac{|f(y)|}{|y|^4} dy \leq C\|f\|_\infty r^3 \int_{2r}^\infty s^{-3} ds \leq C\|f\|_\infty r.$$

Termwise integration is justified by the integrable majorant $C|f(y)|/|y|$ on the far region. At $z = 0$ the velocity vanishes by rotational covariance and continuity. $\square$

Lemma 1.2 (Exit controlled only by radial drift). *Let a diffusion satisfy*

$$dX_s = v(s, X_s) ds + \sqrt{2\nu} dB_s, \quad X_0 = x,$$

and let σ be its first exit from B_b , where $|x| < b$. Suppose, before exit,

$$|v(s, y) \cdot y/|y|| \leq V(s) \quad (0 < |y| < b), \quad V \geq 0, \quad D := \int_0^T V(s) ds < b - |x|.$$

Then

$$\mathbb{P}(\sigma \leq T) \leq 2 \exp\left(-\frac{(b - |x| - D)^2}{8\nu T}\right). \quad (2)$$

There is no bound on the tangential component of v .

Proof. For $\lambda > 0$ set $\phi(y) = \cosh(\lambda|y|)$. Its even power series shows that it is smooth at zero. In dimension two,

$$\Delta\phi = \lambda^2 \cosh(\lambda r) + \frac{\lambda}{r} \sinh(\lambda r) \leq 2\lambda^2 \phi, \quad v \cdot \nabla \phi \leq \lambda V(s)\phi,$$

using $\sinh a \leq a \cosh a$ for $a \geq 0$. Therefore Itô's formula shows that

$$e^{-2\nu\lambda^2(s\wedge\sigma) - \lambda \int_0^{s\wedge\sigma} V(\tau) d\tau} \phi(X_{s\wedge\sigma})$$

is a nonnegative supermartingale. Stopping at T gives

$$\mathbb{P}(\sigma \leq T) \leq \frac{\cosh(\lambda|x|)}{\cosh(\lambda b)} e^{2\nu\lambda^2 T + \lambda D} \leq 2e^{-\lambda(b-|x|-D) + 2\nu\lambda^2 T}.$$

Choose $\lambda = (b - |x| - D)/(4\nu T)$. The stopped calculation remains valid for bounded drifts smooth for positive times: stop first before their time endpoint and then pass to that endpoint. No approximation of a radial drift bound is used. $\square$

Consider real, fourfold symmetric colours solving

$$\partial_t w_l + u^\nu \cdot \nabla w_l = \nu \Delta w_l, \quad u^\nu = K * \sum_l w_l, \quad (3)$$

with geometric scales

$$A_l = A_1 \beta^{l-1}, \quad R_l = R_1 q^{l-1}, \quad 1 < \beta_{\min} \leq \beta \leq \beta_{\max}, \quad q \leq \frac{1}{4}, \quad \beta q \leq \frac{1}{2}.$$

Assume

$$\|w_l(0)\|_1 \leq C_0 A_l R_l^2, \quad \|w_l(0)\|_\infty \leq C_0 A_l. \quad (4)$$

In the countable case assume that the initial series converges absolutely in $L^1 \cap L^p$, $p > 2$, and use its mild Navier–Stokes solution and the associated linear coloured evolutions. The common velocity is bounded and smooth for positive times. Norm contraction identifies its vorticity with the sum of the colours, and (1) implies uniform convergence of the velocity series because $\sum_l A_l R_l < \infty$.

Fix an index i , a window $T = H/A_i$, and

$$\kappa_i = \frac{\nu}{A_i R_i^2}.$$

Suppose that for some fixed $a_0 > 0$ every $l \leq i$ has an initial constant core,

$$w_l(0) = c_l \quad \text{on } B_{a_0 R_l}, \quad |c_l| \leq C_0 A_l. \quad (5)$$

Write $h_l = (I - P)w_l$ and impose the actual density hypothesis

$$\sup_{0 \leq t \leq T} \|h_l(t)\|_1 \leq \delta A_l R_l^2 \quad (l \leq i), \quad 0 < \delta \leq 1. \quad (6)$$

No nonradial hypothesis is imposed on $l > i$.

The scalar stochastic representation used below is

$$w_l(t, x) = \mathbb{E}[w_l(0, X_t)], \quad dX_s = -u^\nu(t-s, X_s) ds + \sqrt{2\nu} dB_s, \quad X_0 = x. \quad (7)$$

Its sign is verified by applying Itô's formula to $w_l(t-s, X_s)$: the drift is $-\partial_t w_l - u^\nu \cdot \nabla w_l + \nu \Delta w_l = 0$. For completeness, first stop at $t - \varepsilon$, where the coefficients are smooth. Boundedness of the drift gives a continuous limiting process at t . Its transition densities away from starting time are bounded by $C/[\nu s]$. This bound follows by L^1 contraction, the Nash inequality $\|f\|_2^4 \leq C \|f\|_1^2 \|\nabla f\|_2^2$, the L^2 energy identity, and the corresponding adjoint estimate, composed at the midpoint. Thus $w_l(\varepsilon) \rightarrow w_l(0)$ in L^1 suffices to pass to the endpoint in expectation. Approximation of the bounded initial colour in L^1 by continuous functions also justifies evaluation at the limiting endpoint. This verifies (7) for the actual colours without smoothing an asserted radial bound.

Proposition 1.3 (Uniform persistence of actual constant cores). *Consider the finite or countable coloured evolution and geometric scales specified above, with $\nu, H > 0$. Assume (4), (5), and (6) on $[0, T]$, where $T = H/A_i$. There is $C \geq 1$, depending only on $C_0, a_0, \beta_{\min}, \beta_{\max}$, with the following property. Set*

$$\mathcal{D} = \sqrt{\delta} + \frac{\delta q^2}{\beta - q^2} + \frac{\beta q}{1 - \beta q}, \quad c_* := \frac{9}{512}, \quad \varepsilon_* := \frac{a_0(\beta - 1)}{16CH}.$$

If

$$CH\mathcal{D} \leq \frac{a_0}{16}, \quad 4C_0 e^{-c_* a_0^2 / (\kappa_i H)} \leq \frac{\varepsilon_*}{2}, \quad (8)$$

then, with $a = a_0/2$, for $l \leq i$ and $0 < t \leq T$,

$$\|w_l(t) - c_l\|_{L^\infty(B_{aR_l})} \leq 4C_0 A_l e^{-c_* a_0^2 R_l^2 / (\nu t)}. \quad (9)$$

All constants are independent of i , viscosity, and the number of colours.

Proof. The maximum principle gives $\|h_l\|_\infty \leq 2C_0 A_l$. The radial average induces a purely tangential velocity, so (1) and (6) imply

$$|(K * w_l) \cdot e_r| \leq \|K * h_l\|_\infty \leq C\sqrt{\delta} A_l R_l \quad (l \leq i).$$

Every unrestricted inner colour has $\|K * w_k\|_\infty \leq C A_k R_k$, whence

$$\sum_{k>l} \|K * w_k\|_\infty \leq C A_l R_l \frac{\beta q}{1 - \beta q}. \quad (10)$$

Temporarily assume that all probe-core defects satisfy

$$\|w_k - c_k\|_{L^\infty(B_{aR_k})} \leq \varepsilon A_k.$$

For $k < l$ and $|z| \leq a_0 R_l$, one has $|z| \leq a R_k/2$. Decompose the older colour into

$$w_k = c_k \mathbf{1}_{B_{aR_k}} + (w_k - c_k) \mathbf{1}_{B_{aR_k}} + w_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}}.$$

The first term gives exactly $(c_k/2)Jz$. The second gives a velocity bounded by $C\varepsilon A_k|z|$ by Lemma 1.1. The radial part of the third has zero velocity inside the disk. Its remaining density is $h_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}}$, whose L^1 norm is at most $\delta A_k R_k^2$. Its exterior series starts at degree three, so

$$|K * (w_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}})(z)| \leq C\delta A_k |z|^3 / R_k^2. \quad (11)$$

Here and below the density is evaluated at its actual viscous time. Summing the self, inner and older bounds gives

$$|u^\nu(t, z) \cdot e_r| \leq CA_l R_l \left(\mathcal{D} + \frac{\varepsilon}{\beta - 1} \right) \quad (|z| \leq a_0 R_l, l \leq i). \quad (12)$$

We used the geometric sums

$$\sum_{k < l} \frac{A_k}{R_k^2} \leq \frac{A_l}{R_l^2} \frac{q^2}{\beta - q^2}, \quad \sum_{k < l} A_k \leq \frac{A_l}{\beta - 1}.$$

Fix C large enough for (12).

On a bootstrap interval with $\varepsilon \leq \varepsilon_*$, the integrated radial drift in the backward representation is at most $a_0 R_l / 8$, because $A_l T \leq H$ and (8) controls both terms. Start at $|x| \leq a_0 R_l / 2$ and stop on exiting $B_{a_0 R_l}$. If the backward path does not exit, its endpoint belongs to the initial constant core. The possible discrepancy outside it is at most $2C_0 A_l$. Lemma 1.2 therefore gives

$$|w_l(t, x) - c_l| \leq 4C_0 A_l \exp\left(-\frac{(3a_0 R_l / 8)^2}{8\nu t}\right),$$

which is (9). Subtracting c_l is legitimate: a constant is itself a bounded scalar solution, and bounded stochastic representation does not require its integrability.

The maximum normalized probe defect over the finite set $l \leq i$ is continuous for positive time. It tends to zero at time zero, since the probe has a fixed margin inside the initial constant disk and the common drift is bounded. If it first attained ε_* , the preceding estimate, $R_l \geq R_i$, $t \leq H/A_i$, and (8) would instead bound it by $\varepsilon_*/2$. This closes the bootstrap. The finite maximum adds no dependence on i ; every infinite inner sum was already controlled by (10). $\square$

Proposition 1.4 (Harmonic exterior field and actual density leak). *Under Proposition 1.3, set*

$$b_i = \sum_{l \leq i} K * w_l, \quad \Omega_i = \frac{1}{2} \sum_{l \leq i} c_l, \quad H_i = \sum_{l \leq i} K * (w_l \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_l}}), \quad L_i = \sum_{l \leq i} K * ((w_l - c_l) \mathbf{1}_{B_{aR_l}}).$$

For $|z| \leq aR_i/2$ one has the exact decomposition

$$b_i(t, z) = \Omega_i Jz + H_i(t, z) + L_i(t, z). \quad (13)$$

The field H_i is divergence free and curl free on B_{aR_i} , and

$$|H_i(t, z)| \leq C\delta A_i |z|^3 / R_i^2, \quad |DH_i(t, z)| \leq C\delta A_i |z|^2 / R_i^2, \quad (14)$$

$$|L_i(t, z)| \leq CA_i e^{-c_* a_0^2 / (\kappa_i H)} |z|. \quad (15)$$

The cubic coefficient of H_i is exactly

$$c_{3,i}(t) = -\frac{i}{2\pi} \sum_{l \leq i} \int_{|y| \geq aR_l} \frac{w_l(t, y)}{\bar{y}^4} dy. \quad (16)$$

With $\eta_l(t) = \|h_l(t)\|_1 / (A_l R_l^2)$,

$$|c_{3,i}(t)| \leq C \frac{A_i}{R_i^2} \sum_{m=0}^{i-1} (q^2/\beta)^m \eta_{i-m}(t) \leq C\delta A_i / R_i^2. \quad (17)$$

Furthermore,

$$|H_i(t, z) - c_{3,i}(t)\bar{z}^3| \leq C\delta A_i |z|^7/R_i^6, \quad (18)$$

$$|D(H_i(t, z) - c_{3,i}(t)\bar{z}^3)| \leq C\delta A_i |z|^6/R_i^6. \quad (19)$$

The derivative denotes the real spatial Jacobian. No derivative bound on L_i is asserted.

Proof. The constant disks induce $(c_l/2)Jz$ in the specified region, proving (13). Exterior densities induce curl free velocities away from their supports, and all Biot–Savart velocities are divergence free. Radial averaging commutes with radial truncation, and an exterior radial density induces zero velocity inside its truncation radius. Consequently, on this region,

$$H_i(t, z) = -\frac{i}{2\pi} \sum_{l \leq i} \sum_{m \geq 0} \bar{z}^{4m+3} \int_{|y| \geq aR_l} \frac{h_l(t, y)}{\bar{y}^{4m+4}} dy.$$

The inverse radial moment is zero, so this gives exactly (16). The coefficient integral is at most

$$\frac{\|h_l\|_1}{(aR_l)^4}.$$

This proves (17).

The series and its first spatial derivatives converge uniformly on $|z| \leq aR_i/2$. Indeed each series ratio is at most $1/16$, and the derivative majorant adds only the factor $4m+3$. Thus, for each colour, its harmonic field and derivative are bounded by $C\delta A_l |z|^3/R_l^2$ and $C\delta A_l |z|^2/R_l^2$, respectively. Removing $m=0$ gives $C\delta A_l |z|^7/R_l^6$ and $C\delta A_l |z|^6/R_l^6$. Summing uses

$$\sum_{l \leq i} \frac{A_l}{R_l^d} \leq \frac{A_i/R_i^d}{1 - q^d/\beta} \quad (d = 2, 6).$$

These prove (14), (18), and (19). Finally Lemma 1.1, (9), and $\sum_{l \leq i} A_l \leq A_i \beta / (\beta - 1)$ prove (15). $\square$

Scope for the localized tail problem. If $R_* = \kappa_i^{1/4} R_i \leq aR_i/2$, then throughout B_{R_*} ,

$$|b_i - \Omega_i Jz| \leq C\delta A_i |z|^3/R_i^2 + CA_i e^{-c_* a_0^2/(\kappa_i H)} |z|.$$

The integrated linear-leak coefficient is at most $CH e^{-c_* a_0^2/(\kappa_i H)}$. The actual density leak is therefore quantified; no material hole has been retained by assumption. The harmonic derivative estimate (14) is available for a coupled transfer or energy calculation, while the core leak has only the growth estimate (15). The parameter margins are compatible with $H \asymp \log(1/q)$ and $\kappa_i = q^{3/2}$ if $\sqrt{\delta}H$ is sufficiently small. This compatibility does not prove (6); obtaining that hypothesis and controlling the coupled inner aggregate over its whole receiver window remain separate nonlinear tasks. In particular no fixed-data nonunique vanishing-viscosity theorem is concluded here.

2 A quantitative parabolic localization estimate

The next estimate concerns an actual convection–diffusion equation, and gives a quantitative substitute for compact material support. Its local cubic drift hypothesis must still be verified for a coupled Navier–Stokes cascade.

Lemma 2.1 (Localized tail exit). *Let f solve*

$$\partial_t f + U \cdot \nabla f = \nu \Delta f, \quad U(t, x) = \Omega(t)x^\perp + V(t, x) + b(t, x), \quad \operatorname{div} U = 0,$$

on $0 \leq t \leq T$, with $\|f(0)\|_1 \leq m$ and $\operatorname{supp} f(0) \subset B_r$. Suppose

$$\begin{aligned} \|V(t)\|_\infty &\leq C_0 m (\nu t)^{-1/2}, \\ |b(t, x)| &\leq L|x| \quad \text{for all } x, \\ |b(t, x)| &\leq C_3 |x|^3 + \ell(t)|x| \quad \text{for } |x| \leq R_*. \end{aligned}$$

Assume $\ell \geq 0$, Ω is integrable, and

$$r \leq R_*/4, \quad C_3 R_*^2 T + \int_0^T \ell(t) dt \leq \frac{1}{4}, \quad C_0 \frac{m}{\nu} \sqrt{\nu T} \leq R_*/8.$$

There are absolute $c, C > 0$ such that

$$\sup_{t \leq T} \int_{|x| \geq R_*} |f(t, x)| dx \leq C m \exp\left(-\frac{c R_*^2}{\nu T}\right).$$

For each fixed integer $k \geq 0$ there are constants $c, C_k > 0$ such that

$$\int_{|x| \geq R_*} |x|^k |f(T, x)| dx \leq C_k m e^{C_k L T} R_*^k \exp\left(-\frac{c R_*^2}{\nu T}\right),$$

provided additionally $\sqrt{\nu T} \leq c R_$. The conclusion holds for smooth drifts and for limits for which the parabolic solution and the corresponding diffusion representation are obtained by smooth approximation.*

Proof. Kato's inequality bounds $|f|$ by the positive scalar solution from $|f(0)|$. Normalize this initial mass and represent the positive solution by the diffusion

$$dX_t = U(t, X_t) dt + \sqrt{2\nu} dB_t.$$

A time-dependent rotation removes $\Omega(t)x^\perp$; the rotated Brownian motion remains standard and all norm assumptions are unchanged. Let σ be the first exit from B_{R_*} . Before that exit, deterministic Gronwall applied to the integral stochastic equation gives

$$\sup_{s \leq t \wedge \sigma} |X_s| \leq e^{C_3 R_*^2 t + \int_0^t \ell} \left[r + 2C_0 \frac{m}{\nu} \sqrt{\nu t} + \sqrt{2\nu} \sup_{s \leq t} |B_s| \right].$$

The stated margins imply that $\sigma \leq T$ requires $\sup_{s \leq T} |B_s| \geq c R_*/\sqrt{\nu}$. The coordinate reflection bound for planar Brownian motion yields

$$\mathbb{P}(\sigma \leq T) \leq C \exp\left(-\frac{c R_*^2}{\nu T}\right).$$

An endpoint outside B_{R_*} requires this exit, proving the mass bound.

For the moment assertion, condition at σ . The remaining drift has linear growth bounded by $L|x|$ plus $\|V(t)\|_\infty$. Its integrated bounded contribution is at most $2C_0(m/\nu)\sqrt{\nu T} \leq R_*/4$. Gronwall and Brownian moments therefore give, uniformly in the exit time,

$$\mathbb{E}[|X_T|^k \mid \mathcal{F}_\sigma] \leq C_k e^{C_k L T} R_*^k \quad \text{on } \{\sigma \leq T\}.$$

Multiply by the indicator of this event and use its probability bound. Finally restore the initial mass, which is at most m . $\square$

For a receiver with $\kappa_i = \nu/(A_i R_i^2)$ and stage duration $T = H/A_i$, the choice $R_* = \kappa_i^{1/4} R_i$ gives

$$\frac{R_*^2}{\nu T} = \frac{1}{\kappa_i^{1/2} H}, \quad C_3 R_*^2 T \leq C x_i \kappa_i^{1/2} H \quad \text{if } C_3 \leq C A_i x_i / R_i^2.$$

A global bound $L \leq C A_i$ costs only $e^{C_k H}$, independent of depth. The exponential exit factor dominates that cost as $\kappa_i \downarrow 0$ for logarithmic stage lengths. This parameter calculation does not verify the required local cubic hypothesis for the actual foreign field.

3 Nonlinear inner-tail suppression through a receiver window

The estimate in this section is conditional on quantitative information about the actual foreign velocity. It closes a longer-time nonlinear subproblem than the isolated-tail and first-diffusion-time estimates. It does not establish the required nonradial-density bound for the active and older outer colours.

Write $J(x_1, x_2) = (-x_2, x_1)$, $z = x_1 + ix_2$, and

$$K(x) = \frac{Jx}{2\pi|x|^2}, \quad B_4(f) = \int_{\mathbb{R}^2} \bar{z}^4 f(z) \, dz.$$

A scalar is fourfold symmetric if $f(Jx) = f(x)$; a vector field has the corresponding covariance if $b(t, Jx) = Jb(t, x)$. Constants denoted by C can change from line to line.

3.1 A Gaussian-energy estimate for a globally weak foreign field

Proposition 3.1 (Weak foreign field through a finite window). *There are universal constants $c, \lambda_0, C > 0$ with the following property. Let $p > 2$, and let $f_0 \in L^1 \cap L^p$ be real, fourfold symmetric, supported in B_1 , and satisfy*

$$\int f_0 = 0, \quad B_4(f_0) = 0, \quad m = \|f_0\|_1.$$

Suppose b is divergence free, has fourfold covariance, and, after removal of a time-dependent solid rotation, satisfies

$$|b(s, x)| \leq \ell(s)|x|, \quad |b(s, x)| \leq d|x|^3, \quad (1+s)\ell(s) \leq \lambda_0.$$

Let f solve

$$\partial_s f + (K * f + b) \cdot \nabla f = \Delta f.$$

For $S \geq 2$, assume the displayed hypotheses on $0 \leq s \leq S-1$ and

$$m S^{1/25} \sqrt{1 + \log S} \leq c. \tag{20}$$

Then

$$\sup_{0 \leq s \leq S-1} |B_4(f(s))| \leq C \left[m^2 S^{7/50} (1 + \log S) + m d S^{102/25} \right]. \tag{21}$$

The constants are independent of a decomposition of the datum into smaller scales. The assertion holds for smooth drifts and for the parabolic limits obtained by approximation with the stated bounds.

Proof. Scalar contraction and the divergence-free Nash estimate give

$$\|f(s)\|_1 \leq m, \quad \|f(s)\|_\infty \leq Cm/s, \quad \|K * f(s)\|_\infty \leq Cm/\sqrt{s}.$$

The last estimate follows by splitting Biot–Savart at a radius and optimizing. Take $s_0 = 1/100$. The positive scalar solution from $|f_0|$ dominates $|f|$. Its diffusion representation, after removing solid rotation, obeys

$$|X_s| \leq e^{\lambda_0 s} \left[1 + Cm\sqrt{s} + \sqrt{2} \sup_{a \leq s} |B_a| \right] \quad (0 \leq s \leq s_0).$$

The Gaussian exponential moment of the Brownian supremum is finite at this fixed small time. With $\beta = 3/10$ it follows that

$$\int e^{\beta|x|^2/(1+s_0)} |f(s_0, x)| \, dx \leq Cm.$$

Multiplication by the Nash L^∞ bound gives a weighted L^2 norm $O(m)$. Fixed polynomial absolute moments are $O(m)$ on $[0, s_0]$. The exact harmonic-moment equation and symmetrization of self-advection therefore yield

$$|B_4(f(s_0))| \leq C(m^2 + md). \quad (22)$$

For clarity, the self contribution is bounded by $Cm \int |x|^2 |f|$, because $|\bar{z}^3 - \bar{y}^3|/|z - y| \leq C(|z|^2 + |y|^2)$; the foreign contribution is bounded by $d \int |x|^6 |f|$. These bounds also prove the result directly if the final time is smaller than s_0 .

For $s \geq s_0$ put

$$\tau = \log(1 + s), \quad y = x/\sqrt{1 + s}, \quad w(\tau, y) = (1 + s)f(s, x), \quad E(\tau, y) = \sqrt{1 + s}b(s, \sqrt{1 + s}y).$$

Then

$$w_\tau = \mathcal{L}w - (K * w + E) \cdot \nabla w, \quad \mathcal{L} = \Delta + \frac{1}{2}y \cdot \nabla + 1,$$

with $|E| \leq \lambda(\tau)|y|$, $\lambda \leq \lambda_0$, and $|E| \leq a(\tau)|y|^3$, where $a(\tau) = de^{2\tau}$. Let

$$F_\beta = \|e^{\beta|y|^2/2}w\|_2, \quad D_\beta = \|e^{\beta|y|^2/2}\nabla w\|_2^2, \quad \beta = 3/10.$$

Before using the full Gaussian weight, we establish membership in its energy space. Put $\tau_0 = \log(1 + s_0)$ and, for $R \geq 1$, set

$$\Phi_R(t) = \frac{R^2 t}{R^2 + t}, \quad \theta_R(t) = \Phi'_R(t) = \frac{R^4}{(R^2 + t)^2}, \quad \rho_R(y) = e^{\beta\Phi_R(|y|^2)}, \quad g_R = \rho_R^{1/2}w.$$

These weights are bounded and smooth, $0 \leq \theta_R \leq 1$, and $\rho_R \uparrow e^{\beta|y|^2}$. The short-time exponential moment and Nash bound proved above give $\int \rho_R w(\tau_0)^2 \leq Cm^2$, uniformly in R . For each bounded weight, spatial cutoffs and integration by parts give the exact identity

$$\langle w, \mathcal{L}w \rangle_{\rho_R} = -\|\nabla g_R\|_2^2 + \frac{1}{2}\|g_R\|_2^2 - \beta \int |y|^2 \theta_R(|y|^2) \left(\frac{1}{2} - \beta \theta_R(|y|^2) \right) g_R^2 \, dy.$$

Here the second derivatives of $\log \rho_R$ cancel after substituting $w = \rho_R^{-1/2}g_R$. With $c_\beta = \beta(1/2 - \beta) > 0$, the last term is at most $-c_\beta \int |y|^2 \theta_R g_R^2$. The divergence-free advection contributes

$$\beta \int (K * w + E) \cdot y \theta_R g_R^2 \, dy.$$

For $\tau \geq \tau_0$, the Nash estimate and the similarity change give $\|K * w\|_\infty \leq C_{s_0} m$. The foreign contribution is at most $\beta \lambda_0 \int |y|^2 \theta_R g_R^2$, while Young's inequality gives

$$\beta \|K * w\|_\infty \int |y| \theta_R g_R^2 \leq \frac{c_\beta}{4} \int |y|^2 \theta_R g_R^2 + C_\beta \|K * w\|_\infty^2 \|g_R\|_2^2.$$

Choose λ_0 so that $\beta \lambda_0 \leq c_\beta/4$. It follows that, with constants independent of R ,

$$\frac{1}{2} \frac{d}{d\tau} \|g_R\|_2^2 \leq -\|\nabla g_R\|_2^2 - \frac{c_\beta}{2} \int |y|^2 \theta_R g_R^2 + C(1+m^2) \|g_R\|_2^2.$$

Gronwall gives a uniform bound on every fixed finite time interval. Monotone convergence and weak lower semicontinuity show that F_β is finite there, and that $\nabla(e^{\beta|y|^2/2}w)$ and $|y|e^{\beta|y|^2/2}w$ are square integrable in space-time. Thus D_β is locally integrable in time. The transformed equation has its time derivative in the dual of the space with norm $\|g\|_2 + \|\nabla g\|_2 + \||y|g\|_2$: the linear drift and resulting quadratic potential define bounded bilinear forms on that space. The resulting energy identity supplies the continuous weighted- L^2 representative. Cutoffs in the now integrable full weight justify the identities and energy inequalities below. This preliminary argument uses no oscillator gap for the bounded weights.

Writing $g = e^{\beta|y|^2/2}w$ gives the exact identities

$$\begin{aligned} \langle w, \mathcal{L}w \rangle_\beta &= \frac{1}{2} \|g\|_2^2 - \|\nabla g\|_2^2 - \beta(1/2 - \beta) \||y|g\|_2^2, \\ D_\beta &= \|\nabla g\|_2^2 + \beta^2 \||y|g\|_2^2 + 2\beta \|g\|_2^2. \end{aligned}$$

For $a_\beta = \sqrt{\beta(1/2 - \beta)}$, completing squares in $\sum_j \|\partial_j g + a_\beta y_j g\|_2^2$ proves

$$\mathcal{H}_\beta := \|\nabla g\|_2^2 + \beta(1/2 - \beta) \||y|g\|_2^2 \geq 2a_\beta \|g\|_2^2 = \frac{\sqrt{6}}{5} \|g\|_2^2.$$

Moreover $D_\beta \leq 3\mathcal{H}_\beta$: one can first bound it by $\frac{3}{2}\mathcal{H}_\beta + \frac{3}{5}\|g\|_2^2$ and then use the last inequality. Splitting $\mathcal{H}_\beta$ into $\frac{3}{50}\mathcal{H}_\beta + \frac{47}{50}\mathcal{H}_\beta$ yields

$$\langle w, \mathcal{L}w \rangle_\beta \leq -\frac{1}{50} D_\beta + \frac{1}{25} F_\beta^2. \quad (23)$$

Here $\frac{1}{2} - \frac{47}{50} \frac{\sqrt{6}}{5} < \frac{1}{25}$.

The Hardy–Littlewood–Sobolev estimate and Gaussian weighted Hölder give

$$\|K * w\|_4 \leq C \|w\|_{4/3} \leq C_\beta F_\beta.$$

The two-dimensional Gagliardo–Nirenberg inequality gives

$$\|g\|_4 \leq C \|g\|_2^{1/2} \|\nabla g\|_2^{1/2}.$$

Since $\||y|g\|_2 + \|\nabla g\|_2 \leq C_\beta \sqrt{D_\beta}$, self-advection contributes at most

$$C F_\beta^{3/2} D_\beta^{3/4} \leq C_\beta F_\beta D_\beta$$

to the weighted energy. The foreign contribution is $\beta \int E \cdot y e^{\beta|y|^2} w^2$, bounded by $C_\beta \lambda D_\beta$. Thus, while F_β is sufficiently small,

$$\frac{1}{2} (F_\beta^2)' \leq -\frac{1}{100} D_\beta + \frac{1}{25} F_\beta^2.$$

The initial weighted norm and a continuity bootstrap using (20) prove

$$F_\beta(\tau) \leq C m e^{\tau/25}, \quad \int_{\log(1+s_0)}^{\log S} D_\beta(\tau) \, d\tau \leq C m^2 S^{2/25}. \quad (24)$$

Let P be angular averaging, $G = Pw$, and $h = (I - P)w$. Every ordinary moment of h of degree at most three vanishes: angular averaging removes radial moments, and fourfold symmetry removes all other moments in these degrees. In

$$X = L^2(e^{|y|^2/4} \, dy), \quad H = \|h\|_X, \quad D_h = \|\nabla h\|_X^2,$$

one consequently has

$$D_h \geq 3H^2. \quad (25)$$

Indeed $\mathcal{L}$ is self-adjoint in X , and its Hermite basis is $\partial^\alpha \mathcal{G}$, with $\mathcal{G} = (4\pi)^{-1} e^{-|y|^2/4}$ and eigenvalue $-|\alpha|/2$. Orthogonality to degrees at most three is equivalent to the stated ordinary-moment cancellations, because the corresponding dual Hermite polynomials span these polynomial degrees. Expansion in this basis gives $\langle h, \mathcal{L}h \rangle_X \leq -2H^2$; since the left side is $-D_h + H^2$, (25) follows. This elementary Hermite expansion also follows from the tensor one-dimensional harmonic-oscillator factorization.

Angular averaging is orthogonal for all the radial weights used here, and commutes with $\mathcal{L}$. Writing $V_a = K * a$, the angular equation is

$$h_\tau = \mathcal{L}h - (I - P) \operatorname{div}[V_G h + V_h G + V_h h + EG + Eh].$$

The radial field V_G is tangential, so its weighted energy contribution is zero. The h - h and E - h contributions are bounded by CHD_h and $C\lambda D_h$. Put $D_G = \|\nabla G\|_X^2 \leq D_\beta$. Weighted integration by parts, unweighted HLS, and GN give

$$|\langle h, -\operatorname{div}(V_h G) \rangle_X| \leq CH F_\beta^{1/2} D_G^{1/4} D_h^{1/2} \leq \eta D_h + C_\eta H^2 F_\beta D_G^{1/2}.$$

The strict Gaussian margin $\beta - 1/4 = 1/20$ absorbs cubic multiplication:

$$\left\| e^{|y|^2/8} EG \right\|_2 \leq Ca(\tau) F_\beta.$$

Consequently the E - G contribution is bounded by $\eta D_h + C_\eta a(\tau)^2 F_\beta^2$. Take η and the bootstrap constants sufficiently small. Using (25) yields

$$(H^2)' \leq -\frac{19}{5} H^2 + C F_\beta D_G^{1/2} H^2 + Ca(\tau)^2 F_\beta^2.$$

By (24) and Cauchy–Schwarz,

$$\int F_\beta D_G^{1/2} \, d\tau \leq C m^2 S^{2/25} \sqrt{1 + \log S} \leq C c^2.$$

Variation of constants therefore gives

$$H(\tau) \leq C m \left[e^{-19\tau/10} + d e^{51\tau/25} \right]. \quad (26)$$

Factors depending on the fixed initial time are absorbed in C .

Finally $B_4(f(s)) = e^{2\tau} B_4(w(\tau))$. Harmonicity cancels diffusion, and differentiating $e^{2\tau}$ cancels the dilation term. Radial self-advection vanishes. HLS and Gaussian weighted Hölder show that

$$\left| \frac{d}{d\tau} B_4(f(e^\tau - 1)) \right| \leq C e^{2\tau} [F_\beta H + a(\tau) F_\beta].$$

Substitution of (24) and (26), followed by integration, gives a first term $Cm^2 S^{1/25+1/10}(1+\log S)$ and remaining terms at most $Cm d S^{102/25}$, using $m \leq 1$. Together with (22), this proves the proposition.

The bounded-weight argument and the subsequent uniform estimates also justify passage from smooth data and drifts to the stated parabolic limits. No initial L^∞ norm enters a constant. $\square$

3.2 Comparison with the actual local foreign field

Proposition 3.2 (Local harmonic field and a rough small leak). *Let $\nu, r, T, R_* > 0$, $R_* \geq 4r$, and $p > 2$. Let $f_0 \in L^1 \cap L^p$ be real, fourfold symmetric, supported in B_r , and satisfy $\int f_0 = 0$ and $B_4(f_0) = 0$. Put $M = \|f_0\|_1$, and let f solve on $[0, T]$*

$$f_t + (K * f + b) \cdot \nabla f = \nu \Delta f.$$

After removal of a solid rotation, suppose b is divergence free, has fourfold covariance, and admits, on B_{2R_} , the decomposition*

$$\begin{aligned} b &= b_h + e, & |b(t, x)| &\leq L|x| \quad \text{globally,} \\ |b_h(t, x)| &\leq C_3|x|^3, & |Db_h(t, x)| &\leq C_3|x|^2, \\ |e(t, x)| &\leq \epsilon L|x|. \end{aligned} \tag{27}$$

where b_h is divergence free with fourfold covariance and $0 \leq \epsilon \leq 1$. All bounds hold on $0 \leq t \leq T$. Its streamfunction is defined on B_{2R_} . Assume*

$$C_3 R_*^2 T + \epsilon L T \leq \frac{1}{4}, \quad C(M/\nu) \sqrt{\nu T} \leq R_*/8, \quad \sqrt{\nu T} \leq c R_*. \tag{28}$$

Take a fixed radial cutoff of that streamfunction, equal to one in B_{R_} and zero outside B_{2R_*} , and denote its velocity by $\tilde{b}$. Let g solve the same self-consistent equation as f , with foreign field $\tilde{b}$ and the same initial datum. Set $\ell = C C_3 R_*^2$. If M/ν and $\ell(r^2/\nu + T)$ are sufficiently small, then*

$$\begin{aligned} \sup_{t \leq T} |B_4(f(t)) - B_4(g(t))| &\leq C M R_*^4 e^{CLT} \left[\epsilon L (T + r \sqrt{T/\nu}) \right. \\ &\quad \left. + (1 + L R_* \sqrt{T/\nu}) e^{-c R_*^2/(\nu T)} \right]. \end{aligned} \tag{29}$$

No derivative bound on the leak e is required. The assertion holds for smooth drifts and for parabolic limits obtained by approximation with the stated bounds.

Proof. Choose the streamfunction ψ_h zero at the origin. Integration along radial segments gives $|\psi_h(x)| \leq C C_3 |x|^4$. A streamfunction cutoff gives

$$|\tilde{b}| \leq C C_3 |x|^3, \quad |\tilde{b}| \leq \ell |x|, \quad \|D\tilde{b}\|_\infty \leq \ell.$$

It is divergence free, has fourfold covariance, and equals b_h in B_{R_*} . A second useful bound is $|\tilde{b}(x)| \leq CL|x|$: from $\epsilon \leq 1$ and (27), $|b_h| \leq 2L|x|$ on B_{2R_*} , so $|\psi_h| \leq CL|x|^2$ there, and cutoff derivatives preserve this linear growth. Thus this bound does not require $C_3 R_*^2 \leq L$.

We first verify the velocity-gradient estimate needed for comparison. Normalize $r = \nu = 1$ and write $m = M/\nu$; the cutoff growth and Lipschitz constant are normalized in the same way. Nash gives

$$\|g(t)\|_\infty \leq Cm/t, \quad \|u_g(t)\|_\infty \leq Cm/\sqrt{t}, \quad u_g = K * g.$$

For $M_k(t) = \int |x|^k |g(t, x)| \, dx$, Kato's inequality gives

$$\begin{aligned} M_2' &\leq 4m + Cmt^{-1/2}\sqrt{mM_2} + 2\ell M_2, \\ M_4' &\leq 16M_2 + Cmt^{-1/2}m^{1/4}M_4^{3/4} + 4\ell M_4. \end{aligned}$$

Since $M_2(0), M_4(0) \leq m$, a continuity bootstrap gives

$$M_2(t) \leq Cm(1+t), \quad M_4(t) \leq Cm(1+t)^2$$

when m and $\ell(1+T)$ are small. On these trial bounds the integrated self terms are respectively $O(m^2(1+t))$ and $O(m^2(1+t)^2)$; $\ell T \ll 1$ absorbs the foreign terms. Hence

$$\|xg(t)\|_1 \leq Cm\sqrt{1+t}, \quad \|xg(t)\|_4 \leq Cmt^{-3/4}\sqrt{1+t}.$$

The second follows from $\|xg\|_4^4 \leq \|g\|_\infty^3 M_4$.

With H_t the heat kernel, scaling and the smooth kernels' exterior decay give

$$\begin{aligned} \|\nabla K * H_t\|_\infty &\leq Ct^{-1}, \\ \|\nabla K * \operatorname{div} H_t\|_\infty &\leq Ct^{-3/2}, \quad \|\nabla K * \operatorname{div} H_t\|_1 \leq Ct^{-1/2}, \quad \|\nabla K * \operatorname{div} H_t\|_{4/3} \leq Ct^{-3/4}. \end{aligned}$$

For the nonlinear differentiated kernel the tail is $O(|x|^{-3})$, so the last three norms are finite before scaling. The self flux $u_g g$ has L^1 norm $Cm^2 t^{-1/2}$ and L^∞ norm $Cm^2 t^{-3/2}$. Split the heat-mild time integral at $t/2$, using the kernel's L^∞ norm first and its L^1 norm second; its contribution to ∇u_g is Cm^2/t . The foreign flux has L^1 norm $C\ell m\sqrt{1+t}$ and L^4 norm $C\ell m t^{-3/4}\sqrt{1+t}$. Use the kernel's L^∞ norm first and its $L^{4/3}$ norm second. This contribution is

$$C\ell m t^{-1/2}\sqrt{1+t} \leq Cm/t,$$

because $\ell(1+T)$ is small. The initial heat contribution is Cm/t . Undoing normalization proves

$$\|\nabla u_g(t)\|_\infty \leq C(M/\nu)/t. \quad (30)$$

Put $\delta = f - g$, $v = K * \delta$, and $F = (b - \tilde{b})f$. Then

$$\delta_t - \nu \Delta \delta + (u_f + \tilde{b}) \cdot \nabla \delta + v \cdot \nabla g = -\operatorname{div} F.$$

Let ψ be a streamfunction of δ . Testing in H^{-1} gives

$$\frac{1}{2}(\|v\|_2^2)' + \nu \|\delta\|_2^2 \leq (\|\nabla u_g\|_\infty + \|D\tilde{b}\|_\infty) \|v\|_2^2 + \|v\|_2 \|F\|_2. \quad (31)$$

Here $v \cdot \nabla \psi = 0$. Thus the term $v \cdot \nabla g$ vanishes when paired with ψ , and the same identity replaces $u_f = u_g + v$ by u_g in the other self transport term. Integration by parts bounds that term by $\|\nabla u_g\|_\infty \|v\|_2^2$. The identical calculation applies to $\tilde{b}$. The circulation of δ is zero, and its polynomial moments are finite, so $v \in L^2$ and these tests are justified by approximation.

Both solutions converge to their common datum in $L^1 \cap L^p$ as $t \downarrow 0$. Interpolation, using $p > 2$, therefore gives $\|\delta(t)\|_2 \rightarrow 0$. Their first absolute moments are uniformly bounded near zero by the Kato moment estimates and the linear-growth bounds. Since $\int \delta = 0$,

$$|\widehat{\delta}(t, \xi)| \leq |\xi| \int |x| |\delta(t, x)| \, dx.$$

In the Fourier identity $\|v\|_2^2 = C \int |\xi|^{-2} |\widehat{\delta}(t, \xi)|^2 \, d\xi$, the part with $|\xi| < a$ is $O(a^2)$ uniformly near zero. The part with $|\xi| \geq a$ is at most $Ca^{-2} \|\delta(t)\|_2^2$. First taking $t \downarrow 0$ and then $a \downarrow 0$ proves $\|v(t)\|_2 \rightarrow 0$.

The true coefficient in (31) is integrable at time zero, although the upper bound (30) is singular there. Indeed $g_0 \in L^p$, $p > 2$, gives from the same mild formula

$$\|\nabla u_g(t)\|_\infty \leq C_0(t^{-1/p} + t^{1/2-1/p}), \quad 0 < t \leq 1.$$

The constant may depend on the actual initial L^p norm and on the bounded cutoff drift. The kernel norms are $Ct^{-1/p}$ and $Ct^{-1/2-1/p}$ in $L^{p'}$; the latter time singularity is integrable, and $(u_g + \widetilde{b})g$ has bounded L^p norm near zero by contraction and the initial L^1 – L^p Biot–Savart bound. Together with the initial trace just proved, this starts the integrating factor in (31) from $v(0) = 0$. Only after writing its source integral do we use (30) on $[s, t]$, obtaining

$$\|v(t)\|_2 \leq \int_0^t (t/s)^{CM/\nu} e^{\ell(t-s)} \|F(s)\|_2 \, ds. \quad (32)$$

Common smooth approximations have bounded initial L^p norms, so no arbitrary initial term at a positive time is discarded in this argument.

The localization estimate, applied at each time $s \leq T$, gives

$$\int_{|x| \geq R_*} |x|^2 |f(s, x)| \, dx \leq CM e^{CLs} R_*^2 e^{-cR_*^2/(\nu s)}.$$

The cutoff field and the actual field have global growth $CL|x|$. Combining this tail estimate with Nash yields

$$\|(b - \widetilde{b})f \mathbf{1}_{|x| \geq R_*}\|_2 \leq CLM \frac{R_*}{\sqrt{\nu s}} e^{CLs} e^{-cR_*^2/(\nu s)}.$$

Inside B_{R_*} the difference of drifts is exactly e . A weighted Kato estimate gives $\int |x|^2 |f(s)| \leq CM(r^2 + \nu s)e^{CLs}$. Consequently

$$\|ef \mathbf{1}_{|x| < R_*}\|_2 \leq C\epsilon LM \sqrt{\frac{r^2 + \nu s}{\nu s}} e^{CLs}.$$

Substitute these bounds into (32). Since $CM/\nu < 1/4$, integration of $s^{-1/2-CM/\nu}$ is legitimate, and yields, uniformly for $t \leq T$,

$$\|v(t)\|_2 \leq CM e^{CLT} \left[\epsilon L(T + r\sqrt{T/\nu}) + LR_* \sqrt{T/\nu} e^{-cR_*^2/(\nu T)} \right]. \quad (33)$$

The bounded factor $e^{\ell T}$ is absorbed into C .

For a radial cutoff χ_* equal to one in B_{R_*} and zero outside B_{2R_*} ,

$$\|\nabla(\chi_* \bar{z}^4)\|_2 \leq CR_*^4, \quad \left| \int \chi_* \bar{z}^4 \delta \right| \leq CR_*^4 \|v\|_2.$$

Both f and g satisfy the localized exit estimate, with the margins (28). Their exterior absolute fourth moments are at most $CMR_*^4 e^{CLT} e^{-cR_*^2/(\nu T)}$. Combining this with (33) proves (29). $\square$

3.3 Physical normalization, the available power margin, and scope

Set

$$m = M/\nu, \quad d = CC_3r^4/\nu, \quad S = 2 + \nu T/r^2.$$

To supply the full time interval required by Proposition 3.1, extend the cutoff comparison field $\tilde{b}$ by zero on $(T, T + r^2/\nu]$ and continue its solution g there. This extension preserves divergence, fourfold covariance, and the cubic and linear growth bounds. It is a bounded measurable-in-time, spatially Lipschitz drift and is covered by the parabolic approximation formulation. On the original interval the normalized weak-field coefficient is $\ell_{\text{norm}} = (r^2/\nu)\ell$ and satisfies

$$(1 + s)\ell_{\text{norm}}(s) \leq \ell(r^2/\nu + T),$$

and it is zero afterwards. Consequently the same smallness hypothesis supplies the normalized drift through $s = S - 1$; the comparison on $[0, T]$ is unchanged. The space-time change $x = ry$, $t = r^2s/\nu$, and $f(t, x) = \nu r^{-2}F(s, y)$ gives these quantities exactly; moreover $B_4(f) = \nu r^4 B_4(F)$. Applying Proposition 3.1 to the cutoff-field solution and Proposition 3.2 to the actual solution gives

$$\begin{aligned} \sup_{t \leq T} |B_4(f(t))| &\leq CMr^4 \left[\frac{M}{\nu} S^{7/50} (1 + \log S) + \frac{C_3 r^4}{\nu} S^{102/25} \right] \\ &\quad + CMR_*^4 e^{CLT} \left[\epsilon L(T + r\sqrt{T/\nu}) + (1 + LR_*\sqrt{T/\nu}) e^{-cR_*^2/(\nu T)} \right]. \end{aligned} \quad (34)$$

This requires (20) and the small normalized linear-growth bound of the cutoff field in addition to the local hypotheses. No cutoff has been applied to the actual nonlinear solution.

For this geometric cascade budget fix β and $C_H > 0$, assume $0 < \delta \leq 1$ and $1 \leq H \leq C_H \log(1/q)$, and take a receiver (A_i, R_i) with

$$r = qR_i, \quad M \leq C\beta A_i q^2 R_i^2, \quad \kappa = \frac{\nu}{A_i R_i^2} = q^{3/2}, \quad T = H/A_i, \quad R_* = \kappa^{1/4} R_i.$$

Suppose the actual outer-colour estimates furnish

$$L \leq CA_i, \quad C_3 \leq C\delta A_i/R_i^2, \quad \epsilon \leq Ce^{-c/(\kappa H)}.$$

Then $m = O(\beta q^{1/2})$, $d = O(\delta q^{5/2})$, and $S = O(q^{-1/2}H)$. Dividing the first line of (34) by $A_i R_i^6$ gives

$$O\left(\beta^2 q^{643/100} H^{7/50} (1 + \log S) + \beta \delta q^{323/50} H^{102/25}\right). \quad (35)$$

The powers are 6.43 and 6.46. The localization terms are smaller than every q -power for logarithmic H , since

$$\frac{R_*^2}{\nu T} = \frac{1}{\sqrt{\kappa}H}, \quad LT = O(H), \quad C_3 R_*^2 T = O(\delta \sqrt{\kappa}H).$$

The remaining tail margins follow from

$$\begin{aligned} \ell(r^2/\nu + T) &= O\left(\delta[q^{5/4} + q^{3/4}H]\right), \\ \frac{(M/\nu)\sqrt{\nu T}}{R_*} &= O(\beta q^{7/8}\sqrt{H}), \quad \frac{\sqrt{\nu T}}{R_*} = q^{3/8}\sqrt{H}, \\ mS^{1/25}\sqrt{1 + \log S} &= O\left(\beta q^{12/25} H^{1/25} \sqrt{1 + \log S}\right). \end{aligned}$$

Also $r/R_* = q^{5/8}$ and $\epsilon LT = O(He^{-c/(q^{3/2}H)})$. All these quantities tend to zero uniformly in the stated horizon range. Writing $L_q = \log(1/q)$, the two terms of (35) are bounded by

$$C\beta^2 q^{643/100} L_q^{57/50} + C\beta\delta q^{323/50} L_q^{102/25} = o(q^6),$$

uniformly for $0 < \delta \leq 1$ and $1 \leq H \leq C_H L_q$. A fixed prefactor in the support radius changes only the constants.

The actual outer-core argument supplies the required harmonic-plus-leak hypothesis conditionally on the density bound

$$\sup_{t \leq T} \|(I - P)w_l(t)\|_1 \leq \delta A_l R_l^2, \quad l \leq i.$$

Its harmonic exterior multipole series can be differentiated to give $|Db_h(x)| \leq CC_3|x|^2$ throughout B_{2R_*} when this disk lies strictly inside the constant-core probe region; the exponentially small core density defect contributes the rough leak. When these inputs are obtained from Proposition 1.3, its core margins (8) must also hold. Thus the present section closes full-window nonlinear inner-tail suppression under this explicit outer-density input. It does not prove that input for the active colour and older seeds, a signed outward transfer, or distinct vanishing-viscosity accumulation paths. The zero initial fourth moment used here is the one of the prescribed initial inner stack: termwise integration is justified by $\sum_l A_l R_l^6 < \infty$. A restart from a later state retains its actual fourth moment. If only $B_4(f_0) = 0$ is dropped while all other hypotheses of Proposition 3.1 remain in force, the same proof adds $|B_4(f_0)|$ to (21). A noncompact restarted state additionally requires a separate Gaussian-space entry estimate.

Remark 3.3 (Selected-cutoff scope). The displayed $q^{6.43}$ and $q^{6.46}$ budget uses $\kappa_i = q^{3/2}$ at the selected cutoff level. At later, larger receivers the same physical viscosity has a smaller normalized value; their aggregate of smaller colours includes the intended active child. The budget is not an erasure estimate for that entire aggregate at every later stage. After a source leaves its small cap neighbourhood, its constant-core probe on a canonical source window requires a separate comparison; it is not a consequence of the small-density hypothesis over that post-cap window.