

1 An isolated nonlinear active stage for a fixed smooth profile

The result of this section concerns one isolated annulus. It replaces the discontinuous reference vorticity by one fixed smooth profile, explicitly. The smoothing length is chosen once and is independent of viscosity and of the small parameter below. In particular, the theorem does not assert the analogous nonlinear estimate for a family of sharp patch boundaries.

Theorem 1.1 (Isolated smooth activation). *There exist a real, radial, compactly supported C^∞ vorticity W_ϵ , with zero circulation, a number $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with $\sigma_\epsilon > 0$, and a nonzero complex number b_ϵ , with the following property. For every $2 < p < \infty$ and $K_* > 0$ there is $q_0 > 0$ such that the conclusions below hold for $0 < q < q_0$. Put*

$$L_q = \log(1/q), \quad \alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad \delta_q = L_q^{-4}, \quad H_q = \frac{1}{\sigma_\epsilon} \log \frac{\delta_q}{\alpha_q}.$$

There is a 2π -periodic family of smooth real seeds Φ_ϑ , supported in two fixed annular collars, such that

$$\int \Phi_\vartheta = 0, \quad B_4(\Phi_\vartheta) = 0, \quad \Phi_\vartheta(ix) = \Phi_\vartheta(x).$$

Let $\omega^{q,\vartheta}$ be the unforced Navier–Stokes solution with viscosity κ_q and initial vorticity $W_\epsilon + \alpha_q \Phi_\vartheta$. Then, uniformly in $\vartheta \in \mathbb{R}$,

$$\sup_{0 \leq t \leq H_q} \|\omega^{q,\vartheta}(t) - e^{\kappa_q t \Delta} W_\epsilon\|_{L^1 \cap L^p} \leq C \delta_q, \quad (1)$$

$$\int_0^{H_q} \|K * (\omega^{q,\vartheta}(t) - e^{\kappa_q t \Delta} W_\epsilon)\|_\infty dt \leq C \delta_q, \quad (2)$$

$$B_4(\omega^{q,\vartheta}(H_q)) = \delta_q e^{i(\vartheta + \tau_\epsilon H_q)} b_\epsilon + o(\delta_q). \quad (3)$$

Here $\|f\|_{L^1 \cap L^p} = \|f\|_1 + \|f\|_p$; all fourth absolute moments in (3) are finite. In particular, (3) is a statement about the whole moment, rather than only a compactly tested moment. The constants depend on the fixed profile, p and K_ , but not on q or ϑ .*

The positive cap δ_q will eventually be fixed by choosing one q . Its logarithmic decrease is useful for the simultaneous condition $H_q \sqrt{\delta_q} \rightarrow 0$. The theorem concerns a family of isolated initial data and does not claim nonuniqueness of any fixed-data vanishing-viscosity limit.

1.1 The fixed profile and its positive angular spectrum

Fix $\rho^2 = 3/5$ and choose a nonnegative, radial, radially decreasing $\chi \in C_c^\infty(B_1)$ with integral one. Define

$$\chi_\epsilon(x) = \epsilon^{-2} \chi(x/\epsilon), \quad W_\epsilon = \chi_\epsilon * (-\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}).$$

The convolution is smooth, radial, supported in $B_{1+\epsilon}$ and has zero integral. It is constant on $B_{\rho-\epsilon}$. Choose $\epsilon_* < (1 - \rho)/4$ and subsequently $0 < \epsilon < \epsilon_*$. Its radial derivative has the form

$$W'_\epsilon(r) = D_{\rho,\epsilon}(r) - \rho^2 D_{1,\epsilon}(r), \quad D_{a,\epsilon} \geq 0, \quad \text{supp } D_{a,\epsilon} \subset [a - \epsilon, a + \epsilon], \quad \int_0^\infty D_{a,\epsilon}(r) dr = 1. \quad (4)$$

To verify the sign, express the radial decreasing mollifier by its superlevel disks. The area of intersection of two disks is nonincreasing in their center separation: along each line parallel to their displacement, the overlap of the two interval sections is nonincreasing. Layer-cake

integration gives radial monotonicity of the disk convolution. Its value is one at the origin and zero at infinity, giving the displayed derivative mass. The support assertion follows because the convolution is exactly constant at points whose distance to the disk boundary exceeds ϵ .

Set

$$\Omega_\epsilon(r) = r^{-2} \int_0^r s W_\epsilon(s) ds.$$

On the two derivative collars, respectively,

$$\Omega_\epsilon(r) = \Omega_0(r_i) + O(\epsilon), \quad (r_1, r_2) = (\rho, 1), \quad (\Omega_0(r_1), \Omega_0(r_2)) = (-1/5, 0). \quad (5)$$

Indeed disk translation changes its L^1 norm by at most a constant times the translation length. Thus $\|W_\epsilon - W_0\|_1 \leq C\epsilon$, where $W_0 = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}$. The radial circulation formula, on radii bounded away from zero, gives $|\Omega_\epsilon - \Omega_0| \leq C\epsilon$. The piecewise smooth Ω_0 is Lipschitz there. Direct integration gives the interface values.

For a positive angular mode $m \geq 1$ use the convention $f(r)e^{im\theta}$. Put

$$h_m(r, s) = \left(\frac{\min(r, s)}{\max(r, s)} \right)^m, \quad \psi_f(r) = -\frac{1}{2m} \int_0^\infty h_m(r, s) f(s) s ds.$$

The function $\psi_f e^{im\theta}$ solves $\Delta\psi = f e^{im\theta}$, is regular at the origin and decays at infinity. This follows either by solving the radial second-order equation on each side of $r = s$, or by checking the derivative jump $1/s$ of the displayed Green kernel. Since $u_r = -im\psi_f/r$, the frozen Euler linearization at W_ϵ is

$$\mathcal{L}_m f = -im\Omega_\epsilon f + im\psi_f W'_\epsilon/r. \quad (6)$$

For $m = 0$ the linearization is zero: two radial vorticities have tangential velocities and do not advect one another. Negative sectors are the complex conjugates of the positive sectors. From this point onward all angular sectors belong to $4\mathbb{Z}$. No spectral assertion is made for other symmetry classes.

Lemma 1.2 (The fixed smooth unstable pair). *One can choose a fixed sufficiently small $\epsilon > 0$ and $0 < \eta < \sigma_\epsilon/4$ so that in the quarter-turn symmetry class: the $m = 4$ generator has one algebraically simple eigenvalue $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with positive real part greater than η ; its growing radial eigenfunction f_ϵ is smooth and supported in the two derivative collars; and all other positive-real-part spectral values, except the conjugate growing eigenvalue in $m = -4$, have real part at most η . Moreover*

$$\lambda_\epsilon \longrightarrow \rho^5/2 + 3i/10 \quad \text{as } \epsilon \downarrow 0.$$

For each finite set of sectors in $4\mathbb{Z}$ and each finite radial Sobolev order k , the corresponding propagator has the decomposition

$$e^{t\mathcal{L}} = e^{\lambda_\epsilon t} P_+ + e^{\overline{\lambda_\epsilon} t} P_- + \mathcal{R}_k(t), \quad \|\mathcal{R}_k(t)\|_{H^k \rightarrow H^k} \leq C_k e^{2\eta t}. \quad (7)$$

The growing terms are included only if their sectors are present.

Proof. First consider any eigenvalue λ with $\Re\lambda > 0$. The eigenvalue equation forces $f = 0$ off the derivative collars and gives

$$f(r)r dr = g(r)W'_\epsilon(r) dr, \quad g(r) = \frac{im\psi_f(r)}{\lambda + im\Omega_\epsilon(r)}.$$

There is no division by W'_ϵ here. In the Hilbert space $L^2(|W'_\epsilon| dr)$ the equation for g is

$$\lambda g = -im\Omega_\epsilon g - \frac{i}{2} \int h_m(r, s) g(s) W'_\epsilon(s) ds.$$

Each same-collar block is skew-adjoint, since the sign of W'_ϵ is constant there and $h_m(r, s)$ is real and symmetric. Angular multiplication is also skew-adjoint. The two opposite-collar blocks have operator norms at most their Hilbert–Schmidt norms, bounded by

$$Ca_*^m, \quad a_* = (\rho + \epsilon_*)/(1 - \epsilon_*) < 1,$$

because their two weighted masses are 1 and ρ^2 and their kernels are bounded by a_*^m . Consequently

$$\Re \lambda \leq Ca_*^m. \quad (8)$$

This fixes a finite angular cutoff in any half-plane $\Re \lambda \geq \eta > 0$, independently of ϵ .

For one of the finitely many positive sectors, eliminate f to obtain the equation on the two collars

$$\psi(r) = -\frac{i}{2} \int h_m(r, s) \frac{W'_\epsilon(s) \psi(s)}{\lambda + im\Omega_\epsilon(s)} ds =: \mathcal{T}_\epsilon(\lambda) \psi(r). \quad (9)$$

Work first in the sup-norm space of continuous functions on the two collars. Let P evaluate at r_1, r_2 and let E extend a vector to constants on the two collars, so $PE = I$. The receiving-radius derivative of the kernel is bounded there for fixed m . Hence

$$\|(I - EP)\mathcal{T}_\epsilon(\lambda)\| \leq C\epsilon$$

uniformly on compact subsets of $\Re \lambda \geq \eta$. For $\psi = Ep + w$, $Pw = 0$, solve the w equation by an analytic Neumann series. The remaining two-dimensional equation has a determinant converging uniformly on these compact sets to the determinant of

$$p_i = \frac{i}{2} \sum_j \frac{c_j h_m(r_i, r_j)}{\lambda + im\Omega_0(r_j)} p_j, \quad (c_1, c_2) = (-1, \rho^2).$$

The sign follows from $W'_0 = -\sum c_j \delta_{r_j}$. Using interface variables $f(r)r dr = \sum c_i r_i^2 q_i \delta_{r_i}$, this equation is equivalent to the matrix eigenvalue problem

$$M_m q = \lambda q, \quad M_m = \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

For $m = 4$ its roots are $\pm \rho^5/2 + 3i/10$, both simple. For $m = 8, 12, \dots$ they are imaginary, since the real-matrix discriminant is

$$\left(\frac{2(m-4)}{5}\right)^2 - 4\rho^{2m+2} > 0.$$

At $m = 8$ the first term is $64/25$ and the second is $4(3/5)^9 < 1$; thereafter the first increases and the second decreases. Large $|\lambda|$ gives no omitted roots: once $|\lambda| > C_m$, the operator in (9) has norm less than one, uniformly in ϵ , because the denominator then has size comparable with $|\lambda|$. Uniform determinant convergence and Rouché's theorem therefore give one simple root near the growing $m = 4$ root and no other roots in $\Re \lambda \geq \eta$. Combined with (8), this proves the claimed positive spectrum after one fixed ϵ is selected. Take η smaller first if necessary, so that $2\eta < \sigma_\epsilon$.

We give the operator-domain and multiplicity details needed for the propagator claim. Choose a closed interval $I \Subset (0, \infty)$ containing both collars strictly in its interior. On $H^k(I)$ define ψ_f by the Green integral against the zero extension of f from I . This extension need not be globally H^k : multiplication of the potential by W'_ϵ restricts output strictly inside I . The radial elliptic equation, or direct differentiation of its Green kernel, gives two local derivatives. Thus the coupling in (6) is compact on $H^k(I)$ and the angular multiplier is bounded there. Its spectrum is imaginary: off the multiplier's value range the inverse is multiplication by $(\lambda + im\Omega_\epsilon)^{-1}$. The compact-operator Fredholm alternative consequently reduces all spectrum in $\Re\lambda > 0$ to eigenfunctions. The equation bootstraps such eigenfunctions to smoothness and forces their support into the collars, so the roots already found are exactly the positive spectrum on every $H^k(I)$.

To check algebraic simplicity independently of a sup-norm determinant, start on $X_0 = L^2(I)$. Write $Af = -im\Omega_\epsilon f$, $Cf = \psi_f|_{\text{collars}}$ and $B\psi = imW'_\epsilon\psi/r$. Then $\mathcal{L}_m = A + BC$ and $\mathcal{T}_\epsilon = C(\lambda - A)^{-1}B$. Off the imaginary multiplier range, the resolvent identity is

$$(\lambda - \mathcal{L}_m)^{-1} = (\lambda - A)^{-1} + (\lambda - A)^{-1}B(I - \mathcal{T}_\epsilon)^{-1}C(\lambda - A)^{-1}. \quad (10)$$

Here $C : X_0 \rightarrow C(\text{collars})$ is bounded by its elliptic gain, and $B : C(\text{collars}) \rightarrow X_0$ is bounded. The invertible complementary Schur factor and the simple two-by-two determinant zero imply a rank-one simple pole in (10). The residue is nonzero, since an eigenfunction exists. On $X_k = H^k(I)$ the resolvent equation and its radial elliptic bootstrap give, locally uniformly for λ off the imaginary multiplier range,

$$\|v\|_{H^k(I)} \leq C_k(\|f\|_{H^k(I)} + \|v\|_{L^2(I)}), \quad (\lambda - \mathcal{L}_m)v = f.$$

The X_k resolvent is the restriction of the X_0 resolvent; the simple pole has no higher-order singularity by this estimate and has the same nonzero smooth one-dimensional range. This proves algebraic simplicity on every space used below. The $m = -4$ conclusion follows by conjugation; $m = 0$ is the identity propagator.

After removing the growing Riesz projections, the remaining bounded generator has spectral bound at most η . Its exponential is bounded by $Ce^{2\eta t}$. For clarity this last fact does not assume a spectral growth theorem for unbounded generators. For a bounded operator, integrate $e^{tz}(z - \mathcal{L})^{-1}$ on a finite rectangular contour enclosing its remaining spectrum and having right edge $\Re z = 2\eta$. Expansion of the resolvent on a large circle and termwise integration gives the exponential series; contour deformation gives the rectangle formula. The resolvent is bounded on this fixed contour, proving the estimate. The finitely many sectors and derivative orders give (7). Coefficients initially smooth and supported in the collars remain so under this propagator: multiplication preserves that support and the coupling has smooth output flat at the collar endpoints. This proves the lemma. $\square$

1.2 An exactly neutral smooth seed

Normalize the growing radial eigenfunction f_ϵ so that its first collar mass tends to $c_1 r_1^2 = -\rho^2$. The finite-dimensional reduction then implies

$$f_\epsilon(r)r \, dr \rightharpoonup -\rho^2 \delta_\rho - i\rho^3 \delta_1.$$

Indeed its interface vector tends to $(1, -i\rho)$, the growing eigenvector of M_4 . Define

$$b_\epsilon = 2\pi \int_0^\infty r^5 f_\epsilon(r) \, dr. \quad (11)$$

The weak convergence on the two compact collars gives

$$b_\epsilon \longrightarrow 2\pi(-\rho^6 - i\rho^3) \neq 0.$$

Reduce the fixed ϵ if necessary. Since $\mathcal{L}_4$ is i times a real operator, $\overline{f_\epsilon}$ is an eigenfunction of $\mathcal{L}_4$ with eigenvalue $-\overline{\lambda_\epsilon} = -\sigma_\epsilon + i\tau_\epsilon$. Define the real seed

$$\Phi_\vartheta(r, \theta) = e^{i\vartheta} \left(f_\epsilon(r) - \frac{b_\epsilon}{\overline{b_\epsilon}} \overline{f_\epsilon(r)} \right) e^{4i\theta} + \text{complex conjugate}. \quad (12)$$

Its circulation is zero and its only initial sectors are ± 4 . For the convention $B_4 = \int \bar{z}^4 \omega$, angular integration gives $B_4(f(r)e^{4i\theta}) = 2\pi \int r^5 f(r) dr$. Therefore $B_4(\Phi_\vartheta) = 0$ exactly, while the growing coefficient is $e^{i\vartheta}$. The stable coefficient has modulus one.

1.3 The finite seed–viscosity expansion

For the rest of the proof abbreviate the fixed profile to W , and write $\sigma = \sigma_\epsilon$, $\tau = \tau_\epsilon$, $W_\kappa(t) = e^{\kappa t} W$ and $U_\kappa = K * W_\kappa$. The latter is an exact radial heat solution. Put $X = L^1 \cap L^p$ with the sum norm. Splitting the Biot–Savart convolution at radius one gives

$$\|K * g\|_\infty \leq C_p \|g\|_X. \quad (13)$$

On the near part use Hölder and $|x|^{-1} \in L^{p'}(B_1)$, which holds because $p' < 2$. On the far part use its bounded kernel and the L^1 norm. This estimate requires no zero-circulation assumption.

The frozen full linear operator is $\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W$. Its sectors are those in (6). Choose integers $M, N \geq 1$, to be fixed below, and construct

$$f_{\text{app}} = \sum_{j=1}^M \sum_{\ell=0}^N \alpha^j \kappa^\ell f_{j,\ell}, \quad \omega_{\text{app}} = W_\kappa + f_{\text{app}}. \quad (14)$$

Set $f_{1,0}(0) = \Phi_\vartheta$ and all other initial coefficients zero. With $W_r(t) = t^r \Delta^r W / r!$, solve the triangular recursion

$$\begin{aligned} (\partial_t - \mathcal{L})f_{j,\ell} &= \mathbf{1}_{\ell \geq 1} \Delta f_{j,\ell-1} - \sum_{r=1}^{\ell} [(K * W_r) \cdot \nabla f_{j,\ell-r} + (K * f_{j,\ell-r}) \cdot \nabla W_r] \\ &\quad - \sum_{\substack{j_1+j_2=j, j_1, j_2 \geq 1 \\ \ell_1+\ell_2=\ell}} (K * f_{j_1,\ell_1}) \cdot \nabla f_{j_2,\ell_2}. \end{aligned} \quad (15)$$

Every right-hand side uses earlier coefficients in the order $j + \ell$. They are smooth and supported in the fixed collars: Laplacians preserve the support of smooth flat functions; $\Delta^r W$, $r \geq 1$, has that support; and every advective factor differentiates such a supported function. The linear source is supported where W' is supported. Each coefficient has zero circulation, because each forcing is a Laplacian or divergence of an integrable vector field. Its frequencies belong to $4\mathbb{Z}$ and satisfy $|m| \leq 4j$.

Lemma 1.3 (Coefficient bounds and seeded residual). *For every finite derivative order k , the finitely many coefficients in (15) satisfy*

$$\|f_{j,0}(t)\|_{H^k} \leq C_{j,k} e^{j\sigma t}, \quad (16)$$

$$\|f_{j,\ell}(t)\|_{H^k} \leq C_{j,\ell,k} (1+t)^D e^{j\sigma t}, \quad \ell > 0. \quad (17)$$

There is a finite D , enlarged as needed, such that if $a(t) = \alpha e^{\sigma t} \leq \delta_0 \leq 1$ on $[0, H]$ and $\kappa(1+H)^D \leq c$, the residual

$$R_{\text{app}} = \partial_t \omega_{\text{app}} + (K * \omega_{\text{app}}) \cdot \nabla \omega_{\text{app}} - \kappa \Delta \omega_{\text{app}}$$

satisfies

$$\|R_{\text{app}}(t)\|_X \leq C[a(t)^{M+1} + \kappa^{N+1}(1+t)^D a(t)]. \quad (18)$$

All constants are uniform in ϑ .

Proof. The first coefficient is the exact growing-plus-decaying solution from (12). For $j \geq 2$, $\ell = 0$, induction and Sobolev multiplication bound its forcing by $Ce^{j\sigma t}$. Convolution against the growing exponent σ , or the remaining exponent $2\eta < \sigma$, is bounded by a constant times $e^{j\sigma t}$, because $j\sigma - \sigma \geq \sigma$. This proves (16) with no polynomial factor. For positive ℓ the same induction applies, with finite polynomial losses from $W_r(t)$ and possible resonance with the first growing mode. This proves (17). Derivative loss causes no infinite iteration: the coefficient set is finite and the profile is smooth in every order. At each induction step apply the preceding coefficient estimates in a sufficiently higher fixed order and the propagator lemma in the requested order. Compact support and zero circulation also control $K * f$ and its needed derivatives, including low frequencies. More explicitly $|\hat{f}(\xi)| \leq C|\xi|$ near zero, from its zero mass and finite first absolute moment, while the high-frequency multiplier has order minus one.

Expand the radial heat base only in its couplings to f_{app} . The integral Taylor formula and heat contraction in L^1, L^p give

$$G_N(t) := W_\kappa(t) - \sum_{r=0}^N \kappa^r W_r(t), \quad \|G_N(t)\|_X + \|\nabla G_N(t)\|_X \leq C\kappa^{N+1}t^{N+1}.$$

Its residual couplings obey

$$\|(K * G_N) \cdot \nabla f_{\text{app}} + (K * f_{\text{app}}) \cdot \nabla G_N\|_X \leq C\kappa^{N+1}(1+t)^D a(t)$$

by (13) and the coefficient bounds. These are the only noncompact factors. Every other coefficient is supported in one fixed bounded set, where its high Sobolev norms control the required L^1, L^p product norms. The recursion cancels exactly all monomials with $1 \leq j \leq M$, $0 \leq \ell \leq N$. Terms of seed order $j > M$ are bounded by Ca^{M+1} : the pure-seed bounds contain no time polynomial, and each positive-viscosity polynomial is absorbed after enlarging D and using $\kappa(1+H)^D \leq c$. Terms of viscosity order $\ell > N$, including the diffusion of $f_{j,N}$, have the second bound in (18); $a^j \leq a$ for $a \leq 1$ absorbs their higher seed powers. Crucially the exact radial heat base is part of (14), so no residual term without a seed factor occurs. Finally rotations of (12) change only unit complex phases; all bounds are uniform in ϑ . $\square$

1.4 Nonlinear closure in the actual vorticity norm

Let ω^κ be the NS solution from $W + \alpha\Phi_\vartheta$ and put $\zeta = \omega^\kappa - \omega_{\text{app}}$. Its exact equation, with zero initial data, is

$$\partial_t \zeta + (K * \omega^\kappa) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta) \cdot \nabla \omega_{\text{app}} - R_{\text{app}}. \quad (19)$$

Kato's inequality gives its L^1 estimate, and multiplication by $|\zeta|^{p-2}\zeta$ gives its L^p estimate; transport disappears by divergence freedom and diffusion is nonpositive in both inequalities. Using (13) and adding gives

$$\frac{d^+}{dt} \|\zeta\|_X \leq C_p \|\nabla \omega_{\text{app}}\|_X \|\zeta\|_X + \|R_{\text{app}}\|_X. \quad (20)$$

These inequalities follow first with smooth convex regularizations of the norm and compact cutoffs, then by their limits. All solutions here are smooth at positive time, have finite X norm, and are continuous in that norm at time zero, so the integrated inequality starts at zero. No derivative bound on the actual NS vorticity enters its coefficient.

We choose parameters without circular dependence. Fix

$$C_1 = C_p(\|\nabla W\|_X + 1), \quad r_1 = (C_1/\sigma - 1)_+,$$

before choosing expansion orders. Choose M with $(M+1)\sigma > C_1$ and N with

$$\frac{3}{2}(N+1) > K_* r_1. \quad (21)$$

Construct the finite coefficients, then fix a cap threshold δ_0 using the pure-seed coefficients alone and reduce the positive-viscosity threshold so that $\|\nabla f_{\text{app}}\|_X \leq 1$ whenever $a \leq \delta_0$ and $\kappa(1+H)^D$ is sufficiently small. This is possible by the finite coefficient bounds: make the pure-seed gradient at most $1/2$ by the cap choice, and the finitely many positive-viscosity contributions at most $1/2$ by the later viscosity threshold. Thus a later increase of N need not reduce a previously fixed pure-seed cap. The exact heat gradient obeys $\|\nabla W_\kappa\|_X \leq \|\nabla W\|_X$, so C_1 is the already fixed exponent in (20). The coefficient constants affect only the later cap threshold; they do not change the previous choice of M, N .

Convolving (18) against $e^{C_1(t-s)}$ gives the all-time estimate

$$\|\zeta(t)\|_X \leq C a(t)^{M+1} + C \kappa^{N+1} (1+t)^{D+1} a(t) e^{(C_1-\sigma)t}. \quad (22)$$

The first integral has denominator $(M+1)\sigma - C_1 > 0$. For the second use a bound by $e^{\max(C_1, \sigma)t}$ and an additional polynomial factor if the two exponents coincide. At $H = \sigma^{-1} \log(\delta/\alpha)$ this yields

$$\|\zeta(H)\|_X \leq C \delta^{M+1} + C \kappa^{N+1} (1+H)^{D+1} \delta (\delta/\alpha)^{r_1}. \quad (23)$$

For the theorem's parameters, (21) implies that the second bound is $o(\delta_q)$: a strict positive power of q beats every fixed power of $\log(1/q)$. Also $\kappa_q(1+H_q)^D \rightarrow 0$, $\delta_q \rightarrow 0$, and $\alpha_q/\delta_q \rightarrow 0$. The coefficient estimates and (22) therefore prove (1), including uniformity through the entire interval. They also give the actual nonradial density bound

$$\|(I - P_{\text{rad}})\omega^\kappa(t)\|_1 \leq C a(t) + C a(t)^{M+1} + C \kappa^{N+1} (1+t)^{D+1} a(t) e^{(C_1-\sigma)t}. \quad (24)$$

Indeed radial averaging contracts L^1 , and the heat base is radial. This estimate concerns the actual density; it is not inferred from a contour coordinate.

Similarly (13), the coefficient bounds and (22) imply

$$\int_0^H \|K * (\omega^\kappa - W_\kappa)\|_\infty dt \leq C \delta + C \delta^{M+1} + C \kappa^{N+1} (1+H)^{D+1} \delta (\delta/\alpha)^{r_1}. \quad (25)$$

To integrate the approximate terms use $\int_0^H a(t)^j dt \leq \delta^j/(j\sigma)$. The error terms grow at a fixed positive exponential rate; their nondecreasing polynomial factor can be evaluated at H , so there is no additional factor H . This proves (2).

1.5 Whole-moment control and the cap

We verify the tails needed to remove the moment cutoff. Write $u^\kappa = U_\kappa + v^\kappa$, with U_κ tangential, and $B_H = \int_0^H \|v^\kappa\|_\infty dt$. By (25), $B_{H_q} = O(\delta_q)$. Let $d(x) = \sqrt{1 + |x|^2}$. Then $|\nabla d| \leq 1$, $\Delta d \leq 2$, and $U_\kappa \cdot \nabla d = 0$. Kato's inequality with $e^{\lambda d}$ gives, for every $\lambda > 0$,

$$\int e^{\lambda d(x)} |\omega^\kappa(t, x)| dx \leq m \exp\{\lambda d_0 + \lambda B_H + 2\kappa\lambda t + \kappa\lambda^2 t\}, \quad (26)$$

where $m = \|\omega^\kappa(0)\|_1$ and $d_0 = \sqrt{1 + (1 + \epsilon)^2}$. No initial exponential tail is presupposed beyond compact support. To justify the unbounded test use

$$F_L(d) = \frac{e^{\lambda d}}{1 + e^{\lambda d}/L}.$$

It has $F'_L \leq \lambda F_L$ and $F''_L \leq \lambda^2 F_L$; therefore $\Delta F_L(d) \leq (\lambda^2 + 2\lambda)F_L(d)$. For fixed L add a compact cutoff and let its radius tend to infinity. Finite mass and bounded drift make the cutoff errors vanish. Gronwall then gives (26) uniformly in L , and monotone convergence removes L . The drift is bounded here by (13) and the scalar L^1, L^p contractions.

Choose a fixed $R > 1 + \epsilon$ with $d_R - d_0 > 2c_* > 0$. For small q , $B_{H_q} < c_*/2$ and $\kappa_q H_q \rightarrow 0$. Exponential Markov and optimization of (26) give

$$\int_{|x|>s} |\omega^\kappa(t, x)| dx \leq m \exp\left\{-\frac{(d_s - d_0 - B_H - 2\kappa t)^2}{4\kappa t}\right\}, \quad s \geq R.$$

Integrating this mass bound in s gives

$$\sup_{0 \leq t \leq H_q} \int_{|x|>R} |x|^4 |\omega^{q, \vartheta}(t, x)| dx \leq C e^{-c/(\kappa_q H_q)}. \quad (27)$$

For example the weighted tail equals R^4 times the mass tail plus $4 \int_R^\infty s^3$ times the lower-radius mass tail. The displayed Gaussian estimate bounds both terms after reducing c . This also proves finiteness of the required absolute moments.

Let ξ be a smooth radial cutoff equal to one on B_R and supported in B_{R+1} . Every formal nonradial coefficient is supported in B_R ; the exact heat base pairs to zero with both $\bar{z}^4$ and $\xi \bar{z}^4$, since it is radial. At the cap the first coefficient has growing moment

$$\alpha e^{\lambda_\epsilon H} e^{i\vartheta} b_\epsilon = \delta e^{i(\vartheta + \tau_\epsilon H)} b_\epsilon$$

and stable moment of size $C\alpha^2/\delta$. The remaining pure-seed coefficients have compactly tested moment $O(\delta^2)$; positive-viscosity coefficients have moment $O(\kappa(1 + H)^D \delta)$. The actual error pairs against $\xi \bar{z}^4$ with bound $\|\xi \bar{z}^4\|_\infty \|\zeta(H)\|_1$. Using (23) and (27) proves

$$\begin{aligned} |B_4(\omega^\kappa(H)) - \delta e^{i(\vartheta + \tau_\epsilon H)} b_\epsilon| &\leq C\delta^2 + C\alpha^2/\delta + C\kappa(1 + H)^D \delta + C\delta^{M+1} \\ &\quad + C\kappa^{N+1}(1 + H)^{D+1} \delta(\delta/\alpha)^{r_1} + C e^{-c/(\kappa H)}. \end{aligned}$$

For $\delta = \delta_q$, every bound divided by δ_q tends to zero. This proves (3) and completes the theorem.

Remark 1.4 (Scaling and scope). For a physical colour $A\omega(x/R)$ and viscosity $\nu = \kappa_q A R^2$, the cap time is H_q/A , the density bound is $CAR^2\delta_q$, and the cap moment is $AR^6\delta_q e^{i(\vartheta + \tau_\epsilon H_q)} b_\epsilon + o(AR^6\delta_q)$. The integrated perturbation velocity bound is $CR\delta_q$. These normalized constants do not depend on the scale or amplitude. They apply to an isolated colour. A foreign-colour field introduces additional terms in the coefficient equations and the error equation; no estimate for those terms follows merely from this scaling. In particular, (24) cannot be asserted for an interacting stack by substituting its contour amplitudes into this isolated theorem.

2 The canonical ancient profile of the isolated smooth activation

We retain the fixed profile, eigenvalue and seeds constructed in Section 1. Abbreviate them to $W = W_\epsilon$, $\lambda = \sigma + i\tau = \lambda_\epsilon$, $b = b_\epsilon \neq 0$, and Φ_ϑ . In this section the cap $\delta > 0$ is fixed and sufficiently small. It does not decrease with the vanishing parameter. An additional Sobolev estimate permits identification of a unique isolated ancient profile. This identification requires no convergence of an infinite formal series.

Theorem 2.1 (The isolated ancient limit). *Fix an integer $k \geq 3$ and $2 < p < \infty$. There exists $\delta_* > 0$ such that, for $0 < \delta < \delta_*$, there is a real quarter-turn invariant Euler solution $\Omega_\delta(s)$ on $-\infty < s < \infty$, with zero circulation, having the following properties. For every fixed $K_* > 0$, put*

$$\alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad H_q = \sigma^{-1} \log(\delta/\alpha_q), \quad \vartheta_q = -\tau H_q.$$

Let ω^q be the isolated Navier–Stokes solution with viscosity κ_q and initial vorticity $W + \alpha_q \Phi_{\vartheta_q}$. For every $J < \infty$ and every $0 < \varepsilon < k$,

$$\omega^q(H_q + \cdot) \longrightarrow \Omega_\delta \quad \text{in } C([-J, 0]; L^1 \cap L^p) \text{ and in } C([-J, 0]; H^{k-\varepsilon}) \quad \text{as } q \downarrow 0. \quad (28)$$

The parameter δ_ and the limiting profile can be chosen independently of K_* ; the threshold for small q may depend on K_* . On $s \leq 0$, Ω_δ is supported in one fixed ball and has a uniform H^k bound. Its whole fourth harmonic moment satisfies*

$$B_4(\Omega_\delta(s)) = \delta b e^{\lambda s} + O(\delta^2 e^{2\sigma s}) \quad (s \leq 0). \quad (29)$$

For $s \geq 0$ it satisfies

$$|B_4(\Omega_\delta(s))| \leq C_\delta (1 + s)^4. \quad (30)$$

These assertions concern the isolated annulus.

2.1 A high Sobolev error estimate without derivative loss

Fix once and for all a radius $R > 1 + \epsilon$ with a positive gap in $d(x) = \sqrt{1 + |x|^2}$, for example $R = 3 + \epsilon$. This radius and the corresponding compact-support norm constants are fixed before the energy exponent or expansion orders. The later cap smallness condition will make the integrated perturbation drift smaller than this fixed gap. For this proof let

$$\|g\|_{Z_k} = \|g\|_1 + \|g\|_{H^k}.$$

The finite expansion from (14) and (15) can be constructed in every finite derivative order. In particular, the residual estimate holds in Z_k . A useful version separates its pure-seed constant from constants depending on the viscosity order:

$$\|R_{\text{app}}(t)\|_{Z_k} \leq [C_M + C_{M,N}\kappa(1+t)^D]a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^D a(t), \quad a(t) = \alpha e^{\sigma t} \leq 1. \quad (31)$$

Here C_M is independent of N . To verify the additional norm and this separation, the pure-seed products are estimated by their compactly supported coefficient Sobolev bounds. Every other omitted seed-order product contains at least one positive power of κ , and its finite time polynomial is incorporated in D . The noncompact heat remainder products are

$$(K * G_N) \cdot \nabla f_{\text{app}} + (K * f_{\text{app}}) \cdot \nabla G_N.$$

The heat Taylor formula gives

$$\|K * G_N\|_{W^{k,\infty}} + \|\nabla G_N\|_1 + \|\nabla G_N\|_{H^k} \leq C_{k,N} \kappa^{N+1} t^{N+1}.$$

For the first term use heat Taylor on the smooth initial velocity $K * W$ and contraction of each bounded derivative. For the other terms use heat contraction on $\nabla \Delta^{N+1} W$ in L^1 and H^k . The coefficient bounds give $\|K * f_{\text{app}}\|_{W^{k,\infty}} + \|\nabla f_{\text{app}}\|_1 + \|\nabla f_{\text{app}}\|_{H^k} \leq Ca(t)$ after the positive-viscosity threshold is made small. Leibniz's rule proves the claimed seeded remainder in both norms.

Let $\zeta = \omega^\kappa - \omega_{\text{app}}$ and $u_{\text{app}} = K * \omega_{\text{app}}$. Rewriting (19) gives

$$\partial_t \zeta + u_{\text{app}} \cdot \nabla \zeta + (K * \zeta) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta) \cdot \nabla \omega_{\text{app}} - R_{\text{app}}. \quad (32)$$

We record the estimates that avoid a quasilinear derivative loss. Sobolev embedding and the near-far kernel split give, for $k \geq 3$,

$$\|K * g\|_\infty \leq C_k (\|g\|_1 + \|g\|_{H^k}), \quad (33)$$

$$\|\nabla K * g\|_{H^j} \leq C_j \|g\|_{H^j}, \quad j \geq 0. \quad (34)$$

The second is Plancherel's theorem for a bounded order-zero Fourier multiplier. In particular $\|\nabla K * g\|_\infty \leq C_k \|g\|_{H^k}$.

For each integer derivative order up to k , the undifferentiated transport term has zero energy pairing. The commutator with $(K * \zeta) \cdot \nabla$ is bounded by

$$C_k (\|\nabla K * \zeta\|_\infty \|\zeta\|_{H^k} + \|\nabla \zeta\|_\infty \|\nabla K * \zeta\|_{H^{k-1}}) \leq C_k \|\zeta\|_{H^k}^2.$$

This commutator inequality follows from Leibniz's rule: the endpoint terms put the first derivative in L^∞ and the last derivative in L^2 , and the intermediate terms follow by Sobolev interpolation in dimension two. The embeddings for both displayed endpoint terms hold because $k > 2$. No derivative beyond order k of ζ occurs. Similarly,

$$\|(K * \zeta) \cdot \nabla \omega_{\text{app}}\|_{H^k} \leq C \left(\|\nabla \omega_{\text{app}}\|_{H^k} + \sum_{j=1}^k \|\nabla^j \omega_{\text{app}}\|_\infty \right) \|\zeta\|_{Z_k}.$$

For the term where no derivative lands on $K * \zeta$, use (33); for every term with a positive number of such derivatives use (34) in L^2 and put the corresponding derivative of the smooth approximation in L^∞ . The H^k transport estimate with u_{app} uses only its fixed $W^{k,\infty}$ bounds. Its coefficients are bounded, independently of α, κ, H , by fixed norms of W plus one, once the approximate perturbation is made small in the finitely many required norms.

Together with Kato's L^1 inequality this gives, for $Y = \|\zeta\|_{Z_k}$,

$$\frac{d^+ Y}{dt} \leq C_k Y + C_k Y^2 + \|R_{\text{app}}\|_{Z_k}, \quad Y(0) = 0. \quad (35)$$

Diffusion has a nonpositive energy contribution. The constant C_k can be fixed before M, N : the pure-seed approximation is made small by choosing its cap after M , and each positive-viscosity coefficient is made small by choosing its viscosity threshold after N .

Set $E_k = 2C_k$, increasing it if necessary to dominate $C_p(\|\nabla W\|_{L^1 \cap L^p} + 1)$ and the compact-support $L^1 \cap L^p$ stability constant used below. Choose one finite M with

$$(M+1)\sigma > E_k. \quad (36)$$

After its pure-seed coefficients are known, choose $\delta_* > 0$ small enough for all their required norms and their pure-seed remainder. For each subsequently prescribed K_* choose N so that, with $r_k = (E_k/\sigma - 1)_+$,

$$\frac{3}{2}(N+1) > K_* r_k. \quad (37)$$

The N -dependent coefficients are controlled by decreasing the threshold for q , not by changing δ_* . In particular arrange $C_{M,N}\kappa(1+H)^D \leq 1$ in (31). On a bootstrap interval $Y \leq 1$, (35) now gives

$$Y(t) \leq C_M a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^{D+1}a(t)e^{(E_k-\sigma)_+t}. \quad (38)$$

At the cap, the second term is

$$C_{M,N}\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_k} = o(1)$$

in the prescribed polynomial window. The first is smaller than $1/2$ after reducing δ_* . Continuity contradicts a first bootstrap exit, so the estimate holds through the full cap interval. It also yields a uniform H^k bound for the actual vorticity. This bound is obtained from the actual error equation, rather than assumed from the formal approximation.

2.2 Finite homological coefficients and compactness

For a real angle θ define the rotation

$$(\mathcal{T}_\theta f)(r, \varphi) = f(r, \varphi + \theta/4), \quad \mathcal{L}_{\text{rot}} = \mathcal{L} - (\tau/4)\partial_\varphi.$$

It acts isometrically on every radial or full-plane Sobolev norm, commutes with the radial linearization, and respects Biot–Savart advection. The pure-seed coefficient $f_{j,0}$ with seed phase zero has a limit

$$e^{-j\sigma t}\mathcal{T}_{-\tau t}f_{j,0}(t) \longrightarrow F_j \quad \text{in every finite } H^m \text{ as } t \rightarrow \infty. \quad (39)$$

For $j = 1$ the decaying part is $O(e^{-2\sigma t})$ after this normalization, so

$$F_1 = f_\epsilon(r)e^{4i\varphi} + \text{complex conjugate}.$$

For $j \geq 2$, put $g_j = e^{-j\sigma t}\mathcal{T}_{-\tau t}f_{j,0}$. Its equation is

$$\partial_t g_j = (\mathcal{L}_{\text{rot}} - j\sigma)g_j - \sum_{j_1+j_2=j} (K * g_{j_1}) \cdot \nabla g_{j_2}.$$

On its finite set of sectors, the homogeneous propagator decays at least as $Ce^{-(j-1)\sigma t}$ by (7). Induction, the coefficient bounds, and dominated convergence in Duhamel's integral prove (39) and the finite homological relations

$$F_j = -(j\sigma - \mathcal{L}_{\text{rot}})^{-1} \sum_{j_1+j_2=j} (K * F_{j_1}) \cdot \nabla F_{j_2}, \quad 2 \leq j \leq M. \quad (40)$$

The inverse is the integral of the exponentially decaying propagator. All these functions are smooth and supported in the fixed collars. Only the finitely many sectors in $4\mathbb{Z}$ with $|m| \leq 4M$ are used. No all-mode Sobolev spectral-growth assertion is needed.

Define the fixed polynomial ancient profile

$$P_M(s) = W + \sum_{j=1}^M \delta^j e^{j\sigma s} \mathcal{T}_{\tau s} F_j, \quad s \leq 0. \quad (41)$$

For the phase $\vartheta_q = -\tau H_q$, (39) shows that every pure-seed term of the cap-shifted approximation converges to the corresponding term in (41). The positive-viscosity terms tend to zero because $\kappa_q(1 + H_q)^D \rightarrow 0$, while $e^{\kappa_q(H_q+s)\Delta}W \rightarrow W$. For each fixed $s \leq 0$, (38) gives the asymptotic error $C_M(\delta e^{\sigma s})^{M+1}$, since its viscosity remainder tends to zero by (37).

We justify actual compactness separately. The kernel bound and (38), integrated as in (25), give

$$\int_0^{H_q} \|K * (\omega^q(t) - W_{\kappa_q}(t))\|_\infty dt \leq C_M \delta + o(1).$$

Choose δ_* so this is less than the positive radial gap fixed before the expansion orders. The exponential Kato proof of (27), with this fixed cap and the already fixed radius R , gives

$$\sup_{0 \leq t \leq H_q} \int_{|x| > R} |x|^4 |\omega^q(t, x)| dx \leq C e^{-c/(\kappa_q H_q)}. \quad (42)$$

Initial velocities have uniformly bounded finite energy, since the initial vorticities are compactly supported and have zero circulation. The NS velocity energy inequality and the uniform vorticity H^k bound therefore bound the velocities in H^{k+1} . The equation bounds $\partial_t \omega^q$ uniformly in H^{k-2} on the cap interval.

On each shifted interval $[-J, 0]$, Rellich compactness and interpolation give local compactness in $C(H^{k-\varepsilon})$. The tail bound and the uniform H^k , hence L^∞ , bound give global L^2 tightness. Interpolation applied to a smooth exterior cutoff then gives global $H^{k-\varepsilon}$ tightness; the cutoff is chosen outside the same fixed B_R . Thus compactness holds globally in that norm. The mass tail gives global L^1 tightness, and local convergence plus uniform L^∞ bounds gives compactness in $C(L^1 \cap L^p)$ for every finite p . A diagonal subsequence over J gives an ancient Euler solution: the transport term passes by the stated strong convergence, and $\kappa_q \Delta \omega^q \rightarrow 0$ on every compact shifted interval. The solution is supported in B_R on $s \leq 0$, since the right side of (42) tends to zero.

By lower semicontinuity of the H^k norm and the strong L^1 convergence, every such limit satisfies, for all $s \leq 0$,

$$\|\Omega(s) - P_M(s)\|_{Z_k} \leq C_M(\delta e^{\sigma s})^{M+1}. \quad (43)$$

One may first take rational times; continuity in the weaker compactness norm and weak H^k continuity extend the bound to every time.

2.3 Uniqueness of the ancient subsequential limit

We use a norm in which no extra derivative of a comparison orbit is required. If Ω_1, Ω_2 are two ancient subsequential limits, their compact supports and uniform H^k bounds imply a uniform bound for $\|\nabla \Omega_j\|_{L^1 \cap L^p}$. Indeed compact support controls the L^1 norm by the L^2 derivative norm, and H^k , $k \geq 3$, controls every finite L^p derivative norm. Their perturbations from W are small in these norms after reducing δ_* , so the stability coefficient is bounded by the fixed constant dominated by E_k above.

For $D = \Omega_1 - \Omega_2$ the exact Euler equation is

$$\partial_s D + (K * \Omega_1) \cdot \nabla D = -(K * D) \cdot \nabla \Omega_2.$$

The same Kato and L^p argument as (20) gives, for $s_0 \leq s_1 \leq 0$,

$$\|D(s_1)\|_{L^1 \cap L^p} \leq e^{E_k(s_1 - s_0)} \|D(s_0)\|_{L^1 \cap L^p}.$$

Both limits have the same polynomial (41). Using (43) at $s_0 = -T$ yields

$$\|D(0)\|_{L^1 \cap L^p} \leq C_M \delta^{M+1} e^{-(M+1)\sigma - E_k T} \longrightarrow 0.$$

The same comparison with any fixed endpoint $s_1 \leq 0$ proves equality throughout the ancient interval. The strict inequality (36) uses one finite M only. Thus every subsequence has the same limit, proving the full convergence in (28). Repeating the argument for any different K_* gives the same limit; only its permissible viscosity expansion order and small- q threshold change.

For clarity, a direct H^k difference stability estimate would require an H^{k+1} bound on the comparison vorticity because it differentiates $(K * D) \cdot \nabla \Omega_2$ k times. The $L^1 \cap L^p$ comparison above deliberately avoids this extra hypothesis. The high Sobolev estimate is used for compactness and for the available first derivatives.

2.4 Forward extension and the whole-moment asymptotics

The ancient limit has a smooth enough, compactly supported initial state at zero. The classical smooth two-dimensional Euler construction extends it uniquely to all $s \geq 0$. We recall the bounds behind global continuation. Its transported vorticity has conserved L^1 and L^∞ norms. The near-far Biot-Savart split gives a time-independent velocity bound. Splitting the gradient kernel at a small radius and using its cancellation gives the logarithmic estimate

$$\|\nabla u\|_\infty \leq C_{\|\omega\|_1, \|\omega\|_\infty} [1 + \log(2 + \|u\|_{H^{k+1}})].$$

The near part uses the Hölder norm of ω supplied by H^k ; the intermediate annuli give the logarithm, and the far part uses L^1 . Choosing the small radius inversely proportional to a fixed positive power of $2 + \|u\|_{H^{k+1}}$ gives the displayed estimate. The velocity Sobolev energy inequality $\frac{d}{ds} \|u\|_{H^{k+1}} \leq C \|\nabla u\|_\infty \|u\|_{H^{k+1}}$ then prevents finite-time blowup. The local smooth solution is obtained by regularization and these same energy bounds; their difference energy bound gives uniqueness. Finite energy at zero follows from compact support and zero circulation, so these velocity norms are legitimate.

The bounded velocity transports its support into B_{R+Vs} . Consequently

$$\int |x|^4 |\Omega_\delta(s, x)| dx \leq \|\Omega_\delta(0)\|_1 (R + Vs)^4 \quad (s \geq 0),$$

which proves (30). On $s \leq 0$, its support is in the fixed B_R . Pairing (43) with $\bar{z}^4$ is therefore bounded by its L^1 error times R^4 . The first homological coefficient has moment b , rotation multiplies it by $e^{i\tau s}$, and every higher term is $O(\delta^2 e^{2\sigma s})$. This proves (29) and completes the theorem.

Corollary 2.2 (Every fixed source window, including after the cap). *For the isolated family in Theorem 2.1, every fixed $J < \infty$ satisfies*

$$\omega^q(H_q + \cdot) \longrightarrow \Omega_\delta \quad \text{in } C([-J, J]; L^1 \cap L^p), \quad \sup_{|s| \leq J} |B_4(\omega^q(H_q + s)) - B_4(\Omega_\delta(s))| \longrightarrow 0.$$

Consequently its source moment converges in every fixed weighted window $\int_{-S}^U e^{-as} |B_q(s) - B_\delta(s)| ds$ with $S, U < \infty$ and $a \in \mathbb{R}$.

Proof. The negative window follows from (28) and (42). On $0 \leq s \leq J$, let $\zeta_q(s) = \omega^q(H_q + s) - \Omega_\delta(s)$. Its exact equation is

$$\partial_s \zeta_q + (K * \omega^q(H_q + s)) \cdot \nabla \zeta_q = \kappa_q \Delta \zeta_q - (K * \zeta_q) \cdot \nabla \Omega_\delta + \kappa_q \Delta \Omega_\delta.$$

The global Euler extension has bounded H^k norm and compact support on this fixed interval. Since $k \geq 3$, its second derivatives belong to H^1 , which embeds in every finite L^p in dimension two; compact support also bounds their L^1 norms. Thus $\|\nabla \Omega_\delta\|_{L^1 \cap L^p} + \|\Delta \Omega_\delta\|_{L^1 \cap L^p} \leq C_J$. Kato's inequality, the L^p inequality and (13) give

$$\sup_{0 \leq s \leq J} \|\zeta_q(s)\|_{L^1 \cap L^p} \leq e^{C_J J} (\|\zeta_q(0)\|_{L^1 \cap L^p} + C_J \kappa_q) \longrightarrow 0.$$

No actual NS derivative bound is used in this comparison.

We remove the moment cutoff separately. Scalar norm contraction gives one uniform post-cap velocity bound V . Choose a fixed radius $R' > R + VJ + d$, also enclosing the Euler support on $[0, J]$, with $d > 0$. The positive post-cap propagator has a forward diffusion with $|X_s - y| \leq VJ + \sqrt{2\kappa_q}|B_s|$. For $|y| \leq R$, its terminal fourth moment outside $B_{R'}$ is at most $C_J e^{-c_J/\kappa_q}$ by Gaussian endpoint moments. For unrestricted y its full fourth moment is at most $C_J(1 + |y|^4)$. The positive domination of the signed solution by the propagation of $|\omega^q(H_q)|$ therefore gives

$$\sup_{0 \leq s \leq J} \int_{|x| > R'} |x|^4 |\omega^q(H_q + s, x)| dx \leq C_J e^{-c_J/\kappa_q} + C_J \int_{|y| > R} (1 + |y|^4) |\omega^q(H_q, y)| dy \longrightarrow 0$$

by (42). A compact radial cutoff of $\bar{z}^4$, the just-proved L^1 convergence, and this tail prove uniform whole-moment convergence. On a fixed weighted time window the weight is bounded, so its final assertion follows by integration. $\square$

Corollary 2.3 (A nonzero bilateral discounted moment). *Define*

$$\mathcal{F}_\delta(z) = \int_{-\infty}^{\infty} e^{-zs} B_4(\Omega_\delta(s)) ds.$$

It is analytic and is not identically zero on $0 < \Re z < \sigma$. More precisely it extends meromorphically to $0 < \Re z < 2\sigma$ as

$$\mathcal{F}_\delta(z) = \frac{\delta b}{\lambda - z} + \mathcal{G}_\delta(z), \tag{44}$$

where $\mathcal{G}_\delta$ is analytic on that larger strip. For every fixed real c , the exceptional set

$$\{\beta > 1 : \mathcal{F}_\delta((\sigma - ic)/\beta) = 0\}$$

is discrete in $(1, \infty)$. In particular one can select arbitrarily large β outside it, including for $c = 1/2$.

Proof. The negative-time estimate (29) and the positive-time polynomial bound give absolute, locally uniform convergence on $0 < \Re z < \sigma$, including after any fixed number of z derivatives. Subtract the leading negative-time exponential. Its integral is

$$\int_{-\infty}^0 \delta b e^{(\lambda - z)s} ds = \frac{\delta b}{\lambda - z}.$$

The remaining negative-time integral is analytic for $\Re z < 2\sigma$, and the positive-time integral is analytic for $\Re z > 0$. This proves (44). Its pole at $z = \lambda$ has nonzero coefficient, so the transform cannot be identically zero on the smaller strip. Its zeros there are isolated. The curve $\beta \mapsto (\sigma - ic)/\beta$ is nonconstant, lies in that strip for $\beta > 1$, and has no interior accumulation of its zeros. Any terminal interval of large β contains values outside the zero set, proving the last assertion. Accumulation as $\beta \rightarrow \infty$ is possible at the boundary $z = 0$ and is not excluded or needed. $\square$

Corollary 2.4 (Changing the canonical cap). *If $0 < \delta' < \delta < \delta_*$ and $s_0 = \sigma^{-1} \log(\delta'/\delta)$, then*

$$\Omega_{\delta'}(s) = \mathcal{T}_{-\tau s_0} \Omega_{\delta}(s + s_0) \quad (-\infty < s < \infty), \quad \mathcal{F}_{\delta'}(z) = e^{(z - i\tau)s_0} \mathcal{F}_{\delta}(z).$$

Thus the zero set of the transform is independent of the canonical cap. At $z = (\sigma - ic)/\beta$ its modulus scales by $(\delta'/\delta)^{1/\beta}$.

Proof. The time-translated and rotated orbit has the polynomial (41) with δ replaced by δ' , since $\delta e^{\sigma s_0} = \delta'$. It has the same order- $M + 1$ remainder. The ancient uniqueness argument therefore identifies it with $\Omega_{\delta'}$ on negative times, and forward Euler uniqueness extends the identity to every time. Rotation $\mathcal{T}_{\theta}$ multiplies B_4 by $e^{i\theta}$. Substituting $v = s + s_0$ into the absolutely convergent bilateral transform gives the stated exponential factor, which never vanishes. Its modulus at the displayed point is $e^{\sigma s_0/\beta}$. $\square$

Remark 2.5 (Scope). The ancient profile, its cap compactness and its nonzero transform have been proved for one isolated fixed smooth annulus. They can supply an isolated source profile and a possible discounted transfer coefficient. This corollary does not identify that coefficient in a coupled receiver, does not estimate the effect of all other colours, and does not construct distinct vanishing-viscosity accumulation points from one fixed datum.