

# 1 Weighted inactive estimates and a two-phase continuation

This section is conditional only on the local full-material flow, interaction, forced-endpoint, packet-transport, and signed-transfer statements listed below. In particular, the signed-transfer statement is not proved by an amplitude upper bound. It uses a source whose cap has already been completed. The deepest source is covered by the separate base-pulse lemma, rather than by the small-noise argument.

Fix  $\beta \in [4, 5]$ ,  $\sigma > 0$ , and

$$A_i = A_1 \beta^{i-1}, \quad R_i = R_1 q^{i-1}, \quad L = \log(1/q), \quad \varepsilon = q^6.$$

We always take  $L \geq 1$ . A receiving stage has current index  $j$ , incoming index  $k = j + 1$ , initial time  $T_k$ , and current time  $\theta = A_j(t - T_k)$ . Its deadline is

$$H = \frac{9}{\sigma} L.$$

For  $j = 1$ , every inactive maximum and inactive feedback below is defined to be zero. All constants are independent of  $j$ , the finite depth  $N$ , and sufficiently small  $q$ .

## The local inputs and the induction data

Let  $E = c^{1,\gamma}(S^1; \mathbb{R}^2)^2$ . In each colour's stationary material gauge and its own frame with the exact larger-colour solid rotation removed, write its perturbation as  $h_i$ , and put  $x_i = \|h_i\|_E$ . Rotations and the label translations used in these gauges are bounded uniformly in this norm. Fix a local neighbourhood of radius  $r_* > 0$ , sufficiently small that its inner domains contain  $B_{r_{\text{pl}}}$  in normalized coordinates for one fixed  $r_{\text{pl}} > 0$ . Choose this inner neighbourhood small enough that every isolated subperiod orbit starting there stays in a larger fixed comparison neighbourhood for  $0 \leq r \leq P$ . The constants below are taken on these fixed neighbourhoods, before choosing  $\delta$  or  $q$ .

The isolated linear flow satisfies, for  $s \geq 0$ ,

$$\|S(s)\| \leq M_\eta e^{(\sigma+\eta)s}, \quad \|S_c(s)\| \leq M(1+s), \quad \|M_c^n\| \leq M(1+n), \quad M_c = S_c(P). \quad (1)$$

The full local equation has a mild form with this linear flow, and its isolated nonlinear remainder has norm at most  $Cx_i^2$ . The full  $E$ -norm interaction coefficients, after removal of solid rotation, are bounded by

$$\begin{cases} CA_l q^{6(l-i)} x_l, & l > i, \\ CA_l q^{2(i-l)} x_l, & l < i. \end{cases} \quad (2)$$

Torus image interactions satisfy the corresponding bounds with an additional factor at most  $q^8$ ; a local self-image term is bounded by  $CA_i q^8 x_i$ . These are bounds for both material contours, including tangential components and the full  $C^{1,\gamma}$  norm.

The current geometric coordinates are  $(a, b)$ , with inverse chart and unstable graph satisfying

$$x_j \leq C_{\text{ch}}(|a| + \|b\|), \quad |a| + \|b\| \leq C_{\text{ch}} x_j, \quad \|g(a)\| \leq C_g |a|^2, \quad y = b - g(a). \quad (3)$$

The exact continuous moment equation, in any of the rotating frames needed here, is

$$a_\theta = \lambda a + N(a, b) + F_a(\theta), \quad \Re \lambda = \sigma, \quad |N(a, b)| \leq C_a(|a|^2 + |a| \|b\| + \|b\|^2). \quad (4)$$

The scalar forcing is bounded by the full material forcing norm.

We use the following form of the separately proved forced-endpoint lemma. For every  $r \in [0, P]$ , its raw and flattened maps satisfy

$$b(\theta + r) = S_c(r)b(\theta) + R_{b,r} + E_{b,r}, \quad \|R_{b,r}\| \leq C_E(|a(\theta)|^2 + |a(\theta)| \|b(\theta)\| + \|b(\theta)\|^2), \quad (5)$$

$$y(\theta + r) = S_c(r)y(\theta) + R_{y,r} + E_{y,r}, \quad \|R_{y,r}\| \leq C_E(|a(\theta)| \|y(\theta)\| + \|y(\theta)\|^2), \quad (6)$$

$$\|E_{b,r}\| + \|E_{y,r}\| \leq C_E \int_{\theta}^{\theta+r} \|F_j(v)\|_E dv. \quad (7)$$

The constants are uniform in  $r$ , and the bounds hold whenever the actual stopped trajectory and the bounded-time isolated comparisons remain in their respective fixed neighbourhoods. The graph is invariant for every isolated subperiod flow. Thus these statements include bounded interpolation and do not differentiate the unbounded label generator through  $g$ .

Let  $d(s)$  be the support radius of the aggregate packet  $l \geq k$ , divided by  $R_k$ , with  $s = A_k(t - T_k) = \beta\theta$ . While the packet is in the central plateaux and all larger colours are local, transport gives the upper Dini inequality

$$D^+ d(s) \leq C_v + C_r q^2 d(s)^3. \quad (8)$$

The packet starts with  $d(0) \leq d_0$ , where  $d_0$  is fixed. If  $d(\beta\theta) \leq D_0(1 + \beta\theta)$  and central separation holds, its full material forcing on the current receiver is at most  $C_p \varepsilon(1 + \theta)^4$ ; here and below  $C_p$  may depend on the fixed  $D_0$ . Its weighted forcing on the inactive system is at most  $C_p \varepsilon(1 + \theta)^4/D$ . These follow from conserved mass, fourfold cancellation, and separated exterior kernel derivatives.

Fix  $D > \beta$ , and define

$$w_i = (D\varepsilon)^{j-i}, \quad I_j(\theta) = \max_{i < j} \frac{x_i(\theta)}{w_i}.$$

The initial induction data, reindexed from the completed cap of  $k$ , are

$$I_j(0) \leq DK\delta\varepsilon, \quad x_j(0) \leq DK\delta\varepsilon, \quad d(0) \leq d_0. \quad (9)$$

All preceding completed stages have their clock, current-time prefix, support, and source-window conclusions. The separate base-pulse lemma supplies these data for the deepest source, uniformly in  $N$ .

Finally, the separately proved signed-transfer lemma is used in the following stopped-interval form. There are fixed  $\Theta_w > 0$  and  $c_s > 0$  such that, if the current receiver remains local with  $|a| < q^3$ , has the raw-centre and upper-amplitude bounds proved below, and satisfies the prefix and packet barriers, then

$$\left| e^{-\lambda\theta} a(\theta) \right| \geq c_s \varepsilon \quad (\Theta_w \leq \theta \text{ before its first } q^3 \text{ crossing}). \quad (10)$$

Here the source is already capped; its negative history is part of the preceding completed-stage data. In the first pulse it is supplied by the direct base-pulse comparison. No cap of the current recipient is an assumption of (10). Without this signed input, the statements below give conditional stability estimates but do not prove that a recipient crosses  $q^3$ .

## The inactive system

**Lemma 1** (Uniform weighted Volterra estimate). *Choose  $0 < \eta < \sigma/32$ . One may choose  $D$ , then a fixed  $\alpha_* < \sigma/2$ , and finally sufficiently small  $q$ , so that on every interval with*

$$I_j \leq 2K\delta, \quad x_i \leq r_* \ (i \leq j), \quad d(\beta\theta) \leq D_0(1 + \beta\theta),$$

*and central separation, there is a common exponent  $0 < \alpha \leq \alpha_*$  for which*

$$I_j(\theta) \leq C_w e^{\alpha\theta} I_j(0) + C_w \int_0^\theta e^{\alpha(\theta-v)} [x_j(v) + \varepsilon(1+v)^4] dv. \quad (11)$$

The normalized current feedback and its discounted integral satisfy

$$\|F_{\text{out}}(\theta)\|_E \leq C_f D q^8 I_j(\theta), \quad (12)$$

$$q^8 \int_0^\theta e^{-\sigma v} I_j(v) dv \leq C q^8 I_j(0) + C q^8 \varepsilon + C q^8 \int_0^\theta e^{-\sigma v} x_j(v) dv. \quad (13)$$

The constants do not depend on the number of inactive colours.

*Proof.* For an inactive receiver  $i < j$ , an inner inactive source  $l = i + d < j$  has weighted coefficient

$$\frac{A_l}{A_j} q^{6d} \frac{w_l}{w_i} = \frac{A_i}{A_j} (\beta/D)^d.$$

An outer source  $l = i - d$  has weighted coefficient

$$\frac{A_l}{A_j} q^{2d} \frac{w_l}{w_i} = \frac{A_i}{A_j} (D q^8 / \beta)^d.$$

Their row sums are bounded by

$$\frac{C}{\beta} \left( \frac{\beta}{D - \beta} + \frac{D q^8}{\beta - D q^8} \right).$$

The weighted self-image row is at most  $C q^8 / \beta$ . Since  $w_i \leq D \varepsilon$ , the local quadratic term divided by  $w_i$  is at most

$$C \frac{A_i}{A_j} w_i I_j^2 \leq \frac{2CDK\delta\varepsilon}{\beta} I_j.$$

Torus cross-image rows are smaller than the already displayed rows. The current source has coefficient  $q^{6(j-i)} / w_i = D^{-(j-i)} \leq 1/D$ . The packet row is bounded as stated in the packet input. Set

$$\nu = \frac{\sigma + \eta}{\beta}, \quad r(D, q) = \frac{C}{\beta} \left( \frac{\beta}{D - \beta} + \frac{D q^8}{\beta - D q^8} + q^8 + 2DK\delta\varepsilon \right).$$

Taking the maximum of the full mild equations, using  $A_i/A_j \leq 1/\beta$ , gives

$$I_j(\theta) \leq M_\eta e^{\nu\theta} I_j(0) + M_\eta \int_0^\theta e^{\nu(\theta-v)} \{r(D, q) I_j(v) + C[x_j(v) + \varepsilon(1+v)^4]/D\} dv.$$

This is a scalar Volterra inequality for a finite maximum; no summation over the number of rows occurs. Multiplying by  $e^{-\nu\theta}$  and applying integral Grönwall gives (11) with  $\alpha = \nu + M_\eta r(D, q)$  and a fixed  $C_w$ . Choose  $D$  so that  $\nu + M_\eta r(D, 0) < \alpha_* < \sigma/2$ ; then choose  $q$  so that  $D q^8 < \beta/2$  and  $\alpha \leq \alpha_*$ .

For feedback into  $j$ , the source  $i = j - d$  has coefficient

$$\frac{A_i}{A_j} q^{2d} w_i = (D q^8 / \beta)^d.$$

The geometric sum proves (12). To prove (13), insert (11) and use Tonelli's theorem for its nonnegative majorants. For any  $v$ ,

$$\int_v^\theta e^{-\sigma u} e^{\alpha(u-v)} du \leq \frac{e^{-\sigma v}}{\sigma - \alpha_*}.$$

The polynomial source has a finite exponentially discounted integral. This gives the asserted bound with constants uniform in  $\alpha$ .  $\square$

**Lemma 2** (Current-time prefix improvement). *Suppose that, at every current time  $\theta$  in a stopped stage,  $|a(\theta)| \leq \delta$ , and that for all  $v \leq \theta$*

$$x_j(v) \leq C_x |a(\theta)| e^{-\sigma_0(\theta-v)} + C_n \varepsilon (1+L)^{12}, \quad \sigma_0 = \frac{3\sigma}{4}. \quad (14)$$

Then

$$I_j(\theta) \leq C_{\text{pr}} |a(\theta)| + C_{\text{rem}} \varepsilon (1+L)^{12} e^{\alpha\theta}, \quad 0 \leq \theta \leq H. \quad (15)$$

Here  $C_{\text{pr}}$  is independent of  $B, C_n, K$ , provided  $C_x$  is fixed in advance. The remainder constant may depend on these fixed choices. Choosing  $K \geq 4C_{\text{pr}}$  and then sufficiently small  $q$  improves the stopped prefix  $I_j \leq 2K\delta$  to  $I_j \leq K\delta/2$ .

*Proof.* Insert (14) into (11). The above-noise convolution is at most

$$\frac{C_w C_x}{\sigma_0 - \alpha_*} |a(\theta)|.$$

The other terms, including  $I_j(0) \leq DK\delta\varepsilon$ , are bounded by  $C_{\text{rem}}\varepsilon(1+L)^{12}e^{\alpha\theta}$ . For the polynomial packet term use  $\alpha \geq \nu > 0$ , so that  $\int_0^\infty e^{-\alpha v}(1+v)^4 dv$  is uniformly bounded. This proves (15). Since

$$\varepsilon(1+L)^{12}e^{\alpha H} \leq q^{6-9\alpha_*/\sigma}(1+L)^{12} \longrightarrow 0,$$

choose  $q$  so that the remainder is at most  $K\delta/4$ . The amplitude term is at most  $K\delta/4$ . Hence  $I_j \leq K\delta/2$ , a strict half-size improvement even relative to the desired output bound  $K\delta$ .  $\square$

## The two continuation phases

**Proposition 1** (Uniform stopped-stage continuation). *Under the stated local inputs and induction data, one can choose fixed  $\delta, D, K, D_0, B, C_*, Y$  and then sufficiently small  $q$ , uniformly in  $j, N$ , such that the current receiver reaches its first cap  $|a| = \delta$  before  $H$ . Its first  $q^3$  crossing occurs by  $3L/\sigma + C$ ; the cap occurs by  $7L/\sigma + C$ . Up to the cap,*

$$I_j \leq K\delta/2, \quad d(\beta\theta) \leq \frac{D_0}{2}(1 + \beta\theta),$$

and all local and central-separation bounds improve strictly. At the cap, with elapsed time measured from the first  $q^3$  crossing,

$$\|y_{\text{cap}}\| \leq Cq^{6-8\eta/\sigma}(1+L)^6 = o(1). \quad (16)$$

Every earlier above-noise amplitude is bounded in terms of the amplitude at the current time, with rate  $3\sigma/4$ . The time from the first  $q^3$  crossing to the cap is also bounded below by  $cL$ , with a fixed  $c > 0$ . Consequently every fixed backward cap window lies inside this completed stage for small  $q$ . No estimate assumes a future cap of this recipient.

*Proof.* We give the choices and both first-exit arguments explicitly. Choose  $0 < \eta < \sigma/32$ , then  $D$  and  $\alpha_*$  as in Lemma 1. Fix  $C_x$  larger than twice the inverse chart constant and choose  $K \geq 4C_{\text{pr}}$ , also enlarging it if necessary for the supplied base-pulse bound. This choice of  $K$  is independent of the later cap size and fixed comparison windows. Shrink  $\delta$  so that its graph lies well inside the local neighbourhood, so that the moment remainder on  $b = g(a) + y$  has  $C\delta < \sigma/8$ , and so that the small coefficient in the flattened period-map estimate below is less than  $\eta P/2$ . Then fix the signed-transfer windows and base ancient time for this cap. Their fixed comparison constants may depend on these windows, but neither  $C_x$  nor  $K$  does.

Choose  $D_0 \geq 4(d_0 + C_v)$ . On the packet barrier  $d(s) \leq D_0(1 + s)$ , (8) implies

$$d(s) \leq d_0 + C_v s + \frac{C_r q^2 D_0^3}{4} [(1 + s)^4 - 1].$$

For  $s \leq \beta H$ , division by  $1 + s$  gives

$$\frac{d(s)}{1 + s} \leq d_0 + C_v + \frac{C_r q^2 D_0^3}{4} (1 + \beta H)^3 \leq \frac{D_0}{2} \quad (17)$$

for small  $q$ . Moreover choose  $q$  so that

$$q D_0 (1 + \beta H) < r_{\text{pl}}/2.$$

Thus the whole packet lies in half the prescribed central plateau radius of the current receiver, and hence in those of every larger local colour. The radius and central-separation barriers improve independently of the signed-transfer calculation.

On all the stopping intervals used below, (12), the packet bound, and the self-image estimate give

$$\|F_j(\theta)\|_E \leq C_p \varepsilon (1 + \theta)^4 + C_f D q^8 I_j(\theta) + C q^8 x_j(\theta) \leq F_* \varepsilon (1 + L)^4. \quad (18)$$

The fixed  $F_*$  exists because  $I_j \leq 2K\delta$ ,  $x_j \leq r_*$ , and  $q^8/\varepsilon = q^2 \rightarrow 0$ . It is a full material forcing bound.

**Phase I: from the inherited state to  $q^3$ .** Set  $a_* = C_* \varepsilon (1 + L)^{12}$ , with fixed  $C_* \geq 1$ . Choose  $B$  below, and stop at the first of the following events:

$$\begin{aligned} |a| &= q^3, \quad \|b\| = 2B\varepsilon(1 + L)^6, \quad I_j = 2K\delta, \\ d(\beta\theta) &= D_0(1 + \beta\theta), \quad \theta = H, \end{aligned}$$

loss of central separation or a local neighbourhood,

The initial data are strictly inside all non-amplitude barriers for small  $q$ , and  $|a(0)| < a_* < q^3$ .

Iterate (5) at period times. There are at most  $CL$  periods, and  $\sum_{m \leq CL} (1 + m) \leq CL^2$ . The stopped bounds and (18), including the final partial period using (5), give

$$\|b\| \leq C_b \varepsilon (1 + L)^6 + C_b q^3 L^2 (2B\varepsilon(1 + L)^6) + C_b L^2 (2B\varepsilon(1 + L)^6)^2. \quad (19)$$

Here the  $a^2$  term is at most  $q^6 = \varepsilon$ , and its convolution is absorbed in  $C_b \varepsilon (1 + L)^6$ . The inherited  $O(\varepsilon)$  centre state is absorbed there as well. Choose  $B \geq 4C_b$ , and then  $q$  so that the last two terms are at most  $B\varepsilon(1 + L)^6/2$ . This improves the centre barrier to

$$\|b(\theta)\| \leq B\varepsilon(1 + L)^6. \quad (20)$$

The choice of  $B$  is fixed before  $q$ .

The exact moment equation and (20) imply that, whenever  $a_* \leq |a| \leq q^3$ ,

$$\frac{|N(a, b)|}{|a|} \leq C(q^3 + B\varepsilon(1 + L)^6 + B^2\varepsilon/C_*), \quad \frac{|F_a|}{|a|} \leq (F_*/C_*)(1 + L)^{-8}.$$

For small  $q$ , their sum is at most  $\sigma/4$ . Since  $\Re \lambda = \sigma$ ,

$$\frac{d}{d\theta} |a| \geq \sigma_0 |a|, \quad \sigma_0 = 3\sigma/4, \quad a_* \leq |a| \leq q^3. \quad (21)$$

This also prevents an above-noise amplitude from returning below  $a_*$ . At each current time, previous above-noise amplitudes are at most the current amplitude times  $e^{-\sigma_0(\theta - v)}$ . Below noise, (3) and (20)

give  $x_j \leq C_n \varepsilon (1+L)^{12}$ . Above noise,  $\|b\|/|a| \leq (B/C_*)(1+L)^{-6}$ , so for small  $q$   $x_j \leq C_x |a|$ , with the fixed  $C_x$  chosen before  $B$ . Consequently (14) holds at every current time in Phase I. Lemma 2 improves the prefix to  $I_j \leq K\delta/2$ .

The current local norm is at most  $C_{\text{ch}}(q^3 + B\varepsilon(1+L)^6) < r_*/2$ , and every inactive norm is at most  $K\delta(D\varepsilon)^{j-i}/2 < r_*/2$ . These strict inequalities prevent local exit. Equation (17) prevents packet or central-separation exit. Thus the only remaining Phase I endpoints are its amplitude target or the deadline.

For completeness, the amplitude upper bound also follows before any signed estimate. Set  $U(\theta) = e^{-\sigma\theta} |a(\theta)|$ . By (13) and  $x_j \leq C_{\text{ch}}(|a| + B\varepsilon(1+L)^6)$ ,

$$q^8 \int_0^\theta e^{-\sigma v} I_j(v) dv \leq Cq^8 \varepsilon (1+L)^6 + Cq^8 \int_0^\theta U(v) dv.$$

The packet term in (18) has bounded discounted integral; the self-image term has bound  $Cq^8 \int U + Cq^8 \varepsilon (1+L)^6$ . Therefore

$$U(\theta) \leq C\varepsilon + C(q^3 + B\varepsilon(1+L)^6 + q^8) \int_0^\theta U(v) dv.$$

The additive  $b^2$  term is  $O(B^2 \varepsilon^2 (1+L)^{12}) = o(\varepsilon)$ . The displayed coefficient times  $H$  tends to zero, giving

$$|a(\theta)| \leq C_u \varepsilon e^{\sigma\theta}. \quad (22)$$

In particular a  $q^3$  crossing cannot occur before  $3L/\sigma - C$ . For small  $q$ , this is later than the fixed source-window endpoint  $\Theta_w$ . All other exits have already been excluded, so that source window is available without assuming a future cap.

Apply the supplied signed input (10) on this stopped interval. At

$$\theta_3 = \frac{3L + \log(2/c_s)}{\sigma}$$

it would give  $|a| \geq 2q^3$  if no earlier crossing occurred. For small  $q$ ,  $\theta_3 < H$ . Thus the first crossing  $\tau$  satisfies  $\tau \leq 3L/\sigma + C$ . At that crossing,

$$\|y(\tau)\| \leq B\varepsilon(1+L)^6 + C_g q^6 \leq (B + C_g)\varepsilon(1+L)^6.$$

**Phase II: from  $q^3$  to the fixed cap.** Restart at  $\theta = \tau$ , and put  $s = \theta - \tau$ . Choose the fixed  $Y$  below. Retain the same prefix, radius, separation, and local barriers, and stop at the first of

$$\begin{aligned} |a| &= \delta, & \|y\| &= 2Y\varepsilon(1+L)^6 e^{2\eta s}, \\ |a| &< q^3 e^{\sigma s}, & \underline{\sigma} &= 2\sigma/3, & \theta &= H. \end{aligned}$$

or a first exit through any retained barrier. The growth condition means a first downward violation for  $s > 0$ ; equality at  $s = 0$  does not stop the phase. Its strict improvement below shows that it holds on a nonempty initial interval and prevents a later downward exit.

Choose  $q$  so that

$$2Y\varepsilon(1+L)^6 e^{2\eta H} < \delta.$$

This is possible since  $6 - 18\eta/\sigma > 0$ . On the stopped interval,  $|a| \leq \delta$  and  $\|y\| \leq \delta$ . At period times counted from  $\tau$ , (6) therefore has remainder at most  $2C_E \delta \|y\|$ , and forcing error at most  $PC_E F_* \varepsilon (1+L)^4$ . Using  $\|M_c^n\| \leq M_\eta e^{\eta n P}$ , multiplication by  $e^{-\eta n P}$  gives a discrete Volterra inequality

$$e^{-\eta n P} \|y_n\| \leq C_\eta (\|y_0\| + \varepsilon(1+L)^4) + \kappa_\eta \delta \sum_{r=0}^{n-1} e^{-\eta r P} \|y_r\|,$$

with fixed  $\kappa_\eta$ . The earlier choice of  $\delta$  makes  $\kappa_\eta \delta \leq \eta P/2$ . Discrete Grönwall and the uniform subperiod endpoint estimate yield

$$\|y(s)\| \leq C_Y \varepsilon (1+L)^6 e^{3\eta s/2}.$$

Choose  $Y \geq C_Y$ , including its dependence on the fixed  $B + C_g$  and interpolation constants. This improves the centre barrier by a factor of at least two:

$$\|y(s)\| \leq Y \varepsilon (1+L)^6 e^{2\eta s}. \quad (23)$$

The initial centre state is strictly inside the enlarged barrier after increasing  $Y$  if needed.

On the weaker growth barrier,

$$\frac{\|y\|}{|a|} \leq 2Y q^3 (1+L)^6 e^{-(\sigma-2\eta)s} \leq 2Y q^3 (1+L)^6, \quad \frac{|F_a|}{|a|} \leq F_* q^3 (1+L)^4.$$

Since  $b = g(a) + y$ , (4) now gives

$$\frac{|N(a, b)|}{|a|} \leq C\delta + C_Y q^3 (1+L)^6.$$

The fixed small cap and then small  $q$  make the sum with the forcing ratio at most  $\sigma/4$ . Thus

$$\frac{d}{ds} |a| \geq \frac{3\sigma}{4} |a| > \sigma |a|. \quad (24)$$

There is also an upper bound from the same exact moment equation:

$$\frac{d}{ds} \log |a| \leq \sigma + C_{\text{rad}} \delta + \zeta(q), \quad \zeta(q) \leq C q^3 (1+L)^6 \longrightarrow 0. \quad (25)$$

Both estimates use only the stopped bounds at the current time. Equation (24) strictly improves the weaker growth barrier. More explicitly,  $e^{-\sigma s} |a(s)|$  has a strictly positive derivative as long as the stopped bounds hold; it starts at  $q^3$ , and cannot have a later downward first exit. This supplies a local growth interval at the restart time as well.

Below noise in Phase I the contour norm is  $O(\varepsilon(1+L)^{12})$ . Above noise in Phase I it is at most  $C_x |a|$ . In Phase II, (3),  $g(a) = O(a^2)$ , and the small  $y/a$  ratio give the same bound  $x_j \leq C_x |a|$ , after the fixed small-cap choice. The rates (21) and (24) concatenate across  $\tau$ . Consequently (14) holds at every current time throughout both phases. Lemma 2 again improves the retained prefix to  $K\delta/2$ . The packet bound improves by (17). Choose  $\delta$  initially and then  $q$  so that

$$C_{\text{ch}}(\delta + C_g \delta^2) + C_{\text{ch}} 2Y \varepsilon (1+L)^6 e^{2\eta H} < r_*/2.$$

The current local norm and all inactive local norms are therefore strictly inside their neighbourhoods. Central separation improves as before. Thus none of the non-amplitude barriers can end Phase II.

By (24), the cap must occur within

$$\frac{1}{\sigma_0} \log(\delta/q^3) = \frac{4}{\sigma} L + C$$

of  $\tau$ . Together with the Phase I clock, this is  $7L/\sigma + C < H$ , so the deadline cannot be the first endpoint. Conversely, if  $s_{\text{cap}}$  is the now established cap time, integrating (25) gives

$$s_{\text{cap}} \geq \frac{\log(\delta/q^3)}{\sigma + C_{\text{rad}} \delta + \zeta(q)} = \frac{3L + \log \delta}{\sigma + C_{\text{rad}} \delta + \zeta(q)} \geq cL$$

for small  $q$ , with, for example,  $c = 2/(\sigma + C_{\text{rad}}\delta + 1) > 0$ . Thus a prescribed finite window  $[-S, 0]$ , relative to this cap in the current colour's normalized time, lies after  $\tau$  for small  $q$ . This conclusion is made only after the cap has been proved; it is not used to obtain that cap. Finally (23) at that cap gives

$$\|y_{\text{cap}}\| \leq CYq^6(1+L)^6e^{2\eta(4L/\sigma+C)} \leq Cq^{6-8\eta/\sigma}(1+L)^6 = o(1),$$

which proves (16).

All choices precede the final choice of  $q$ . The only sums over colours were geometric row sums, and the only time sums had at most  $CL$  periods. Hence every conclusion is uniform in  $j, N$ .  $\square$

## Outputs for the next stage and the remaining proof inputs

At the completed cap of  $j$ , reindexing to the next receiver  $j-1$  is exact:

$$I_{j-1}(T_j) \leq D\varepsilon I_j(T_j) \leq DK\delta\varepsilon/2, \quad x_{j-1}(T_j) \leq DK\delta\varepsilon/2.$$

Thus the next inherited full contour norm is  $O(\varepsilon)$ , without an inherited power of  $L$ . The packet  $l \geq j+1$  has radius  $O(qLR_j)$ . Adding the newly capped colour  $j$ , whose local outer radius is uniformly bounded in units of  $R_j$ , gives a fixed initial radius bound for the next aggregate packet. The weaker reset (16), together with the supplied bounded-time forced comparison lemma, gives convergence to a rotated canonical cap on each fixed preceding and following source window. Its use for a following window requires the usual continuation bootstrap on that compact canonical trajectory and the central hole there; it does not assert a long-time shadowing estimate.

The fixed following window is available: for the next recipient the Phase I upper bound forces its first  $q^3$  crossing to occur only after  $3L/\sigma - C$ , which exceeds every previously fixed window for small  $q$ . The backward source window comes from the already completed stage: (25) ensures that it lies after that stage's first  $q^3$  crossing, and the cap reset and fixed-time inverse comparison control it there. For the deepest source, replace this backward argument by direct comparison from its prescribed ancient initial time  $-S_0$ .

The module does not replace the proofs of the full interaction estimates, forced endpoint and interpolation lemma, canonical compact-window comparison, or signed-transfer estimate with its negative-history bounds. Each has been identified above as an input at its exact point of use. In particular, applying the raw-centre bound to the deepest source at fixed ancient time would be false: that source starts with a fixed nonzero  $a$  and raw centre  $g(a)$ , rather than with an  $O(\varepsilon)$  state. The separate base-pulse lemma is essential.