1 Nonlinear inner-tail suppression through a receiver window

The estimate in this section is conditional on quantitative information about the actual foreign velocity. It closes a longer-time nonlinear subproblem than the isolated-tail and first-diffusion-time estimates. It does not establish the required nonradial-density bound for the active and older outer colours.

Write $J(x_1, x_2) = (-x_2, x_1)$, $z = x_1 + ix_2$, and

$$K(x) = \frac{Jx}{2\pi|x|^2}, \qquad B_4(f) = \int_{\mathbb{R}^2} \bar{z}^4 f(z)\, \mathrm{d}z.$$

A scalar is fourfold symmetric if $f(Jx) = f(x)$; a vector field has the corresponding covariance if $b(t, Jx) = Jb(t, x)$. Constants denoted by C can change from line to line.

1.1 A Gaussian-energy estimate for a globally weak foreign field

Proposition 1.1 (Weak foreign field through a finite window)**.** *There are universal constants $c, \lambda_0, C > 0$ with the following property. Let $p > 2$, and let $f_0 \in L^1 \cap L^p$ be real, fourfold symmetric, supported in B_1, and satisfy*

$$\int f_0 = 0, \qquad B_4(f_0) = 0, \qquad m = \|f_0\|_1 .$$

Suppose b is divergence free, has fourfold covariance, and, after removal of a time-dependent solid rotation, satisfies

$$|b(s, x)| \le \ell(s)|x|, \qquad |b(s, x)| \le d|x|^3, \qquad (1 + s)\ell(s) \le \lambda_0.$$

Let f solve

$$\partial_s f + (K * f + b) \cdot \nabla f = \Delta f.$$

For $S \ge 2$, assume the displayed hypotheses on $0 \le s \le S - 1$ and

$$mS^{1/25}\sqrt{1 + \log S} \le c. \tag{1}$$

Then

$$\sup_{0 \le s \le S-1} |B_4(f(s))| \le C\left[m^2 S^{7/50}(1 + \log S) + mdS^{102/25}\right]. \tag{2}$$

The constants are independent of a decomposition of the datum into smaller scales. The assertion holds for smooth drifts and for the parabolic limits obtained by approximation with the stated bounds.

Proof. Scalar contraction and the divergence-free Nash estimate give

$$\|f(s)\|_1 \le m, \qquad \|f(s)\|_\infty \le Cm/s, \qquad \|K * f(s)\|_\infty \le Cm/\sqrt{s}.$$

The last estimate follows by splitting Biot–Savart at a radius and optimizing. Take $s_0 = 1/100$. The positive scalar solution from $|f_0|$ dominates $|f|$. Its diffusion representation, after removing solid rotation, obeys

$$|X_s| \le e^{\lambda_0 s}\left[1 + Cm\sqrt{s} + \sqrt{2}\sup_{a \le s}|B_a|\right] \quad (0 \le s \le s_0).$$

The Gaussian exponential moment of the Brownian supremum is finite at this fixed small time. With $\beta = 3/10$ it follows that

$$\int e^{\beta|x|^2/(1+s_0)}|f(s_0,x)|\,\mathrm{d}x \le Cm.$$

Multiplication by the Nash L^∞ bound gives a weighted L^2 norm $O(m)$. Fixed polynomial absolute moments are $O(m)$ on $[0,s_0]$. The exact harmonic-moment equation and symmetrization of self-advection therefore yield

$$|B_4(f(s_0))| \le C(m^2 + md). \tag{3}$$

For clarity, the self contribution is bounded by $Cm\int|x|^2|f|$, because $|\bar z^3 - \bar y^3|/|z-y| \le C(|z|^2 + |y|^2)$; the foreign contribution is bounded by $d\int|x|^6|f|$. These bounds also prove the result directly if the final time is smaller than s_0.

For $s \ge s_0$ put

$$\tau = \log(1+s), \quad y = x/\sqrt{1+s}, \quad w(\tau,y) = (1+s)f(s,x), \quad E(\tau,y) = \sqrt{1+s}\,b(s,\sqrt{1+s}\,y).$$

Then

$$w_\tau = \mathcal{L}w - (K*w+E)\cdot\nabla w, \quad \mathcal{L} = \Delta + \tfrac{1}{2}y\cdot\nabla + 1,$$

with $|E| \le \lambda(\tau)|y|$, $\lambda \le \lambda_0$, and $|E| \le a(\tau)|y|^3$, where $a(\tau) = de^{2\tau}$. Let

$$F_\beta = \left\|e^{\beta|y|^2/2}w\right\|_2, \qquad D_\beta = \left\|e^{\beta|y|^2/2}\nabla w\right\|_2^2, \qquad \beta = 3/10.$$

Writing $g = e^{\beta|y|^2/2}w$ gives the exact identities

$$\langle w, \mathcal{L}w\rangle_\beta = \tfrac{1}{2}\|g\|_2^2 - \|\nabla g\|_2^2 - \beta(1/2-\beta)\,\||y|g\|_2^2,$$
$$D_\beta = \|\nabla g\|_2^2 + \beta^2\,\||y|g\|_2^2 + 2\beta\,\|g\|_2^2.$$

For $a_\beta = \sqrt{\beta(1/2-\beta)}$, completing squares in $\sum_j \|\partial_j g + a_\beta y_j g\|_2^2$ proves

$$\mathcal{H}_\beta := \|\nabla g\|_2^2 + \beta(1/2-\beta)\,\||y|g\|_2^2 \ge 2a_\beta\,\|g\|_2^2 = \frac{\sqrt{6}}{5}\,\|g\|_2^2.$$

Moreover $D_\beta \le 3\mathcal{H}_\beta$: one can first bound it by $\frac{3}{2}\mathcal{H}_\beta + \frac{3}{5}\|g\|_2^2$ and then use the last inequality. Splitting $\mathcal{H}_\beta$ into $\frac{3}{50}\mathcal{H}_\beta + \frac{47}{50}\mathcal{H}_\beta$ yields

$$\langle w, \mathcal{L}w\rangle_\beta \le -\tfrac{1}{50}D_\beta + \tfrac{1}{25}F_\beta^2. \tag{4}$$

Here $\frac{1}{2} - \frac{47}{50}\frac{\sqrt{6}}{5} < \frac{1}{25}$.

The Hardy–Littlewood–Sobolev estimate and Gaussian weighted Hölder give

$$\|K*w\|_4 \le C\,\|w\|_{4/3} \le C_\beta F_\beta.$$

The two-dimensional Gagliardo–Nirenberg inequality gives

$$\|g\|_4 \le C\,\|g\|_2^{1/2}\,\|\nabla g\|_2^{1/2}.$$

Since $\||y|g\|_2 + \|\nabla g\|_2 \le C_\beta\sqrt{D_\beta}$, self-advection contributes at most

$$CF_\beta^{3/2}D_\beta^{3/4} \le C_\beta F_\beta D_\beta$$

to the weighted energy. The foreign contribution is $\beta\int E\cdot y\, e^{\beta|y|^2}w^2$, bounded by $C_\beta\lambda D_\beta$. Thus, while F_β is sufficiently small,

$$\tfrac{1}{2}(F_\beta^2)' \le -\tfrac{1}{100}D_\beta + \tfrac{1}{25}F_\beta^2.$$

The initial weighted norm and a continuity bootstrap using (1) prove

$$F_\beta(\tau) \le Cme^{\tau/25}, \qquad \int_{\log(1+s_0)}^{\log S} D_\beta(\tau)\,\mathrm{d}\tau \le Cm^2S^{2/25}. \tag{5}$$

Let P be angular averaging, $G = Pw$, and $h = (I-P)w$. Every ordinary moment of h of degree at most three vanishes: angular averaging removes radial moments, and fourfold symmetry removes all other moments in these degrees. In

$$X = L^2(e^{|y|^2/4}\,\mathrm{d}y), \qquad H = \|h\|_X, \qquad D_h = \|\nabla h\|_X^2,$$

one consequently has

$$D_h \ge 3H^2. \tag{6}$$

Indeed $\mathcal{L}$ is self-adjoint in X, and its Hermite basis is $\partial^\alpha\mathcal{G}$, with $\mathcal{G} = (4\pi)^{-1}e^{-|y|^2/4}$ and eigenvalue $-|\alpha|/2$. Orthogonality to degrees at most three is equivalent to the stated ordinary-moment cancellations, because the corresponding dual Hermite polynomials span these polynomial degrees. Expansion in this basis gives $\langle h, \mathcal{L}h\rangle_X \le -2H^2$; since the left side is $-D_h + H^2$, (6) follows. This elementary Hermite expansion also follows from the tensor one-dimensional harmonic-oscillator factorization.

Angular averaging is orthogonal for all the radial weights used here, and commutes with $\mathcal{L}$. Writing $V_a = K * a$, the angular equation is

$$h_\tau = \mathcal{L}h - (I-P)\operatorname{div}[V_G h + V_h G + V_h h + EG + Eh].$$

The radial field V_G is tangential, so its weighted energy contribution is zero. The h–h and E–h contributions are bounded by CHD_h and $C\lambda D_h$. Put $D_G = \|\nabla G\|_X^2 \le D_\beta$. Weighted integration by parts, unweighted HLS, and GN give

$$|\langle h, -\operatorname{div}(V_h G)\rangle_X| \le CHF_\beta^{1/2}D_G^{1/4}D_h^{1/2} \le \eta D_h + C_\eta H^2 F_\beta D_G^{1/2}.$$

The strict Gaussian margin $\beta - 1/4 = 1/20$ absorbs cubic multiplication:

$$\left\|e^{|y|^2/8}EG\right\|_2 \le Ca(\tau)F_\beta.$$

Consequently the E–G contribution is bounded by $\eta D_h + C_\eta a(\tau)^2F_\beta^2$. Take η and the bootstrap constants sufficiently small. Using (6) yields

$$(H^2)' \le -\tfrac{19}{5}H^2 + CF_\beta D_G^{1/2}H^2 + Ca(\tau)^2F_\beta^2.$$

By (5) and Cauchy–Schwarz,

$$\int F_\beta D_G^{1/2}\,\mathrm{d}\tau \le Cm^2S^{2/25}\sqrt{1+\log S} \le Cc^2.$$

Variation of constants therefore gives

$$H(\tau) \le Cm\left[e^{-19\tau/10} + de^{51\tau/25}\right]. \tag{7}$$

Factors depending on the fixed initial time are absorbed in C.

Finally $B_4(f(s)) = e^{2\tau}B_4(w(\tau))$. Harmonicity cancels diffusion, and differentiating $e^{2\tau}$ cancels the dilation term. Radial self-advection vanishes. HLS and Gaussian weighted Hölder show that

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}B_4(f(e^\tau - 1))\right| \le Ce^{2\tau}[F_\beta H + a(\tau)F_\beta].$$

Substitution of (5) and (7), followed by integration, gives a first term $Cm^2S^{1/25+1/10}(1+\log S)$ and remaining terms at most $CmdS^{102/25}$, using $m \le 1$. Together with (3), this proves the proposition.

All energy tests can first be made with truncated weights and smooth data. The short-time exponential moment starts the weighted evolution; the uniform weighted estimates justify removal of the truncations and passage to the parabolic limit. No initial L^∞ norm enters a constant. □

1.2 Comparison with the actual local foreign field

Proposition 1.2 (Local harmonic field and a rough small leak)**.** *Let $\nu, r, T, R_* > 0$, $R_* \ge 4r$. Let f_0 have the properties of the preceding proposition at support radius r, with physical mass $M = \|f_0\|_1$. Let f solve*

$$f_t + (K * f + b)\cdot\nabla f = \nu\Delta f.$$

After removal of a solid rotation, suppose b is divergence free, has fourfold covariance, and admits, on B_{2R_*}, the decomposition

$$\begin{aligned} b = b_{\mathrm{h}} + e, \qquad & |b(t,x)| \le L|x| \quad \textit{globally},\\ |b_{\mathrm{h}}(t,x)| \le C_3|x|^3, \quad & |Db_{\mathrm{h}}(t,x)| \le C_3|x|^2,\\ |e(t,x)| \le \epsilon L|x|. & \end{aligned} \tag{8}$$

where b_{h} is divergence free with fourfold covariance and $0 \le \epsilon \le 1$. Its streamfunction is defined on B_{2R_*}. Assume

$$C_3R_*^2T + \epsilon LT \le \tfrac{1}{4}, \qquad C(M/\nu)\sqrt{\nu T} \le R_*/8, \qquad \sqrt{\nu T} \le cR_*. \tag{9}$$

Take a fixed radial cutoff of that streamfunction, equal to one in B_{R_*} and zero outside B_{2R_*}, and denote its velocity by $\widetilde{b}$. Let g solve the same self-consistent equation as f, with foreign field $\widetilde{b}$ and the same initial datum. Set $\ell = CC_3R_*^2$. If M/ν and $\ell(r^2/\nu + T)$ are sufficiently small, then

$$\begin{aligned} \sup_{t\le T}|B_4(f(t)) - B_4(g(t))| \le CMR_*^4e^{CLT}\Big[&\epsilon L\big(T + r\sqrt{T/\nu}\big)\\ &+ (1 + LR_*\sqrt{T/\nu})e^{-cR_*^2/(\nu T)}\Big]. \end{aligned} \tag{10}$$

No derivative bound on the leak e is required.

Proof. Choose the streamfunction ψ_{h} zero at the origin. Integration along radial segments gives $|\psi_{\mathrm{h}}(x)| \le CC_3|x|^4$. A streamfunction cutoff gives

$$|\widetilde{b}| \le CC_3|x|^3, \qquad |\widetilde{b}| \le \ell|x|, \qquad \left\|D\widetilde{b}\right\|_\infty \le \ell.$$

It is divergence free, has fourfold covariance, and equals b_{h} in B_{R_*}. A second useful bound is $|\widetilde{b}(x)| \le CL|x|$: from $\epsilon \le 1$ and (8), $|b_{\mathrm{h}}|le2L|x|$ on B_{2R_*}, so $|\psi_{\mathrm{h}}| \le CL|x|^2$ there, and cutoff derivatives preserve this linear growth. Thus this bound does not require $C_3R_*^2 \le L$.

We first verify the velocity-gradient estimate needed for comparison. Normalize $r = \nu = 1$ and write $m = M/\nu$; the cutoff growth and Lipschitz constant are normalized in the same way. Nash gives

$$\|g(t)\|_\infty \le Cm/t, \qquad \|u_g(t)\|_\infty \le Cm/\sqrt{t}, \qquad u_g = K * g.$$

For $M_k(t) = \int |x|^k|g(t,x)|\,\mathrm{d}x$, Kato's inequality gives

$$\begin{aligned} M_2' &\le 4m + Cmt^{-1/2}\sqrt{mM_2} + 2\ell M_2, \\ M_4' &\le 16M_2 + Cmt^{-1/2}m^{1/4}M_4^{3/4} + 4\ell M_4. \end{aligned}$$

Since $M_2(0), M_4(0) \le m$, a continuity bootstrap gives

$$M_2(t) \le Cm(1+t), \qquad M_4(t) \le Cm(1+t)^2$$

when m and $\ell(1+T)$ are small. On these trial bounds the integrated self terms are respectively $O(m^2(1+t))$ and $O(m^2(1+t)^2)$; $\ell T \ll 1$ absorbs the foreign terms. Hence

$$\|xg(t)\|_1 \le Cm\sqrt{1+t}, \qquad \|xg(t)\|_4 \le Cmt^{-3/4}\sqrt{1+t}.$$

The second follows from $\|xg\|_4^4 \le \|g\|_\infty^3 M_4$.

With H_t the heat kernel, scaling and the smooth kernels' exterior decay give

$$\begin{aligned} \|\nabla K * H_t\|_\infty &\le Ct^{-1}, \\ \|\nabla K * \operatorname{div} H_t\|_\infty \le Ct^{-3/2}, \quad \|\nabla K * \operatorname{div} H_t\|_1 &\le Ct^{-1/2}, \quad \|\nabla K * \operatorname{div} H_t\|_{4/3} \le Ct^{-3/4}. \end{aligned}$$

For the nonlinear differentiated kernel the tail is $O(|x|^{-3})$, so the last three norms are finite before scaling. The self flux $u_g g$ has L^1 norm $Cm^2t^{-1/2}$ and L^∞ norm $Cm^2t^{-3/2}$. Split the heat-mild time integral at $t/2$, using the kernel's L^∞ norm first and its L^1 norm second; its contribution to ∇u_g is Cm^2/t. The foreign flux has L^1 norm $C\ell m\sqrt{1+t}$ and L^4 norm $C\ell mt^{-3/4}\sqrt{1+t}$. Use the kernel's L^∞ norm first and its $L^{4/3}$ norm second. This contribution is

$$C\ell mt^{-1/2}\sqrt{1+t} \le Cm/t,$$

because $\ell(1+T)$ is small. The initial heat contribution is Cm/t. Undoing normalization proves

$$\|\nabla u_g(t)\|_\infty \le C(M/\nu)/t. \tag{11}$$

Put $\delta = f - g$, $v = K * \delta$, and $F = (b - \widetilde{b})f$. Then

$$\delta_t - \nu\Delta\delta + (u_f + \widetilde{b}) \cdot \nabla\delta + v \cdot \nabla g = -\operatorname{div} F.$$

Let ψ be a streamfunction of δ. Testing in H^{-1} gives

$$\tfrac{1}{2}(\|v\|_2^2)' + \nu\,\|\delta\|_2^2 \le (\|\nabla u_g\|_\infty + \left\|D\widetilde{b}\right\|_\infty)\,\|v\|_2^2 + \|v\|_2\,\|F\|_2\,. \tag{12}$$

Here $v\cdot\nabla\psi = 0$. Thus the term $v\cdot\nabla g$ vanishes when paired with ψ, and the same identity replaces $u_f = u_g + v$ by u_g in the other self transport term. Integration by parts bounds that term by $\|\nabla u_g\|_\infty \|v\|_2^2$. The identical calculation applies to $\widetilde{b}$. The circulation of δ is zero, and its polynomial moments are finite, so $v \in L^2$ and these tests are justified by approximation.

The true coefficient in (12) is integrable at time zero, although the upper bound (11) is singular there. Indeed $g_0 \in L^p$, $p > 2$, gives from the same mild formula

$$\|\nabla u_g(t)\|_\infty \le C_0(t^{-1/p} + t^{1/2-1/p}), \qquad 0 < t \le 1.$$

The constant may depend on the actual initial L^p norm and on the bounded cutoff drift. The kernel norms are $Ct^{-1/p}$ and $Ct^{-1/2-1/p}$ in $L^{p'}$; the latter time singularity is integrable, and $(u_g + \widetilde{b})g$ has bounded L^p norm near zero by contraction and the initial L^1–L^p Biot–Savart bound. Therefore the integrating factor in (12) starts from $v(0) = 0$. Only after writing its source integral do we use (11) on $[s,t]$, obtaining

$$\|v(t)\|_2 \le \int_0^t (t/s)^{CM/\nu} e^{\ell(t-s)}\,\|F(s)\|_2\ \mathrm{d}s. \tag{13}$$

Common smooth approximations have bounded initial L^p norms, so no arbitrary initial term at a positive time is discarded in this argument.

The localization estimate, applied at each time $s \le T$, gives

$$\int_{|x|\ge R_*} |x|^2 |f(s,x)|\ \mathrm{d}x \le CMe^{CLs}R_*^2 e^{-cR_*^2/(\nu s)}.$$

The cutoff field and the actual field have global growth $CL|x|$. Combining this tail estimate with Nash yields

$$\left\|(b-\widetilde{b})f\,\mathbf{1}_{|x|\ge R_*}\right\|_2 \le CLM\frac{R_*}{\sqrt{\nu s}} e^{CLs} e^{-cR_*^2/(\nu s)}.$$

Inside B_{R_*} the difference of drifts is exactly e. A weighted Kato estimate gives $\int |x|^2|f(s)| \le CM(r^2+\nu s)e^{CLs}$. Consequently

$$\left\|ef\,\mathbf{1}_{|x|<R_*}\right\|_2 \le C\epsilon LM\sqrt{\frac{r^2+\nu s}{\nu s}}\,e^{CLs}.$$

Substitute these bounds into (13). Since $CM/\nu < 1/4$, integration of $s^{-1/2-CM/\nu}$ is legitimate, and yields, uniformly for $t \le T$,

$$\|v(t)\|_2 \le CMe^{CLT}\left[\epsilon L(T + r\sqrt{T/\nu}) + LR_*\sqrt{T/\nu}\,e^{-cR_*^2/(\nu T)}\right]. \tag{14}$$

The bounded factor $e^{\ell T}$ is absorbed into C.

For a radial cutoff χ_* equal to one in B_{R_*} and zero outside B_{2R_*},

$$\left\|\nabla(\chi_*\bar{z}^4)\right\|_2 \le CR_*^4, \qquad \left|\int \chi_*\bar{z}^4\delta\right| \le CR_*^4\,\|v\|_2\,.$$

Both f and g satisfy the localized exit estimate, with the margins (9). Their exterior absolute fourth moments are at most $CMR_*^4 e^{CLT}e^{-cR_*^2/(\nu T)}$. Combining this with (14) proves (10). □

1.3 Physical normalization, the available power margin, and scope

Set

$$m = M/\nu, \qquad d = CC_3 r^4/\nu, \qquad S = 2 + \nu T/r^2.$$

The space-time change $x = ry$, $t = r^2 s/\nu$, and $f(t,x) = \nu r^{-2} F(s,y)$ gives these quantities exactly; moreover $B_4(f) = \nu r^4 B_4(F)$. Applying Proposition 1.1 to the cutoff-field solution and Proposition 1.2 to the actual solution gives

$$\begin{aligned}\sup_{t\le T} |B_4(f(t))| \le CMr^4 \left[\frac{M}{\nu} S^{7/50}(1+\log S) + \frac{C_3 r^4}{\nu} S^{102/25}\right] \\ + CMR_*^4 e^{CLT}\Big[\epsilon L(T + r\sqrt{T/\nu}) + (1 + LR_*\sqrt{T/\nu}) e^{-cR_*^2/(\nu T)}\Big]. \qquad (15)\end{aligned}$$

This requires (1) and the small normalized linear-growth bound of the cutoff field in addition to the local hypotheses. No cutoff has been applied to the actual nonlinear solution.

For the geometric cascade budget take a receiver (A_i, R_i) and

$$r = qR_i, \quad M \le C\beta A_i q^2 R_i^2, \quad \kappa = \frac{\nu}{A_i R_i^2} = q^{3/2}, \quad T = H/A_i, \quad R_* = \kappa^{1/4} R_i.$$

Suppose the actual outer-colour estimates furnish

$$L \le CA_i, \qquad C_3 \le C\delta A_i/R_i^2, \qquad \epsilon \le Ce^{-c/(\kappa H)}.$$

Then $m = O(\beta q^{1/2})$, $d = O(\delta q^{5/2})$, and $S = O(q^{-1/2}H)$. Dividing the first line of (15) by $A_i R_i^6$ gives

$$O\Big(\beta^2 q^{643/100} H^{7/50}(1+\log S) + \beta\delta q^{323/50} H^{102/25}\Big). \qquad (16)$$

The powers are 6.43 and 6.46. The localization terms are smaller than every q-power for logarithmic H, since

$$\frac{R_*^2}{\nu T} = \frac{1}{\sqrt{\kappa} H}, \qquad LT = O(H), \qquad C_3 R_*^2 T = O(\delta\sqrt{\kappa} H).$$

The other margins, including $(M/\nu)\sqrt{\nu T} \ll R_*$ and $mS^{1/25}\sqrt{1+\log S} \ll 1$, hold for sufficiently small q. Both terms in (16) are consequently $o(q^6)$. A fixed prefactor in the support radius changes only the constants.

The actual outer-core argument supplies the required harmonic-plus-leak hypothesis conditionally on the density bound

$$\sup_{t\le T} \|(I-P)w_l(t)\|_1 \le \delta A_l R_l^2, \qquad l \le i.$$

Its harmonic exterior multipole series can be differentiated to give $|Db_{\mathrm{h}}(x)| \le CC_3|x|^2$ throughout B_{2R_*} when this disk lies strictly inside the constant-core probe region; the exponentially small core density defect contributes the rough leak. Thus the present section closes full-window nonlinear inner-tail suppression under this explicit outer-density input. It does not prove that input for the active colour and older seeds, a signed outward transfer, or distinct vanishing-viscosity accumulation paths.