

Unforced two-dimensional Euler on the torus

An annular cascade and the nonlinear transfer estimate

Research proof draft for Umberto Pappalettera

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Status. This document gives a proposed full argument, including the nonlinear estimates that the earlier note left open. It is a research proof draft, not a published or independently established result. Its central assertion is the uniform finite-stack induction in Sections 6–9; the Euler-specific hypotheses and the simultaneous stopping argument are stated explicitly. Algebraic checks and separate adversarial audits found no counterexample to the revised estimates. This status should not be confused with formal certification of a new nonuniqueness theorem.

Proposed theorem 1. *For every $2 < p < \infty$, the square torus admits a mean-zero $\omega_0 \in L^p(\mathbb{T}^2)$, with zero-mean associated velocity, and two different solutions of*

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad u = \nabla^\perp \Delta^{-1} \omega, \quad \int_{\mathbb{T}^2} u = 0,$$

in $C([0, T]; L^p)$ with initial value ω_0 . Both solutions are renormalized. The proposed construction gives a stationary solution and a nonstationary one.

The construction uses shielded, linearly unstable annuli and a fixed amplitude ratio. A nonlinear pulse can have oscillatory late forcing. Its *whole* discounted fourth moment is used, rather than a pointwise sign condition. Nonvanishing of that transform is obtained by choosing the amplitude ratio generically. Only a fixed time window is compared with a canonical orbit; the remaining evolution is estimated by transport, support and moment bounds.

1 Parameters and the exact stationary datum

Identify $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ and put the origin at the centre of a fundamental square. Throughout, C denotes a finite constant independent of the truncation depth N and of sufficiently small q . It can depend on fixed parameters and reference windows. Set

$$\rho^2 = \frac{3}{5}, \quad \sigma = \frac{\rho^5}{2}, \quad \tau = \frac{3}{10}, \quad W = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}.$$

Thus $W = -2/5$ on B_ρ , $W = 3/5$ on $B_1 \setminus B_\rho$, and $\int W = 0$. The planar velocity of $AW(|x|/R)$ is tangential and vanishes outside B_R . Its inner solid rotation is $-A/5$. Choose

$$A_j = A_1 \beta^{j-1}, \quad R_j = R_1 q^{j-1}, \quad \beta \in [4, 5], \quad R_1 = q, \quad L = \log(1/q), \quad \varepsilon = q^6.$$

The choices of β and q will be specified below; $A_1 > 0$ is arbitrary. Require $\beta q^{2/p} < 1$ and $\beta q < 1$. Then

$$\bar{\omega} = \sum_{j \geq 1} A_j W(|x|/R_j) \in L^p, \quad \sum_j \|u_j\|_\infty \leq C \sum_j A_j R_j < \infty.$$

Choose a compactly supported streamfunction for each zero-circulation radial summand and extend it periodically. Its mean-zero velocity is exactly its planar velocity. Hence $\bar{\omega}$ is an exact stationary torus solution: the velocity is tangential wherever the vorticity is radial. The same observation applies to every renormalization of $\bar{\omega}$. On the gaps $R_{j+1} < r < \rho R_j$, $\bar{\omega}(r) = -(2/5) \sum_{i \leq j} A_i$; the datum is unbounded. Write $\bar{\omega}_N = \sum_{j \leq N} A_j W(|x|/R_j)$.

2 Local contours, coordinates and the isolated orbit

2.1 An analytic material equation

Fix $0 < \gamma < \gamma_0 < 1$. Use the little Hölder space

$$E = C^{1,\gamma}(S^1; \mathbb{R}^2)^2$$

of two material embeddings, with fourfold symmetry, near $X_i^0(\theta) = r_i e^{i\theta}$, $r_1 = \rho, r_2 = 1$. The planar velocity field V on these contours is analytic on a fixed chord-arc neighbourhood. Here is an explicit justification of the differentiability needed below.

For one reference disk D , extend boundary perturbations harmonically: $Hh \in C^{1,\gamma}(\overline{D})$. Dacorogna–Moser’s divergence right inverse [2, Theorem 2] gives

$$B : C_{\int=0}^\gamma(D) \longrightarrow C_0^{1,\gamma}(D; \mathbb{R}^2), \quad \operatorname{div} Bf = f.$$

Put $J(h) = \operatorname{Area}(X^0 + h) / |D|$. The map

$$(h, f) \longmapsto \det D(\operatorname{Id} + Hh + Bf) - J(h)$$

takes values in mean-zero C^γ , by the determinant null-Lagrangian identity and its boundary trace, and its f derivative at zero is the identity. The analytic implicit-function theorem supplies $f(h)$, and an extension $E(h)$ with the prescribed boundary and $\det DE(h) = J(h)$.

Hernandez’s complex near-identity bound [1, Definition 1.1, Proposition 1.3], applied to the convex disk D , makes

$$F_D(X)(a) = \int_D K(X(a) - X(b)) db$$

Banach analytic in $C^{1,\gamma}$: local complex boundedness and pointwise holomorphy give Cauchy coefficients and Fréchet holomorphy. His bound does not impose a Jacobian-one constraint on X . Change of variables therefore gives the *physical* self-interface velocity as

$$J(h) \operatorname{Tr} F_D(E(h)).$$

The factor $J(h)$ is essential. The two self terms are treated on their own disks. Cross-interface contour integrals have separated boundaries and nonsingular logarithmic kernels, and are analytic by composition. The torus regular kernel is analytic as well. This proves, after normalizing $x = Rz$, $t = s/A$,

$$\left\| V(X^0 + h) - V(X^0) - DV(X^0)h \right\|_E \leq C \|h\|_E^2. \quad (1)$$

Smooth approximation gives the little-space version. There is no nonlinear radial-graph ODE in this argument.

Remove the known solid rotation from all larger colours. Then the planar reference inner and outer material rotations are $\Omega_1 = -1/5, \Omega_2 = 0$. Independent constant label shifts commute with the material equation. The stationary label gauge is the semilinear flow

$$\partial_s X_i = V_i(X) - \Omega_i \partial_\theta X_i.$$

Label translation is a strongly continuous isometric group on the little Hölder space. Fixed-time flow maps are C^2 by variation of constants and (1). This gauged flow fixes X^0 . At $P = 10\pi$ the label translations are identities, so its time- P map is the material time- P map.

2.2 The full spectral split

For a positive geometric normal Fourier mode m , the linear matrix is

$$L_m = \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

Fourfold symmetry allows $m = 4, 8, \dots$. At $m = 4$ the exponents are $\pm\sigma + i\tau$, the growing normal vector is

$$v_+ = (1, -i\rho), \quad \ell = (1/2, i/(2\rho)), \quad \ell v_+ = 1.$$

For $m \geq 8$ the frequencies are

$$m/5 - 1/2 - \delta_m, \quad 3/10 + \delta_m, \quad 0 < \delta_m = O(\rho^{2m}/m).$$

The normal period multipliers are $-e^{\pm i P \delta_m}$, uniformly separated from 1. Their diagonalizers differ from the identity by exponentially summable Fourier coefficients. For $n \geq 1$,

$$\sum_{m=8,12,\dots} |e^{in P \delta_m} - 1| \leq C \sum_m \min\{2, n \rho^{2m}/m\} \leq C(1 + \log(1 + n)).$$

This bounds the centre powers on the Hölder space.

On the nonzero normal modes, the full material derivative is $\begin{pmatrix} M & 0 \\ B_0 & I \end{pmatrix}$. The bounded shear $\xi \mapsto \xi - B_0(M - I)^{-1}z$ diagonalizes it. Tangential relabelings remain neutral. One may restrict the two fixed enclosed areas and remove the two normal means. Equivalently, retain area variations in the ambient space: their normal means are constant and their tangential means have a finite-dimensional Jordan block of order at most two. We use this latter option. It adds at most linear growth and avoids an unnecessary nonlinear area chart. Thus, with P_u, P_c the full unstable and complementary projections,

$$\|S_c(s)\| \leq C(1 + s), \quad \|S(s)\| \leq C_\eta e^{(\sigma+\eta)s}. \quad (2)$$

The continuous bounds follow by bounded interpolation over one period.

The C^2 period map has a two-real-dimensional local strong unstable manifold. For completeness, apply the backward-sequence fixed-point construction with weight $e^{-\kappa n P}$, $0 < \kappa < \sigma$; the inverse unstable multiplier contracts, and the complementary polynomial powers are summable against that weight. The quadratic Taylor remainder is a contraction on a small ball. The fixed point depends C^2 on its unstable coordinate and gives a graph tangent to the full growing eigenspace. Uniqueness of backward exponential decay makes this graph invariant under the continuous gauged flow. Every enclosed-area invariant equals its value at the backward limit, so the graph automatically has the two reference areas.

2.3 Exact geometric moments and the canonical cap

For the two enclosed domains define

$$Q_i(X) = \frac{1}{2\pi r_i^6} \int_{D_i(X)} \bar{z}^4 dz, \quad a(X) = \ell Q(X).$$

Stokes expresses Q_i as a polynomial functional of X_i, X'_i . It is analytic and invariant under independent relabeling. If z_i is the positive fourth Fourier coefficient of relative normal displacement, then $Q_i = z_i + O(\|h\|^2)$. Consequently $Da(X^0)$ is the full unstable left dual. Pair a with the fixed linear complementary coordinate $b = P_c h$. The inverse-function theorem recovers h analytically from (a, b) . Write the unstable graph as $b = g(a)$, $g(a) = O(|a|^2)$.

Only the moment coordinate is differentiated continuously:

$$a_s = DQ(X)V(X) \cdot \ell.$$

The label generator contributes zero. This is a bounded analytic finite-dimensional functional, with linear term $(\sigma + i\tau)a$ and quadratic remainder. Centre Duhamel estimates can be carried out for the C^2 time- P maps; no unbounded generator is conjugated by a nonlinear chart. Between periods, isolated invariance of the graph and bounded flow maps give the same error bounds up to a fixed factor.

The action $X_i(\theta) \mapsto e^{i\phi} X_i(\theta - \phi)$ fixes the circles and commutes with the planar gauged flow. It sends $a \mapsto e^{-4i\phi} a$. Hence the graph is equivariant and its $|a| = \delta$ physical cap is one rotation family. Choose one ancient orbit $w_*(s)$, with cap at $s = 0$. Near the backward limit,

$$B_*(s) := \int \bar{z}^4 w_*(s, z) dz = b_* e^{(\sigma+i\tau)s} + O(e^{(\sigma+\kappa)s}), \quad b_* \neq 0,$$

because

$$B_* = 2\pi[-\rho^6 Q_1 + \rho^2 Q_2] = 2\pi(-\rho^6 - i\rho^3)a + O(|a|^2 + \|b - g(a)\|). \quad (3)$$

Chemin's general tangential-regularity theorem [3, Theorem 1, p. 522] supplies global finite-time continuation of this signed two-interface patch. Indeed choose a divergence-free C^{γ_0} field tangent and nonzero on both interfaces, using local defining functions on disjoint tubes. Its derivative of the signed piecewise constant vorticity vanishes distributionally; the vorticity is smooth off the tubes. The theorem has no positivity assumption. Constructing the local manifold first at exponent γ_0 and using uniqueness in the lower space gives the same orbit in little $c^{1,\gamma}$ on every compact time interval. Conservation of L^1, L^∞ gives

$$\|Kw_*\|_\infty \leq C, \quad \text{supp } w_*(s) \subset B_{C(1+s)}, \quad |B_*(s)| \leq C(1+s)^4 \quad (s \geq 0).$$

3 A nonlinear transfer coefficient that cannot vanish identically

Define the bilateral transform

$$F(z) = \int_{\mathbb{R}} e^{-zs} B_*(s) ds, \quad 0 < \text{Re } z < \sigma.$$

The bounds just proved give holomorphy, including differentiation under the integral on compact substrips. It is not identically zero: on a vertical line it is the Fourier transform of the nonzero L^1 function $e^{-cs} B_*(s)$. The identity theorem therefore lets us fix $\beta \in [4, 5]$ such that

$$F(z_\beta) \neq 0, \quad z_\beta = \frac{\sigma - i/2}{\beta}. \quad (4)$$

If necessary shrink the cap by translating the same ancient orbit. For $B_\delta(s) = B_*(s + t_\delta)$,

$$F_\delta(z) = e^{zt_\delta} F(z), \quad |F_\delta(z_\beta)| \asymp \delta^{1/\beta}.$$

Thus nonvanishing survives any sufficiently small cap, and $\delta/|F_\delta(z_\beta)| \rightarrow 0$. Choose the cap small enough for all local inequalities below and retain the notation B_*, F .

For any fourfold zero-circulation source inside both receiving interfaces, write $B_{\text{phys}} = \int \bar{z}^4 \omega_{\text{in}} dz$. The logarithmic potential expansion gives the relative normal fourth Fourier forcing at a receiving circle of radius r :

$$f_4(r) = \frac{iB_{\text{phys}}}{4\pi r^6}.$$

These are Fourier coefficients, not real cosine amplitudes. The growing projection is

$$D_4 \frac{B_{\text{phys}}}{R_j^6}, \quad D_4 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0.$$

Let $k = j + 1$. The larger-colour background is $\Omega_j^{\text{ext}} = -(1/5) \sum_{i < j} A_i$, and $\Omega_k^{\text{ext}} = \Omega_j^{\text{ext}} - A_j/5$. In the incoming rotating frame, the receiver exponent is

$$A_j(\sigma + i\tau - 4i/5) = A_j(\sigma - i/2).$$

In incoming time $s = A_k(t - T_k)$, its leading equation is therefore

$$a_s = z_\beta a + D_4 q^6 B_*(s).$$

The full discounted seed is $q^6 D_4 F(z_\beta)$. Set $c_0 = |D_4 F(z_\beta)| > 0$.

Choose a fixed S large enough that the estimates for both canonical tails, and the actual tails in Section 8, are below $c_0/100$. Choose a fixed cap tube whose finite-window continuity error is below $c_0/100$. The actual reset error will in fact tend to zero; a fixed tube alone would give $O(\text{tube})$, not $o(1)$. The canonical inner boundary has positive distance from the origin on $[-S, S]$. Fourfold symmetry fixes the origin, and the transported domain continues to contain it. This supplies a fixed central hole for the small deeper packet.

4 Exact interaction estimates and torus images

Write each transported colour as

$$w_l = A_l[-\mathbf{1}_{D_{l,1}} + \rho^2 \mathbf{1}_{D_{l,2}}].$$

Each domain has its reference area. Every colour separately preserves zero circulation, fourfold symmetry, $L^1 \leq C A_l R_l^2$ and $L^\infty \leq C A_l$. For a colour in its local contour neighbourhood let $x_l = \|h_l\|_E$. Exterior harmonic expansion, using the vanishing moments of degrees one through three and zero circular exterior field, gives

$$\begin{aligned} l > i : \quad & \|\text{relative field on core } i\| \leq C A_l q^{6(l-i)} x_l, \\ l < i : \quad & \|\text{field after subtracting solid rotation on } i\| \leq C A_l q^{2(i-l)} x_l. \end{aligned} \tag{5}$$

The norms include the full $C^{1,\gamma}$ evaluation on a perturbed receiving contour. To see the powers directly, an exterior degree-four moment is $O(A_l R_l^6 x_l)$, and relative velocity is that moment divided by R_i^6 . In an interior constant-vorticity disk the nonradial harmonic velocity starts at cubic order: $O(A_l x_l |x|^3 / R_l^2)$. Derivative bounds have the same powers after normalization; higher harmonics improve them. These bounds are analytic in the contour perturbations. For an arbitrary spent packet the first bound is replaced by its conserved mass times the fourth power of its radius.

Near zero the regular part of the square torus Green function is

$$H(z) = c + c_2 |z|^2 + \text{Re} \sum_{n \geq 1} c_{4n} z^{4n}.$$

Convolve with a *complete constrained colour*. Its circulation and first three holomorphic moments vanish. The quadratic term gives zero velocity; the degree-four term gives only a constant potential. The first possible nonzero image velocity has source degree four and target degree three, from the degree-eight term. Its relative size is

$$C A_l R_l^6 R_i^2 x_l.$$

Relative to (5) this gains R_i^8 for an inner source, R_l^8 for an outer source, and R_i^8 for self interaction. All are at most $R_1^8 = q^8$. This cancellation would be false for unconstrained individual disks. For actual colours it is exact.

5 The active packet needs no shape control

For every transported colour, the torus velocity bound is

$$\|K_{\mathbb{T}} w_l\|_{\infty} \leq C \sqrt{\|w_l\|_1 \|w_l\|_{\infty}} + C \|w_l\|_1 \leq C A_l R_l.$$

Consequently the velocity induced by all colours $l \geq k$ is bounded by $C A_k R_k / (1 - \beta q)$, independently of N . All initial domains are nested; the common Yudovich homeomorphism preserves every inclusion.

Suppose the larger contours remain inside fixed small caps, and the packet remains in their central plateaux. Their exact constant rotations do not change its radius. Fourfold symmetry eliminates linear harmonic strain, leaving the cubic estimate from (5). If d is the packet radius divided by R_k and $s = A_k(t - T_k)$, the upper Dini derivative obeys

$$d'(s) \leq C + C\beta^{-1}q^2d(s)^3. \quad (6)$$

For $0 \leq s \leq C_{\beta}L$, bootstrap $d \leq D_0(1 + s)$ and integrate: the cubic contribution divided by $1 + s$ is at most $Cq^2(1 + L)^3$. Choose D_0 first, then q , to improve the bootstrap. Thus

$$d(s) \leq C(1 + s), \quad |B_{\geq k}(s)| \leq C A_k R_k^6 (1 + s)^4.$$

In particular

$$\int_S^{C_{\beta}L} e^{-\sigma s/\beta} \frac{|B_{\geq k}(s)|}{A_k R_k^6} ds \leq C_{\beta} e^{-\sigma S/\beta} (1 + S)^4. \quad (7)$$

At the next receiving cap its radius is $CqLR_{k-1}$. The deeper packet $l > k$, throughout the history before T_k , has radius at most $CqLR_k$. Before the previous cap this follows from the earlier stage; afterwards apply (6) with the bounded summed waiting time. Its moment in incoming units is

$$O(q^6 L^4).$$

On the fixed window $[-S, S]$, it is inside the canonical inner hole for sufficiently small q . Its full contour forcing is $O(q^6 L^4)$. Later it can be treated as part of the aggregate.

6 Inactive contours: a weighted estimate uniform in depth

During the stage receiving at j , put

$$w_i = (Dq^6)^{j-i} \quad (i < j), \quad I_j = \max_{i < j} x_i / w_i.$$

Take $D > \beta$ sufficiently large. The weighted row coefficients from (5) are

$$A_i(\beta/D)^d \quad \text{outward}, \quad A_i(Dq^8/\beta)^d \quad \text{inward}.$$

Their sums are bounded by

$$C A_i \frac{\beta}{D - \beta}, \quad C A_i \frac{Dq^8}{\beta - Dq^8}.$$

The local nonlinear term, divided by w_i , is bounded by $C A_i Dq^6 I_j^2$. Image terms fit the same rows. Use (2) with a fixed small η , and weighted Volterra Grönwall. First choose D large, then q small, so the inactive subsystem has an upper exponent $a_i A_j$ with

$$a_i \leq \frac{\sigma + \eta}{\beta} + \frac{C_{\eta}}{D - \beta} + C_{\eta} Dq^8 + C_{\eta} Dq^6 K\delta < \frac{\sigma}{2}.$$

Here $I_j \leq 2K\delta$ is a stopped bootstrap, not an assumption for an already completed stage.

At the preceding cap T_k , $k = j + 1$, reindexing gives

$$I_j(T_k) \leq Dq^6 I_k(T_k), \quad x_j(T_k) \leq Dq^6 I_k(T_k).$$

With $\theta = A_j(t - T_k)$, the exact Duhamel consequence is

$$I_j(\theta) \leq C\varepsilon\delta e^{a_i\theta} + C \int_0^\theta e^{a_i(\theta-v)} [x_j(v) + \varepsilon(1+v)^4] dv. \quad (8)$$

The source from the current core has weighted coefficient bounded by CA_j/D ; losing the harmless $1/D$ gives this formula. The older packet has a further q^6 moment factor. Conversely the feedback into the recipient is

$$\|f_{\text{outer}}\| \leq CA_j Dq^8 I_j. \quad (9)$$

An unweighted cap bound would give $q^2\delta$, which is larger than the desired q^6 seed. Estimate (8) is the essential stronger control.

7 Closing the nonlinear estimate before q^3

Start a stage at T_k , and normalize by the receiver amplitude A_j . Its inherited full contour norm is $O(\varepsilon)$. Stop at the first of: $|a| = q^3$, loss of a local contour or packet neighbourhood, $I_j = 2K\delta$, or the deadline

$$H = \frac{9}{\sigma} L.$$

The packet bounds and the stopped prefix bound give actual external forcing $O(\varepsilon L^4)$; $q^8 I_j$ and the self-image term are smaller. All polynomial exponents here are fixed independently of N .

In the raw coordinate b , analytic period-map Taylor bounds have quadratic terms $C(|a|^2 + |a| \|b\| + \|b\|^2)$, and complementary powers at most $C(1+n)$. On this stopped interval $|a|^2 \leq q^6 = \varepsilon$. Bootstrapping $\|b\| \leq B\varepsilon L^6$ gives

$$\|b\| \leq C\varepsilon L^6 + Cq^3 L^2 B\varepsilon L^6 + CB^2 \varepsilon^2 L^{14} + Cq^8 K\delta L^2.$$

Choose B first and q afterwards. Every nonconstant bootstrap coefficient tends to zero; the last term is smaller than εL^6 . Hence

$$\|b(\theta)\| \leq C\varepsilon L^6. \quad (10)$$

Bounded interpolation over one period proves the continuous bound. This proof requires neither a signed seed nor a future cap time.

The moment equation, in any fixed rotating frame, is

$$a_\theta = \lambda a + \varepsilon f(\theta) + N_a + f_{\text{outer}} + f_{\text{image}} + f_{\text{higher}}, \quad \text{Re } \lambda = \sigma,$$

where the leading forcing has polynomial growth and its reference value is $f_*(\theta) = \beta D_4 B_*(\beta\theta)$ in the incoming rotating frame, with $\lambda = \sigma - i/2$. The factor β cancels under $s = \beta\theta$ in the discounted integral. In particular

$$|N_a| \leq C(q^3 + \varepsilon L^6) |a| + C\varepsilon^2 L^{12}.$$

Moment evaluation on a perturbed receiving contour changes the circular-source projection by $C\varepsilon(1+\theta)^4(|a| + \|b\|)$. For an upper bound it fits the same small terms.

Set $U(\theta) = e^{-\sigma\theta} |a(\theta)|$. Inserting (8) into the scalar Duhamel inequality and interchanging integrals, using $\sigma - a_i > 0$, gives

$$q^8 \int_0^\theta e^{-\sigma v} I_j(v) dv \leq Cq^8 \varepsilon L^6 + Cq^8 \int_0^\theta U(v) dv.$$

The polynomial source has a bounded exponentially discounted integral. Therefore

$$U(\theta) \leq C\varepsilon + C(q^3 + \varepsilon L^{12} + q^8) \int_0^\theta U(v) dv.$$

The coefficient times $H = O(L)$ tends to zero. Grönwall yields

$$|a(\theta)| \leq C\varepsilon e^{\sigma\theta}, \quad I_j(\theta) \leq C\varepsilon e^{\sigma\theta} + C\varepsilon L^6 e^{a_i\theta}. \quad (11)$$

For the signed lower bound keep the complex interaction-picture coefficient. Until the first q^3 crossing, the discounted errors are

$$\begin{aligned} \int e^{-\sigma\theta} |a|^2 d\theta &\leq C\varepsilon q^3 L, \\ \int e^{-\sigma\theta} (|a| \|b\| + \|b\|^2) d\theta &\leq C\varepsilon^2 L^{13}, \\ \int e^{-\sigma\theta} |f_{\text{outer}}| d\theta &\leq C\varepsilon q^8 L^7, \\ \int e^{-\sigma\theta} |\text{moving-contour correction}| d\theta &\leq C\varepsilon q^3 \text{poly}(L) + C\varepsilon^2 \text{poly}(L). \end{aligned} \quad (12)$$

All are $o(\varepsilon)$. Higher harmonics and negative history are handled in Section 8; images give $O(\varepsilon q^8 \text{poly}(L))$. This is the explicit closure of the unresolved nonlinear estimate. Absolute values have been used for upper bounds, not to infer a sign of the pulse.

8 The full signed pulse and its negative history

Assume the earlier completed stages have supplied their pointwise prefix bounds, cap reset, packet confinement and $9L/\sigma A_i$ clocks. Fix incoming $k = j + 1$. The total history, in incoming units, is at most

$$A_k T_k \leq \frac{9\beta}{\sigma(\beta-1)} L + C(S_0).$$

The recipient's backward discount thus gains at most

$$Cq^{-9/(\beta-1)} \leq Cq^{-3}. \quad (13)$$

Throughout every earlier stage the recipient j was inactive; its pointwise prefix estimate gives $x_j \leq C\varepsilon\delta$. Its entire own nonlinear prehistory is consequently

$$O(q^{12}L \cdot q^{-3}) = o(q^6).$$

Its moving-contour forcing error has the same type of bound. Reverse-prefix forcing before T_k is smaller still: in a prior stage $l \geq k$, every outer $i < j$ has $x_i \leq K\delta(D\varepsilon)^{l-i}$. The nearest outer core contributes $O(q^2\varepsilon^2)$; summing the geometric rows and discounting gives $O(q^{14} \text{poly}(L)q^{-3}) = o(q^6)$. The recipient self-image remainder is also $O(q^{14} \text{poly}(L))$ before this discount. At T_k , the whole state is $O(\varepsilon)$; fixed-time contour continuity keeps it so on the incoming window. No early-growth dichotomy or assumption about its future cap is necessary.

For the incoming core, introduce a noise level

$$a_* = C_*\varepsilon L^{12}.$$

Before its first q^3 crossing, (10) applies. At $|a| = a_*$, the ratio of actual forcing to $|a|$ is $O(L^{-8})$, and $\|b\|/|a| = O(L^{-6})$. The exact analytic moment equation therefore gives, for a small fixed cap and small q ,

$$\frac{d}{ds} |a| \geq \sigma_0 |a|, \quad \sigma_0 = \frac{3\sigma}{4}, \quad (14)$$

until q^3 . Section 9 extends this inequality to the cap. The crossing is one-way. At any current time, its previous above-noise amplitudes are bounded by backward exponential growth; this is not an estimate assumed from an unknown future cap. Relative to the completed incoming cap at $s = 0$,

$$x_k(s) \leq C\delta e^{\sigma_0 s} + C\varepsilon \text{poly}(L), \quad s \leq 0.$$

Before the receiving stage for k , the pointwise earlier prefix bound is even $O(\varepsilon)$.

The older packet has normalized moment $O(q^6 L^4)$ throughout this negative history. Below-noise moments have the same polynomial bound. By (13), their weighted integral is

$$O(q^{6-9/(\beta-1)} \text{poly}(L)) = o(1).$$

The above-noise negative tail before $-S$ is bounded by

$$C\delta e^{-(\sigma_0 - \sigma/\beta)S},$$

since $\sigma_0 > \sigma/\beta$. The future tail after S is (7). On $[-S, S]$, the incoming cap is $o(1)$ from a rotation of the canonical cap, its deeper packet is inside its central hole, and outer nonradial/image fields are small. Local Lipschitz contour continuation along this compact canonical trajectory compares the actual moment with the canonical one. Beyond the near-circle chart use the usual chord-arc contour dynamics theorem; the reference trajectory is globally regular by Chemin.

Accordingly the circular-receiver leading coefficient, integrated from the actual initial time to any time after S before q^3 , differs from

$$q^6 e^{i\phi} D_4 F(z_\beta)$$

by at most $c_0 q^6/10 + o(q^6)$, after the fixed tails and tube have been chosen. The first pulse omits canonical history before $-S_0$; choose fixed $S_0 \geq S$ to put that tail within the same margin. The phase ϕ is arbitrary and harmless.

Higher source harmonics start at q^{10} . A deliberately crude bound over negative history costs $q^{-3} \text{poly}(L)$, leaving $O(q^7 \text{poly}(L)) = o(q^6)$. Positive tails have a fixed exponential discount and polynomial moments. The same calculation applies to image errors, which have an extra q^8 . Together with Section 7's recipient remainders and its own prehistory, this gives

$$\left| e^{-\lambda\theta} a(\theta) \right| \geq \frac{c_0}{2} q^6$$

after the fixed source window, as long as $|a| < q^3$. Thus the first q^3 crossing occurs no later than

$$\theta = \frac{3}{\sigma} L + C, \tag{15}$$

with a harmless fixed adjustment for the source window. If the stopped interval reached that time without crossing, the displayed lower bound would contradict $|a| < q^3$.

9 Reset, prefix improvement and the simultaneous stopping argument

At the first q^3 time, flatten the graph: $y = b - g(a)$. By (10), $y = O(\varepsilon L^6)$. The time- P map in these coordinates has complementary linear powers bounded by $C(1 + n)$ and nonlinear remainder

$$C(|a| \|y\| + \|y\|^2).$$

Choose $0 < \eta < \sigma/16$, use the bound $C_\eta e^{\eta s}$ for the polynomial powers, and then choose δ so $C_\eta \delta$ is sufficiently small. With actual forcing $O(\varepsilon L^4)$, Duhamel and Grönwall give

$$\|y(s)\| \leq C\varepsilon \text{poly}(L) e^{2\eta s}.$$

Meanwhile, if $|a| \geq q^3 e^{\sigma_0 s}$,

$$\frac{\|y(s)\|}{|a(s)|} \leq Cq^3 \text{poly}(L), \quad \frac{|f(s)|}{|a(s)|} \leq Cq^3 \text{poly}(L).$$

These inequalities improve the growth bootstrap in the exact moment equation, because $g(a) = O(|a|^2)$. Hence (14) holds until $|a| = \delta$, requiring at most $4L/\sigma + C$ further normalized time. Combining with (15), the cap occurs before $7L/\sigma + C < 9L/\sigma$. Moreover

$$\|y_{\text{cap}}\| \leq Cq^{6-8\eta/\sigma} \text{poly}(L) = o(1). \quad (16)$$

Bounded interpolation permits a continuous first cap. Its physical shape is $o(1)$ from a rotation of the canonical cap. The preceding and following fixed windows have the same property. This weaker reset is sufficient; a uniform $q^6 \text{poly}(L)$ bound after q^3 is not needed.

The same bounds show, before cap and at every current time,

$$x_j(v) \leq C|a_j(v)| + C\varepsilon \text{poly}(L).$$

Below a_* , the amplitude itself is $O(\varepsilon \text{poly}(L))$. Above it, (14) gives $|a_j(v)| \leq |a_j(\theta)| e^{-\sigma_0(\theta-v)}$ for earlier above-noise v . Insert these two parts into (8):

$$I_j(\theta) \leq C|a_j(\theta)| + C\varepsilon \text{poly}(L)e^{a_i\theta}.$$

Since $\theta \leq 9L/\sigma$ and $a_i < \sigma/2$, the second term is

$$O(q^{6-9a_i/\sigma} \text{poly}(L)) = o(1).$$

Choose K larger than the fixed convolution constant; for small q this improves $I_j \leq 2K\delta$ to $I_j \leq K\delta$. Every inactive actual norm is at most $K\delta(Dq^6)^{j-i}$, so its neighbourhood cannot fail first.

Here is the order of the simultaneous stopping argument. At the start of a stage assume inherited $O(\varepsilon)$ current norm, $O(\varepsilon\delta)$ reindexed prefix, and the packet radius bound. Stop on the first loss of any of these enlarged bounds, local current neighbourhood, or the deadline. The stopped prefix and packet bounds give actual forcing $O(\varepsilon L^4)$. That forcing proves the raw centre estimate, the noise-level radial inequality, and, after q^3 , the weak reset inequality. These imply a backward amplitude bound at *every current time*, which improves the prefix convolution. The radius inequality independently improves the packet bound. The signed transfer calculation then forces q^3 , and the one-way inequality forces the cap before the deadline. Thus no exit precedes the cap and all output bounds improve their stopped assumptions. The argument never uses a future cap to prove the existence of that cap.

The fixed incoming window is available: at a preceding cap the new recipient has $O(\varepsilon)$ full norm. Bounded local evolution under $O(\varepsilon L^4)$ forcing prevents its reaching a fixed cap in any fixed normalized time as $q \rightarrow 0$. Equivalently the amplitude upper bound shows a delay at least cL . This exceeds any previously fixed S . Thus the source's $[-S, S]$ window lies in the adjacent completed/local stages.

10 Finite stacks and passage to a second solution

For each N , initialize the deepest colour on the isolated ancient orbit at $-S_0$, scaled by A_N, R_N ; initialize every other colour circular. The domains are nested and fourfold, and each has its fixed area. The finite total vorticity is bounded, so it has a global Yudovich solution on the torus. Transport the colours by its common flow.

The deepest first cap occurs after $(S_0 + o(1))/A_N$. On this fixed interval the outer response is small by (5), while its feedback and images are still smaller; local contour continuity gives the canonical

cap. The weighted prefix convolution, with incoming ancient growth, gives $I_N \leq K\delta$ throughout this initialization. There is no older packet. Choose q after all fixed windows and constants, so the same base tube and prefix assertions hold for every N .

Apply Sections 5–9 successively to $j = N - 1, \dots, 1$. At each cap the prefix reindexes with a factor Dq^6 ; the next full recipient norm is $O(q^6)$, with no inherited power of L . All noise and centre exponents are consequently fixed, not enlarged at successive levels. The reset bound (16) is uniform and does not multiply the preceding cap error: its forcing estimate uses only packet mass/support and the prefix, not long-time trajectory shadowing. The packet is in the next plateau, since $CqL \ll \rho$. Every stage therefore reproduces its hypotheses with constants independent of N . The final cap times t_N are bounded above by

$$T \leq \frac{S_0 + C}{A_N} + \sum_{j=1}^{N-1} \frac{9L}{\sigma A_j} \leq C + \frac{9L}{\sigma A_1} \frac{\beta}{\beta - 1}.$$

At the outer cap $a_1 = \delta e^{i\vartheta_N}$ and $y_1 = o(1)$. By (3),

$$|B_1(t_N)| \geq cA_1 R_1^6 \delta$$

for a small fixed δ and then sufficiently small q . The deeper packet's moment is at most $CA_1 R_1^6 \beta q^6 L^4$, and every circular reference colour has zero fourth moment. Take a smooth cutoff $\chi = 1$ on B_2 , supported in B_3 , and the periodic compactly supported test

$$\varphi(x) = \chi(x/R_1)(\bar{x}/R_1)^4.$$

It equals the fourth-moment test on the entire outer colour and inner packet, and $\|\varphi\|_{p'} \leq CR_1^{2/p'}$. The total pairing has magnitude at least $cA_1 R_1^2 \delta$. Hölder gives the fixed, depth-independent separation

$$\|\omega_N(t_N) - \bar{\omega}_N\|_p \geq cA_1 R_1^{2/p} \delta > 0. \quad (17)$$

No material-coordinate norm is substituted for an L^p norm.

The only initial change is the deepest colour, and

$$\|\omega_N(0) - \bar{\omega}_N\|_p \leq CA_N R_N^{2/p} \longrightarrow 0.$$

Together with the geometric tail this gives $\omega_N(0) \rightarrow \bar{\omega}$ strongly in L^p .

Lemma 1 (Strong compactness for these exact approximations). *If exact bounded torus Euler solutions have initial data converging strongly in L^p , $p > 2$, a subsequence converges strongly in $C([0, T]; L^p)$ to a renormalized Euler solution.*

Proof. Transport bounds ω_N uniformly in $L_t^\infty L_x^p$. Biot–Savart bounds u_N uniformly in $L_t^\infty W_x^{1,p}$; the velocity equation bounds $\partial_t u_N$ in $L_t^\infty L_x^p$. Interpolation and $p > 2$ yield strong compactness of u_N in $C_t C_x$, and the vorticity equation yields compactness in $C_t L_{\text{weak}}^p$. The flux passes to the limit because the velocity converges uniformly. For $u \in L_t^\infty W_x^{1,p}$, $\omega \in L_t^\infty L_x^p$, the DiPerna–Lions commutator is integrable since $2/p < 1$. Thus the limit transport equation is renormalized and has its initial trace; in particular its L^p norm is conserved and it is strongly time-continuous.

Every approximating norm equals its initial norm, which converges to that conserved limiting norm. For any $s_N \rightarrow s$, weak convergence of $\omega_N(s_N)$ to $\omega(s)$ and convergence of norms imply strong convergence by uniform convexity. A contrary sequence of times would contradict this conclusion, so the convergence is uniform in time. $\square$

Apply this lemma on a common interval containing all cap times. Pass also to $t_N \rightarrow t_*$. Strong convergence and (17) imply $t_* > 0$ and $\omega(t_*) \neq \bar{\omega}$. The limiting initial datum is $\bar{\omega}$, so this branch and the stationary branch give the proposed nonuniqueness conclusion.

What changed from the earlier incomplete route

The amplitude ratio is fixed before the radius ratio. The entire nonlinear incoming fourth moment has a nonzero bilateral transform for a generic ratio. A polynomial support bound replaces long-time trajectory shadowing. The inactive hierarchy is estimated in a weighted contour norm, giving q^8 reverse feedback. The recipient nonlinear error is closed only up to q^3 ; thereafter a one-way growth and a weaker, sufficient reset give the cap. Crude $9L/\sigma A_j$ deadlines suffice for every negative-history loss. These are mathematical changes to the proof, not numerical evidence for a conjectured cascade.

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