

# Rigorous Mathematical Referee Report on *euler-plane-revised-1.pdf*

This report assesses only mathematical validity. It does not evaluate novelty, significance, exposition, or style. The conclusions below distinguish between proven errors, unresolved proof gaps, and steps that were successfully checked. [cite:1]

## Overall assessment

The paper cannot presently be certified as mathematically correct on the basis of the checked material. At least one serious defect appears in a stated lemma, and several later central arguments rely on estimates or compactness/semigroup steps that are not adequately justified in the manuscript. [cite:1]

## Proven error

### Lemma 5.1 (page 54)

Lemma 5.1 claims that for every fourfold-symmetric function  $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ , one has a global linear bound  $|Kf(z)| \leq C|z|$ . The proof expands the Biot-Savart kernel on the far region  $|y| > 2|z|$ , observes that quarter-turn symmetry kills the low harmonics, and then uses the first surviving term of order  $z^3y^{-4}$ . [cite:1]

That part is structurally plausible, but the displayed estimate is not justified as written. The relevant far-field contribution is of the form

$$|z|^3 \int_{|y| > 2|z|} |f(y)| |y|^{-4} dy,$$

and the manuscript bounds it by using  $\|f\|_\infty$  inside a radial integral. This only controls the near portion of the far region; it does not by itself justify the full tail bound as presented, nor does it establish the claimed global estimate with the stated hypotheses and proof. [cite:1]

More importantly, the lemma is then used as a uniform origin-growth estimate in later arguments. The manuscript does not clearly state what the constant  $C$  may depend on, and the proof does not provide a dependence strong enough for the subsequent cascade estimates. As written, the lemma is not established in the form later needed. [cite:1]

## Consequences of the error

Lemma 5.1 is used later in Proposition 5.3, Proposition 6.4, Lemma 6.7, and Lemma 7.1 for origin-growth and foreign-field estimates. Those later arguments therefore are not proved by the text unless Lemma 5.1 is replaced by a correct statement together with a proof supplying exactly the required uniformity. [cite:1]

## Unresolved proof gaps

### Theorem 2.12 (pages 27–30)

Theorem 2.12 proves strong vanishing-viscosity precompactness in  $C([0, T]; L^1 \cap L^p)$  for compactly supported zero-circulation data. The argument uses renormalization for the limiting Euler transport equation with velocity in  $L_t^\infty W_x^{1,p}$ , followed by exact conservation of limiting  $L^p$  and  $L^1$  norms and then a Clarkson-type argument for strong convergence. [cite:1]

This strategy is reasonable, but the manuscript does not supply enough detail at several nontrivial points: the commutator convergence is stated locally; endpoint renormalized identities are asserted for every prescribed time; spatial cutoffs are used to pass to global conservation; and the weak initial trace is converted into the precise trace required by renormalization. These steps are delicate, and the proof given is too compressed to verify rigorously in its present form. Since the exact norm conservation is then used to deduce strong convergence, this is a substantive gap in the theorem. [cite:1]

### Lemma 3.3 (pages 32–35)

Lemma 3.3 is a central spectral statement. It asserts that after removing the growing Riesz projections, the remainder propagator obeys an estimate of the form

$$\|R_k, S(t)\|_{\ell^2 C_k, Se^{2t}, q} \leq C e^{-\delta t}.$$

The proof appeals to a contour argument for a bounded operator after discussing a compact perturbation of angular multiplication and a finite-dimensional Schur reduction. [cite:1]

What is not fully established is the spectral placement and resolvent control needed to justify the contour deformation and the explicit semigroup bound on the complement. Excluding unstable eigenvalues is not, by itself, enough to obtain the stated estimate when the background multiplication operator contributes continuous spectrum on the imaginary axis. Because Lemma 3.4 and the subsequent nonlinear activation constructions use this remainder estimate essentially, the proof gap is central. [cite:1]

### Proposition 5.3 (pages 55–57)

Proposition 5.3 is stated for a finite or countable coloured evolution. The proof closes a bootstrap using continuity of a maximum probe defect over a finite set and then says the countable case follows because all infinite sums were already controlled. [cite:1]

That is insufficient. Passing from finite families to a countable family requires either a finite-prefix approximation argument with constants uniform in the prefix, or a direct continuity/upper-semicontinuity argument for the supremum over all indices. Neither is supplied, so the countable-colour version is not rigorously established by the proof as written. [cite:1]

## Steps checked successfully

### Lemma 2.6

The Biot–Savart bounds in Lemma 2.6 are mathematically consistent with standard arguments. The  $L^\infty$  estimate follows from near/far splitting and Hölder, the  $L^p$  bound for  $Kw$  follows from Young’s inequality, and the  $L^p$  bound for  $\nabla Kw$  is consistent with Calderón–Zygmund theory. The zero-circulation  $L^2$ -velocity conclusion for compactly supported  $w \in L^1 \cap L^p$  is also plausible from low-frequency cancellation plus high-frequency  $L^2$  control. [cite:1]

### Lemma 5.2

Lemma 5.2 appears correct. The radial test function  $\cosh(\alpha|x|)$  makes the tangential drift irrelevant, and the stopped Itô computation yields the advertised exponential exit estimate, provided the standard localization/approximation justifications are inserted. [cite:1]

### Lemma 4.10

Lemma 4.10 also appears sound in principle. The stated negative-time asymptotic produces a nonremovable simple pole in the meromorphic continuation of the bilateral transform, so the transform cannot vanish identically; analyticity then implies that zeros on a real interval are discrete. [cite:1]

### Final verdict

The mathematics has not been verified. The present draft contains at least one concrete defect and several major unresolved proof gaps in arguments that appear structurally essential to the later cascade and nonuniqueness conclusions. [cite:1]