

A fixed-data viscous cascade in the plane

Mathematical research draft

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Abstract

We study the actual interacting coloured evolution of the unforced planar Navier–Stokes equation. The scales and initial data are fixed independently of viscosity. Positive scalar envelopes close the inactive-density estimate without an assumed packet force. Actual harmonic histories are retained in a finite nonlinear expansion; a separate pointwise moment estimate protects the first activation crossing. Recentring at a fixed reference state controls the entire signed source history and its future tail. A normalized finite coefficient calculation removes dependence on the incoming source waveform from the outgoing state at its exact cap. Uniform descending induction, phase degree and a sparse phase selection then give one compactly supported, zero-circulation datum in $L^1 \cap L^p$, $2 < p < \infty$, with two distinct strong vanishing-viscosity accumulation points. The preliminary conditional lemmas retain their hypotheses; the actual coupled estimates in the later sections establish those hypotheses for the construction.

1 Equation and fixed initial data

The equation is

$$\partial_t \omega^\nu + (K * \omega^\nu) \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu, \quad K(x) = \frac{x^\perp}{2\pi|x|^2}, \quad x^\perp = (-x_2, x_1). \quad (1)$$

Identify x with $z = x_1 + ix_2$, and write

$$B_4(f) = \int_{\mathbb{R}^2} \bar{z}^4 f(x) \, dx.$$

An entire moment is used only after its fourth absolute moment is proved finite. Fourfold symmetry is $f(ix) = f(x)$. The symbol P_{rad} denotes angular averaging; a local symbol P denotes the same operation. We use the standard global mild well-posedness and positive-time smoothing of (1) for integrable initial vorticity, as stated in [1, Theorem 2.2], after rescaling viscosity. The L^r scalar contractions follow directly from the convex energy inequalities. The Biot–Savart L^p gradient and HLS bounds used below are the usual singular-integral inequalities, also stated in [1, Lemma 2.1]. The comparison, moment, core, and selection estimates are proved here; no periodic construction is used as an input.

Lemma 1.1 (Admissibility of a fixed infinite smooth stack). *Let W be a fixed real smooth compactly supported radial density with zero integral, and let Φ_θ be a bounded smooth family with common compact support, zero integral, and $B_4(\Phi_\theta) = 0$. Suppose Φ_θ is fourfold symmetric. Fix*

$\alpha > 0$, $A_j = A_1 \beta^{j-1}$, $R_j = R_1 q^{j-1}$, and $A_1, R_1 > 0$, $\beta > 1$, $0 < q < 1$, and $\beta q^{2/p} < 1$, where $2 < p < \infty$. For any phases θ_j , the single datum

$$\omega_0(x) = \sum_{j \geq 1} A_j [W(x/R_j) + \alpha \Phi_{\theta_j}(x/R_j)]$$

converges absolutely in $L^1 \cap L^p$, has compact support and zero circulation, and satisfies $B_4(\omega_0) = 0$. Changing only phase θ_N changes its $L^1 \cap L^p$ norm by at most $C\alpha A_N(R_N^2 + R_N^{2/p})$, which tends to zero.

Proof. For either summand its L^r norm is at most $CA_j R_j^{2/r}$, uniformly in phase, for $r = 1, p$. The geometric ratio is less than one for both norms. All supports are in a common ball. Integral and phase claims follow by absolute convergence. The absolute fourth moment of each summand is bounded by $CA_j R_j^6$; this series converges since $\beta q^6 < 1$. Radial symmetry gives $B_4(W) = 0$, so termwise integration proves the whole-moment assertion. $\square$

The lemma verifies the datum class; it does not establish the desired behaviour of its solutions. For this construction q , α , the profile and all phases must be fixed before viscosity is sent to zero. In the isolated theorem below q varies to prove a quantitative estimate; that theorem by itself is not a fixed-data selection statement.

The actual interaction estimates are proved in Sections 13–19. The final fixed-data conclusion is Theorem 19.3. Earlier conditional estimates are used only with the actual hypotheses verified in those later sections.

2 An isolated nonlinear active stage for a fixed smooth profile

The result of this section concerns one isolated annulus. It replaces the discontinuous reference vorticity by one fixed smooth profile, explicitly. The smoothing length is chosen once and is independent of viscosity and of the small parameter below. In particular, the theorem does not assert the analogous nonlinear estimate for a family of sharp patch boundaries.

Theorem 2.1 (Isolated smooth activation). *There exist a real, radial, compactly supported C^∞ vorticity W_ϵ , with zero circulation, a number $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with $\sigma_\epsilon > 0$, and a nonzero complex number b_ϵ , with the following property. For every $2 < p < \infty$ and $K_* > 0$ there is $q_0 > 0$ such that the conclusions below hold for $0 < q < q_0$. Put*

$$L_q = \log(1/q), \quad \alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad \delta_q = L_q^{-4}, \quad H_q = \frac{1}{\sigma_\epsilon} \log \frac{\delta_q}{\alpha_q}.$$

There is a 2π -periodic family of smooth real seeds Φ_ϑ , supported in two fixed annular collars, such that

$$\int \Phi_\vartheta = 0, \quad B_4(\Phi_\vartheta) = 0, \quad \Phi_\vartheta(ix) = \Phi_\vartheta(x).$$

Let $\omega^{q,\vartheta}$ be the unforced Navier–Stokes solution with viscosity κ_q and initial vorticity $W_\epsilon + \alpha_q \Phi_\vartheta$. Then, uniformly in $\vartheta \in \mathbb{R}$,

$$\sup_{0 \leq t \leq H_q} \|\omega^{q,\vartheta}(t) - e^{\kappa_q t \Delta} W_\epsilon\|_{L^1 \cap L^p} \leq C\delta_q, \quad (2)$$

$$\int_0^{H_q} \|K * (\omega^{q,\vartheta}(t) - e^{\kappa_q t \Delta} W_\epsilon)\|_\infty dt \leq C\delta_q, \quad (3)$$

$$B_4(\omega^{q,\vartheta}(H_q)) = \delta_q e^{i(\vartheta + \tau_\epsilon H_q)} b_\epsilon + o(\delta_q). \quad (4)$$

Here $\|f\|_{L^1 \cap L^p} = \|f\|_1 + \|f\|_p$; all fourth absolute moments in (4) are finite. In particular, (4) is a statement about the whole moment, rather than only a compactly tested moment. The constants depend on the fixed profile, p and K_* , but not on q or ϑ .

The positive cap δ_q will eventually be fixed by choosing one q . Its logarithmic decrease is useful for the simultaneous condition $H_q \sqrt{\delta_q} \rightarrow 0$. The theorem concerns a family of isolated initial data and does not claim nonuniqueness of any fixed-data vanishing-viscosity limit.

2.1 The fixed profile and its positive angular spectrum

Fix $\rho^2 = 3/5$ and choose a nonnegative, radial, radially decreasing $\chi \in C_c^\infty(B_1)$ with integral one. Define

$$\chi_\epsilon(x) = \epsilon^{-2} \chi(x/\epsilon), \quad W_\epsilon = \chi_\epsilon * (-\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}).$$

The convolution is smooth, radial, supported in $B_{1+\epsilon}$ and has zero integral. It is constant on $B_{\rho-\epsilon}$. Choose $\epsilon_* < (1 - \rho)/4$ and subsequently $0 < \epsilon < \epsilon_*$. Its radial derivative has the form

$$W'_\epsilon(r) = D_{\rho,\epsilon}(r) - \rho^2 D_{1,\epsilon}(r), \quad D_{a,\epsilon} \geq 0, \quad \text{supp } D_{a,\epsilon} \subset [a - \epsilon, a + \epsilon], \quad \int_0^\infty D_{a,\epsilon}(r) dr = 1. \quad (5)$$

To verify the sign, express the radial decreasing mollifier by its superlevel disks. The area of intersection of two disks is nonincreasing in their center separation: along each line parallel to their displacement, the overlap of the two interval sections is nonincreasing. Layer-cake integration gives radial monotonicity of the disk convolution. Its value is one at the origin and zero at infinity, giving the displayed derivative mass. The support assertion follows because the convolution is exactly constant at points whose distance to the disk boundary exceeds ϵ .

Set

$$\Omega_\epsilon(r) = r^{-2} \int_0^r s W_\epsilon(s) ds.$$

On the two derivative collars, respectively,

$$\Omega_\epsilon(r) = \Omega_0(r_i) + O(\epsilon), \quad (r_1, r_2) = (\rho, 1), \quad (\Omega_0(r_1), \Omega_0(r_2)) = (-1/5, 0). \quad (6)$$

Indeed disk translation changes its L^1 norm by at most a constant times the translation length. Thus $\|W_\epsilon - W_0\|_1 \leq C\epsilon$, where $W_0 = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}$. The radial circulation formula, on radii bounded away from zero, gives $|\Omega_\epsilon - \Omega_0| \leq C\epsilon$. The piecewise smooth Ω_0 is Lipschitz there. Direct integration gives the interface values.

For a positive angular mode $m \geq 1$ use the convention $f(r)e^{im\theta}$. Put

$$h_m(r, s) = \left(\frac{\min(r, s)}{\max(r, s)} \right)^m, \quad \psi_f(r) = -\frac{1}{2m} \int_0^\infty h_m(r, s) f(s) s ds.$$

The function $\psi_f e^{im\theta}$ solves $\Delta \psi = f e^{im\theta}$, is regular at the origin and decays at infinity. This follows either by solving the radial second-order equation on each side of $r = s$, or by checking the derivative jump $1/s$ of the displayed Green kernel. Since $u_r = -im\psi_f/r$, the frozen Euler linearization at W_ϵ is

$$\mathcal{L}_m f = -im\Omega_\epsilon f + im\psi_f W'_\epsilon/r. \quad (7)$$

For $m = 0$ the linearization is zero: two radial vorticities have tangential velocities and do not advect one another. Negative sectors are the complex conjugates of the positive sectors. From this point onward all angular sectors belong to $4\mathbb{Z}$. No spectral assertion is made for other symmetry classes.

Lemma 2.2 (The fixed smooth unstable pair). *One can choose a fixed sufficiently small $\epsilon > 0$ and $0 < \eta < \sigma_\epsilon/4$ so that in the quarter-turn symmetry class: the $m = 4$ generator has one algebraically simple eigenvalue $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with positive real part greater than η ; its growing radial eigenfunction f_ϵ is smooth and supported in the two derivative collars; and all other positive-real-part spectral values, except the conjugate growing eigenvalue in $m = -4$, have real part at most η . Moreover*

$$\lambda_\epsilon \longrightarrow \rho^5/2 + 3i/10 \quad \text{as } \epsilon \downarrow 0.$$

For each finite set of sectors in $4\mathbb{Z}$ and each finite radial Sobolev order k , work on the direct sum of the spaces $H^k(I)$, where $I \Subset (0, \infty)$ contains both fixed derivative collars. The propagator restricted to smooth collar-supported functions preserves their support and has the decomposition

$$e^{t\mathcal{L}} = e^{\lambda_\epsilon t} P_+ + e^{\bar{\lambda}_\epsilon t} P_- + \mathcal{R}_k(t), \quad \|\mathcal{R}_k(t)\|_{H^k \rightarrow H^k} \leq C_k e^{2\eta t}. \quad (8)$$

The growing terms are included only if their sectors are present.

Proof. First consider any eigenvalue λ with $\Re \lambda > 0$. The eigenvalue equation forces $f = 0$ off the derivative collars and gives

$$f(r)r \, dr = g(r)W'_\epsilon(r) \, dr, \quad g(r) = \frac{im\psi_f(r)}{\lambda + im\Omega_\epsilon(r)}.$$

There is no division by W'_ϵ here. In the Hilbert space $L^2(|W'_\epsilon| \, dr)$ the equation for g is

$$\lambda g = -im\Omega_\epsilon g - \frac{i}{2} \int h_m(r, s)g(s)W'_\epsilon(s) \, ds.$$

Each same-collar block is skew-adjoint, since the sign of W'_ϵ is constant there and $h_m(r, s)$ is real and symmetric. Angular multiplication is also skew-adjoint. The two opposite-collar blocks have operator norms at most their Hilbert–Schmidt norms, bounded by

$$Ca_*^m, \quad a_* = (\rho + \epsilon_*)/(1 - \epsilon_*) < 1,$$

because their two weighted masses are 1 and ρ^2 and their kernels are bounded by a_*^m . Consequently

$$\Re \lambda \leq Ca_*^m. \quad (9)$$

This fixes a finite angular cutoff in any half-plane $\Re \lambda \geq \eta > 0$, independently of ϵ .

For one of the finitely many positive sectors, eliminate f to obtain the equation on the two collars

$$\psi(r) = -\frac{i}{2} \int h_m(r, s) \frac{W'_\epsilon(s)\psi(s)}{\lambda + im\Omega_\epsilon(s)} \, ds =: \mathcal{T}_\epsilon(\lambda)\psi(r). \quad (10)$$

Work first in the sup-norm space of continuous functions on the two collars. Let P evaluate at r_1, r_2 and let E extend a vector to constants on the two collars, so $PE = I$. The receiving-radius derivative of the kernel is bounded there for fixed m . Hence

$$\|(I - EP)\mathcal{T}_\epsilon(\lambda)\| \leq C\epsilon$$

uniformly on compact subsets of $\Re \lambda \geq \eta$. For $\psi = Ep + w$, $Pw = 0$, solve the w equation by an analytic Neumann series. The remaining two-dimensional equation has a determinant converging uniformly on these compact sets to the determinant of

$$p_i = \frac{i}{2} \sum_j \frac{c_j h_m(r_i, r_j)}{\lambda + im\Omega_0(r_j)} p_j, \quad (c_1, c_2) = (-1, \rho^2).$$

The sign follows from $W'_0 = -\sum c_j \delta_{r_j}$. Using interface variables $f(r)r dr = \sum c_i r_i^2 q_i \delta_{r_i}$, this equation is equivalent to the matrix eigenvalue problem

$$M_m q = \lambda q, \quad M_m = \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

For $m = 4$ its roots are $\pm \rho^5/2 + 3i/10$, both simple. For $m = 8, 12, \dots$ they are imaginary, since the real-matrix discriminant is

$$\left(\frac{2(m-4)}{5}\right)^2 - 4\rho^{2m+2} > 0.$$

At $m = 8$ the first term is $64/25$ and the second is $4(3/5)^9 < 1$; thereafter the first increases and the second decreases. Large $|\lambda|$ gives no omitted roots: once $|\lambda| > C_m$, the operator in (10) has norm less than one, uniformly in ϵ , because the denominator then has size comparable with $|\lambda|$. Uniform determinant convergence and Rouché's theorem therefore give one simple root near the growing $m = 4$ root and no other roots in $\Re \lambda \geq \eta$. Combined with (9), this proves the claimed positive spectrum after one fixed ϵ is selected. Take η smaller first if necessary, so that $2\eta < \sigma_\epsilon$.

We give the operator-domain and multiplicity details needed for the propagator claim. Choose a closed interval $I \Subset (0, \infty)$ containing both collars strictly in its interior. On $H^k(I)$ define ψ_f by the Green integral against the zero extension of f from I . This extension need not be globally H^k : multiplication of the potential by W'_ϵ restricts output strictly inside I . The radial elliptic equation, or direct differentiation of its Green kernel, gives two local derivatives. Thus the coupling in (7) is compact on $H^k(I)$ and the angular multiplier is bounded there. Its spectrum is imaginary: off the multiplier's value range the inverse is multiplication by $(\lambda + im\Omega_\epsilon)^{-1}$. The compact-operator Fredholm alternative consequently reduces all spectrum in $\Re \lambda > 0$ to eigenfunctions. The equation bootstraps such eigenfunctions to smoothness and forces their support into the collars, so the roots already found are exactly the positive spectrum on every $H^k(I)$.

To check algebraic simplicity independently of a sup-norm determinant, start on $X_0 = L^2(I)$. Write $Af = -im\Omega_\epsilon f$, $Cf = \psi_f|_{\text{collars}}$ and $B\psi = imW'_\epsilon \psi/r$. Then $\mathcal{L}_m = A + BC$ and $\mathcal{T}_\epsilon = C(\lambda - A)^{-1}B$. Off the imaginary multiplier range, the resolvent identity is

$$(\lambda - \mathcal{L}_m)^{-1} = (\lambda - A)^{-1} + (\lambda - A)^{-1}B(I - \mathcal{T}_\epsilon)^{-1}C(\lambda - A)^{-1}. \quad (11)$$

Here $C : X_0 \rightarrow C(\text{collars})$ is bounded by its elliptic gain, and $B : C(\text{collars}) \rightarrow X_0$ is bounded. The invertible complementary Schur factor and the simple two-by-two determinant zero imply a rank-one simple pole in (11). The residue is nonzero, since an eigenfunction exists. On $X_k = H^k(I)$ the resolvent equation and its radial elliptic bootstrap give, locally uniformly for λ off the imaginary multiplier range,

$$\|v\|_{H^k(I)} \leq C_k(\|f\|_{H^k(I)} + \|v\|_{L^2(I)}), \quad (\lambda - \mathcal{L}_m)v = f.$$

The X_k resolvent is the restriction of the X_0 resolvent; the simple pole has no higher-order singularity by this estimate and has the same nonzero smooth one-dimensional range. This proves algebraic simplicity on every space used below. The $m = -4$ conclusion follows by conjugation; $m = 0$ is the identity propagator.

After removing the growing Riesz projections, the remaining bounded generator has spectral bound at most η . Its exponential is bounded by $Ce^{2\eta t}$. For clarity this last fact does not assume a spectral growth theorem for unbounded generators. For a bounded operator, integrate $e^{tz}(z - \mathcal{L})^{-1}$

on a finite rectangular contour enclosing its remaining spectrum and having right edge $\Re z = 2\eta$. Expansion of the resolvent on a large circle and termwise integration gives the exponential series; contour deformation gives the rectangle formula. The resolvent is bounded on this fixed contour, proving the estimate. The finitely many sectors and derivative orders give (8). Coefficients initially smooth and supported in the collars remain so under this propagator: multiplication preserves that support and the coupling has smooth output flat at the collar endpoints. This proves the lemma. $\square$

2.2 An exactly neutral smooth seed

Normalize the growing radial eigenfunction f_ϵ so that its first collar mass tends to $c_1 r_1^2 = -\rho^2$. The finite-dimensional reduction then implies

$$f_\epsilon(r)r \, dr \rightarrow -\rho^2 \delta_\rho - i\rho^3 \delta_1.$$

Indeed its interface vector tends to $(1, -i\rho)$, the growing eigenvector of M_4 . Define

$$b_\epsilon = 2\pi \int_0^\infty r^5 f_\epsilon(r) \, dr. \quad (12)$$

The weak convergence on the two compact collars gives

$$b_\epsilon \rightarrow 2\pi(-\rho^6 - i\rho^3) \neq 0.$$

Reduce the fixed ϵ if necessary. Since $\mathcal{L}_4$ is i times a real operator, $\overline{f_\epsilon}$ is an eigenfunction of $\mathcal{L}_4$ with eigenvalue $-\overline{\lambda_\epsilon} = -\sigma_\epsilon + i\tau_\epsilon$. Define the real seed

$$\Phi_\vartheta(r, \theta) = e^{i\vartheta} \left(f_\epsilon(r) - \frac{b_\epsilon}{\overline{b_\epsilon}} \overline{f_\epsilon(r)} \right) e^{4i\theta} + \text{complex conjugate}. \quad (13)$$

Its circulation is zero and its only initial sectors are ± 4 . For the convention $B_4 = \int \bar{z}^4 \omega$, angular integration gives $B_4(f(r)e^{4i\theta}) = 2\pi \int r^5 f(r) \, dr$. Therefore $B_4(\Phi_\vartheta) = 0$ exactly, while the growing coefficient is $e^{i\vartheta}$. The stable coefficient has modulus one.

2.3 The finite seed–viscosity expansion

For the rest of the proof abbreviate the fixed profile to W , and write $\sigma = \sigma_\epsilon$, $\tau = \tau_\epsilon$, $W_\kappa(t) = e^{\kappa t \Delta} W$ and $U_\kappa = K * W_\kappa$. The latter is an exact radial heat solution. Put $X = L^1 \cap L^p$ with the sum norm. Splitting the Biot–Savart convolution at radius one gives

$$\|K * g\|_\infty \leq C_p \|g\|_X. \quad (14)$$

On the near part use Hölder and $|x|^{-1} \in L^{p'}(B_1)$, which holds because $p' < 2$. On the far part use its bounded kernel and the L^1 norm. This estimate requires no zero-circulation assumption.

The frozen full linear operator is $\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W$. Its sectors are those in (7). Choose integers $M, N \geq 1$, to be fixed below, and construct

$$f_{\text{app}} = \sum_{j=1}^M \sum_{\ell=0}^N \alpha^j \kappa^\ell f_{j,\ell}, \quad \omega_{\text{app}} = W_\kappa + f_{\text{app}}. \quad (15)$$

Set $f_{1,0}(0) = \Phi_\vartheta$ and all other initial coefficients zero. With $W_r(t) = t^r \Delta^r W / r!$, solve the triangular recursion

$$\begin{aligned} (\partial_t - \mathcal{L})f_{j,\ell} = & \mathbf{1}_{\ell \geq 1} \Delta f_{j,\ell-1} - \sum_{r=1}^{\ell} [(K * W_r) \cdot \nabla f_{j,\ell-r} + (K * f_{j,\ell-r}) \cdot \nabla W_r] \\ & - \sum_{\substack{j_1+j_2=j, j_1, j_2 \geq 1 \\ \ell_1+\ell_2=\ell}} (K * f_{j_1,\ell_1}) \cdot \nabla f_{j_2,\ell_2}. \end{aligned} \quad (16)$$

Every right-hand side uses earlier coefficients in the order $j + \ell$. They are smooth and supported in the fixed collars: Laplacians preserve the support of smooth flat functions; $\Delta^r W$, $r \geq 1$, has that support; and every advective factor differentiates such a supported function. The linear source is supported where W' is supported. Each coefficient has zero circulation, because each forcing is a Laplacian or divergence of an integrable vector field. Its frequencies belong to $4\mathbb{Z}$ and satisfy $|m| \leq 4j$.

Lemma 2.3 (Coefficient bounds and seeded residual). *For every finite derivative order k , the finitely many coefficients in (16) satisfy*

$$\|f_{j,0}(t)\|_{H^k} \leq C_{j,k} e^{j\sigma t}, \quad (17)$$

$$\|f_{j,\ell}(t)\|_{H^k} \leq C_{j,\ell,k} (1+t)^D e^{j\sigma t}, \quad \ell > 0. \quad (18)$$

There is a finite D , enlarged as needed, such that if $a(t) = \alpha e^{\sigma t} \leq \delta_0 \leq 1$ on $[0, H]$ and $\kappa(1+H)^D \leq c$, the residual

$$R_{\text{app}} = \partial_t \omega_{\text{app}} + (K * \omega_{\text{app}}) \cdot \nabla \omega_{\text{app}} - \kappa \Delta \omega_{\text{app}}$$

satisfies

$$\|R_{\text{app}}(t)\|_X \leq C[a(t)^{M+1} + \kappa^{N+1}(1+t)^D a(t)]. \quad (19)$$

All constants are uniform in ϑ .

Proof. The first coefficient is the exact growing-plus-decaying solution from (13). For $j \geq 2$, $\ell = 0$, induction and Sobolev multiplication bound its forcing by $Ce^{j\sigma t}$. Convolution against the growing exponent σ , or the remaining exponent $2\eta < \sigma$, is bounded by a constant times $e^{j\sigma t}$, because $j\sigma - \sigma \geq \sigma$. This proves (17) with no polynomial factor. For positive ℓ the same induction applies, with finite polynomial losses from $W_r(t)$ and possible resonance with the first growing mode. This proves (18). Derivative loss causes no infinite iteration: the coefficient set is finite and the profile is smooth in every order. At each induction step apply the preceding coefficient estimates in a sufficiently higher fixed order and the propagator lemma in the requested order. Compact support and zero circulation also control $K * f$ and its needed derivatives, including low frequencies. More explicitly $|\hat{f}(\xi)| \leq C|\xi|$ near zero, from its zero mass and finite first absolute moment, while the high-frequency multiplier has order minus one.

Expand the radial heat base only in its couplings to f_{app} . The integral Taylor formula and heat contraction in L^1, L^p give

$$G_N(t) := W_\kappa(t) - \sum_{r=0}^N \kappa^r W_r(t), \quad \|G_N(t)\|_X + \|\nabla G_N(t)\|_X \leq C\kappa^{N+1}t^{N+1}.$$

Its residual couplings obey

$$\|(K * G_N) \cdot \nabla f_{\text{app}} + (K * f_{\text{app}}) \cdot \nabla G_N\|_X \leq C\kappa^{N+1}(1+t)^D a(t)$$

by (14) and the coefficient bounds. These are the only noncompact factors. Every other coefficient is supported in one fixed bounded set, where its high Sobolev norms control the required L^1, L^p product norms. The recursion cancels exactly all monomials with $1 \leq j \leq M$, $0 \leq \ell \leq N$. Terms of seed order $j > M$ are bounded by Ca^{M+1} : the pure-seed bounds contain no time polynomial, and each positive-viscosity polynomial is absorbed after enlarging D and using $\kappa(1+H)^D \leq c$. Terms of viscosity order $\ell > N$, including the diffusion of $f_{j,N}$, have the second bound in (19); $a^j \leq a$ for $a \leq 1$ absorbs their higher seed powers. Crucially the exact radial heat base is part of (15), so no residual term without a seed factor occurs. Finally rotations of (13) change only unit complex phases; all bounds are uniform in ϑ . $\square$

2.4 Nonlinear closure in the actual vorticity norm

Let ω^κ be the NS solution from $W + \alpha\Phi_\vartheta$ and put $\zeta = \omega^\kappa - \omega_{\text{app}}$. Its exact equation, with zero initial data, is

$$\partial_t \zeta + (K * \omega^\kappa) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta) \cdot \nabla \omega_{\text{app}} - R_{\text{app}}. \quad (20)$$

Kato's inequality gives its L^1 estimate, and multiplication by $|\zeta|^{p-2}\zeta$ gives its L^p estimate; transport disappears by divergence freedom and diffusion is nonpositive in both inequalities. Using (14) and adding gives

$$\frac{d^+}{dt} \|\zeta\|_X \leq C_p \|\nabla \omega_{\text{app}}\|_X \|\zeta\|_X + \|R_{\text{app}}\|_X. \quad (21)$$

These inequalities follow first with smooth convex regularizations of the norm and compact cutoffs, then by their limits. All solutions here are smooth at positive time, have finite X norm, and are continuous in that norm at time zero, so the integrated inequality starts at zero. No derivative bound on the actual NS vorticity enters its coefficient.

We choose parameters without circular dependence. Fix

$$C_1 = C_p(\|\nabla W\|_X + 1), \quad r_1 = (C_1/\sigma - 1)_+,$$

before choosing expansion orders. Choose M with $(M+1)\sigma > C_1$ and N with

$$\frac{3}{2}(N+1) > K_* r_1. \quad (22)$$

Construct the finite coefficients, then fix a cap threshold δ_0 using the pure-seed coefficients alone and reduce the positive-viscosity threshold so that $\|\nabla f_{\text{app}}\|_X \leq 1$ whenever $a \leq \delta_0$ and $\kappa(1+H)^D$ is sufficiently small. This is possible by the finite coefficient bounds: make the pure-seed gradient at most $1/2$ by the cap choice, and the finitely many positive-viscosity contributions at most $1/2$ by the later viscosity threshold. Thus a later increase of N need not reduce a previously fixed pure-seed cap. The exact heat gradient obeys $\|\nabla W_\kappa\|_X \leq \|\nabla W\|_X$, so C_1 is the already fixed exponent in (21). The coefficient constants affect only the later cap threshold; they do not change the previous choice of M, N .

Convolving (19) against $e^{C_1(t-s)}$ gives the all-time estimate

$$\|\zeta(t)\|_X \leq Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)t}. \quad (23)$$

The first integral has denominator $(M+1)\sigma - C_1 > 0$. For the second use a bound by $e^{\max(C_1, \sigma)t}$ and an additional polynomial factor if the two exponents coincide. At $H = \sigma^{-1} \log(\delta/\alpha)$ this yields

$$\|\zeta(H)\|_X \leq C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \quad (24)$$

For the theorem's parameters, (22) implies that the second bound is $o(\delta_q)$: a strict positive power of q beats every fixed power of $\log(1/q)$. Also $\kappa_q(1 + H_q)^D \rightarrow 0$, $\delta_q \rightarrow 0$, and $\alpha_q/\delta_q \rightarrow 0$. The coefficient estimates and (23) therefore prove (2), including uniformity through the entire interval. They also give the actual nonradial density bound

$$\|(I - P_{\text{rad}})\omega^\kappa(t)\|_1 \leq Ca(t) + Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)t}. \quad (25)$$

Indeed radial averaging contracts L^1 , and the heat base is radial. This estimate concerns the actual density; it is not inferred from a contour coordinate.

Similarly (14), the coefficient bounds and (23) imply

$$\int_0^H \|K * (\omega^\kappa - W_\kappa)\|_\infty dt \leq C\delta + C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \quad (26)$$

To integrate the approximate terms use $\int_0^H a(t)^j dt \leq \delta^j/(j\sigma)$. The error terms grow at a fixed positive exponential rate; their nondecreasing polynomial factor can be evaluated at H , so there is no additional factor H . This proves (3).

2.5 Whole-moment control and the cap

We verify the tails needed to remove the moment cutoff. Write $u^\kappa = U_\kappa + v^\kappa$, with U_κ tangential, and $B_H = \int_0^H \|v^\kappa\|_\infty dt$. By (26), $B_{H_q} = O(\delta_q)$. Let $d(x) = \sqrt{1 + |x|^2}$. Then $|\nabla d| \leq 1$, $\Delta d \leq 2$, and $U_\kappa \cdot \nabla d = 0$. Kato's inequality with $e^{\lambda d}$ gives, for every $\lambda > 0$,

$$\int e^{\lambda d(x)} |\omega^\kappa(t, x)| dx \leq m \exp\{\lambda d_0 + \lambda B_H + 2\kappa\lambda t + \kappa\lambda^2 t\}, \quad (27)$$

where $m = \|\omega^\kappa(0)\|_1$ and $d_0 = \sqrt{1 + (1 + \epsilon)^2}$. No initial exponential tail is presupposed beyond compact support. To justify the unbounded test use

$$F_L(d) = \frac{e^{\lambda d}}{1 + e^{\lambda d}/L}.$$

It has $F'_L \leq \lambda F_L$ and $F''_L \leq \lambda^2 F_L$; therefore $\Delta F_L(d) \leq (\lambda^2 + 2\lambda)F_L(d)$. For fixed L add a compact cutoff and let its radius tend to infinity. Finite mass and bounded drift make the cutoff errors vanish. Gronwall then gives (27) uniformly in L , and monotone convergence removes L . The drift is bounded here by (14) and the scalar L^1, L^p contractions.

Choose a fixed $R > 1 + \epsilon$ with $d_R - d_0 > 2c_* > 0$. For small q , $B_{H_q} < c_*/2$ and $\kappa_q H_q \rightarrow 0$. Exponential Markov and optimization of (27) give

$$\int_{|x|>s} |\omega^\kappa(t, x)| dx \leq m \exp\left\{-\frac{(d_s - d_0 - B_H - 2\kappa t)^2}{4\kappa t}\right\}, \quad s \geq R.$$

Integrating this mass bound in s gives

$$\sup_{0 \leq t \leq H_q} \int_{|x|>R} |x|^4 |\omega^{q,\vartheta}(t, x)| dx \leq C e^{-c/(\kappa_q H_q)}. \quad (28)$$

For example the weighted tail equals R^4 times the mass tail plus $4 \int_R^\infty s^3$ times the lower-radius mass tail. The displayed Gaussian estimate bounds both terms after reducing c . This also proves finiteness of the required absolute moments.

Let ξ be a smooth radial cutoff equal to one on B_R and supported in B_{R+1} . Every formal nonradial coefficient is supported in B_R ; the exact heat base pairs to zero with both $\bar{z}^4$ and $\xi\bar{z}^4$, since it is radial. At the cap the first coefficient has growing moment

$$\alpha e^{\lambda_\epsilon H} e^{i\vartheta} b_\epsilon = \delta e^{i(\vartheta+\tau_\epsilon H)} b_\epsilon$$

and stable moment of size $C\alpha^2/\delta$. The remaining pure-seed coefficients have compactly tested moment $O(\delta^2)$; positive-viscosity coefficients have moment $O(\kappa(1+H)^D\delta)$. The actual error pairs against $\xi\bar{z}^4$ with bound $\|\xi\bar{z}^4\|_\infty\|\zeta(H)\|_1$. Using (24) and (28) proves

$$\begin{aligned} |B_4(\omega^\kappa(H)) - \delta e^{i(\vartheta+\tau_\epsilon H)} b_\epsilon| &\leq C\delta^2 + C\alpha^2/\delta + C\kappa(1+H)^D\delta + C\delta^{M+1} \\ &\quad + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1} + Ce^{-c/(\kappa H)}. \end{aligned}$$

For $\delta = \delta_q$, every bound divided by δ_q tends to zero. This proves (4) and completes the theorem.

Remark 2.4 (Scaling and scope). For a physical colour $A\omega(x/R)$ and viscosity $\nu = \kappa_q AR^2$, the cap time is H_q/A , the density bound is $CAR^2\delta_q$, and the cap moment is $AR^6\delta_q e^{i(\vartheta+\tau_\epsilon H_q)} b_\epsilon + o(AR^6\delta_q)$. The integrated perturbation velocity bound is $CR\delta_q$. These normalized constants do not depend on the scale or amplitude. They apply to an isolated colour. A foreign-colour field introduces additional terms in the coefficient equations and the error equation; no estimate for those terms follows merely from this scaling. In particular, (25) cannot be asserted for an interacting stack by substituting its contour amplitudes into this isolated theorem.

3 The canonical ancient profile of the isolated smooth activation

We retain the fixed profile, eigenvalue and seeds constructed in Section 2. Abbreviate them to $W = W_\epsilon$, $\lambda = \sigma + i\tau = \lambda_\epsilon$, $b = b_\epsilon \neq 0$, and Φ_ϑ . In this section the cap $\delta > 0$ is fixed and sufficiently small. It does not decrease with the vanishing parameter. An additional Sobolev estimate permits identification of a unique isolated ancient profile. This identification requires no convergence of an infinite formal series. Here δ specifies the ancient growing amplitude and the linear preparation level. It does not assert that $s = 0$ is the exact first hitting time of $|\mathcal{A}(\Omega_\delta)| = \delta$; at that time $\mathcal{A}(\Omega_\delta(0)) = \delta + O(\delta^2)$.

Theorem 3.1 (The isolated ancient limit). *Fix an integer $k \geq 3$ and $2 < p < \infty$. There exists $\delta_* > 0$ such that, for $0 < \delta < \delta_*$, there is a real quarter-turn invariant Euler solution $\Omega_\delta(s)$ on $-\infty < s < \infty$, with zero circulation, having the following properties. For every fixed $K_* > 0$, put*

$$\alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad H_q = \sigma^{-1} \log(\delta/\alpha_q), \quad \vartheta_q = -\tau H_q.$$

Let ω^q be the isolated Navier–Stokes solution with viscosity κ_q and initial vorticity $W + \alpha_q \Phi_{\vartheta_q}$. For every $J < \infty$ and every $0 < \varepsilon < k$,

$$\omega^q(H_q + \cdot) \longrightarrow \Omega_\delta \quad \text{in } C([-J, 0]; L^1 \cap L^p) \text{ and in } C([-J, 0]; H^{k-\varepsilon}) \quad \text{as } q \downarrow 0. \quad (29)$$

The parameter δ_ and the limiting profile can be chosen independently of K_* ; the threshold for small q may depend on K_* . On $s \leq 0$, Ω_δ is supported in one fixed ball and has a uniform H^k bound. Its whole fourth harmonic moment satisfies*

$$B_4(\Omega_\delta(s)) = \delta b e^{\lambda s} + O(\delta^2 e^{2\sigma s}) \quad (s \leq 0). \quad (30)$$

For $s \geq 0$ it satisfies

$$|B_4(\Omega_\delta(s))| \leq C_\delta (1+s)^4. \quad (31)$$

These assertions concern the isolated annulus.

3.1 A high Sobolev error estimate without derivative loss

Fix once and for all a radius $R > 1 + \epsilon$ with a positive gap in $d(x) = \sqrt{1 + |x|^2}$, for example $R = 3 + \epsilon$. This radius and the corresponding compact-support norm constants are fixed before the energy exponent or expansion orders. The later cap smallness condition will make the integrated perturbation drift smaller than this fixed gap. For this proof let

$$\|g\|_{Z_k} = \|g\|_1 + \|g\|_{H^k}.$$

The finite expansion from (15) and (16) can be constructed in every finite derivative order. In particular, the residual estimate holds in Z_k . A useful version separates its pure-seed constant from constants depending on the viscosity order:

$$\|R_{\text{app}}(t)\|_{Z_k} \leq [C_M + C_{M,N}\kappa(1+t)^D]a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^D a(t), \quad a(t) = \alpha e^{\sigma t} \leq 1. \quad (32)$$

Here C_M is independent of N . To verify the additional norm and this separation, the pure-seed products are estimated by their compactly supported coefficient Sobolev bounds. Every other omitted seed-order product contains at least one positive power of κ , and its finite time polynomial is incorporated in D . The noncompact heat remainder products are

$$(K * G_N) \cdot \nabla f_{\text{app}} + (K * f_{\text{app}}) \cdot \nabla G_N.$$

The heat Taylor formula gives

$$\|K * G_N\|_{W^{k,\infty}} + \|\nabla G_N\|_1 + \|\nabla G_N\|_{H^k} \leq C_{k,N}\kappa^{N+1}t^{N+1}.$$

For the first term use heat Taylor on the smooth initial velocity $K * W$ and contraction of each bounded derivative. For the other terms use heat contraction on $\nabla \Delta^{N+1}W$ in L^1 and H^k . The coefficient bounds give $\|K * f_{\text{app}}\|_{W^{k,\infty}} + \|\nabla f_{\text{app}}\|_1 + \|\nabla f_{\text{app}}\|_{H^k} \leq Ca(t)$ after the positive-viscosity threshold is made small. Leibniz's rule proves the claimed seeded remainder in both norms.

Let $\zeta = \omega^\kappa - \omega_{\text{app}}$ and $u_{\text{app}} = K * \omega_{\text{app}}$. Rewriting (20) gives

$$\partial_t \zeta + u_{\text{app}} \cdot \nabla \zeta + (K * \zeta) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta) \cdot \nabla \omega_{\text{app}} - R_{\text{app}}. \quad (33)$$

We record the estimates that avoid a quasilinear derivative loss. Sobolev embedding and the near-far kernel split give, for $k \geq 3$,

$$\|K * g\|_\infty \leq C_k(\|g\|_1 + \|g\|_{H^k}), \quad (34)$$

$$\|\nabla K * g\|_{H^j} \leq C_j \|g\|_{H^j}, \quad j \geq 0. \quad (35)$$

The second is Plancherel's theorem for a bounded order-zero Fourier multiplier. In particular $\|\nabla K * g\|_\infty \leq C_k \|g\|_{H^k}$.

For each integer derivative order up to k , the undifferentiated transport term has zero energy pairing. The commutator with $(K * \zeta) \cdot \nabla$ is bounded by

$$C_k(\|\nabla K * \zeta\|_\infty \|\zeta\|_{H^k} + \|\nabla \zeta\|_\infty \|\nabla K * \zeta\|_{H^{k-1}}) \leq C_k \|\zeta\|_{H^k}^2.$$

This commutator inequality follows from Leibniz's rule: the endpoint terms put the first derivative in L^∞ and the last derivative in L^2 , and the intermediate terms follow by Sobolev interpolation

in dimension two. The embeddings for both displayed endpoint terms hold because $k > 2$. No derivative beyond order k of ζ occurs. Similarly,

$$\|(K * \zeta) \cdot \nabla \omega_{\text{app}}\|_{H^k} \leq C \left(\|\nabla \omega_{\text{app}}\|_{H^k} + \sum_{j=1}^k \|\nabla^j \omega_{\text{app}}\|_{\infty} \right) \|\zeta\|_{Z_k}.$$

For the term where no derivative lands on $K * \zeta$, use (34); for every term with a positive number of such derivatives use (35) in L^2 and put the corresponding derivative of the smooth approximation in L^∞ . The H^k transport estimate with u_{app} uses only its fixed $W^{k,\infty}$ bounds. Its coefficients are bounded, independently of α, κ, H , by fixed norms of W plus one, once the approximate perturbation is made small in the finitely many required norms.

Together with Kato's L^1 inequality this gives, for $Y = \|\zeta\|_{Z_k}$,

$$\frac{d^+ Y}{dt} \leq C_k Y + C_k Y^2 + \|R_{\text{app}}\|_{Z_k}, \quad Y(0) = 0. \quad (36)$$

Diffusion has a nonpositive energy contribution. The constant C_k can be fixed before M, N : the pure-seed approximation is made small by choosing its cap after M , and each positive-viscosity coefficient is made small by choosing its viscosity threshold after N .

Set $E_k = 2C_k$, increasing it if necessary to dominate $C_p(\|\nabla W\|_{L^1 \cap L^p} + 1)$ and the compact-support $L^1 \cap L^p$ stability constant used below. Choose one finite M with

$$(M+1)\sigma > E_k. \quad (37)$$

After its pure-seed coefficients are known, choose $\delta_* > 0$ small enough for all their required norms and their pure-seed remainder. For each subsequently prescribed K_* choose N so that, with $r_k = (E_k/\sigma - 1)_+$,

$$\frac{3}{2}(N+1) > K_* r_k. \quad (38)$$

The N -dependent coefficients are controlled by decreasing the threshold for q , not by changing δ_* . In particular arrange $C_{M,N}\kappa(1+H)^D \leq 1$ in (32). On a bootstrap interval $Y \leq 1$, (36) now gives

$$Y(t) \leq C_M a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^{D+1} a(t) e^{(E_k - \sigma)t}. \quad (39)$$

At the cap, the second term is

$$C_{M,N}\kappa^{N+1}(1+H)^{D+1} \delta(\delta/\alpha)^{r_k} = o(1)$$

in the prescribed polynomial window. The first is smaller than $1/2$ after reducing δ_* . Continuity contradicts a first bootstrap exit, so the estimate holds through the full cap interval. It also yields a uniform H^k bound for the actual vorticity. This bound is obtained from the actual error equation, rather than assumed from the formal approximation.

3.2 Finite homological coefficients and compactness

For a real angle θ define the rotation

$$(\mathcal{T}_\theta f)(r, \varphi) = f(r, \varphi + \theta/4), \quad \mathcal{L}_{\text{rot}} = \mathcal{L} - (\tau/4)\partial_\varphi.$$

It acts isometrically on every radial or full-plane Sobolev norm, commutes with the radial linearization, and respects Biot–Savart advection. The pure-seed coefficient $f_{j,0}$ with seed phase zero has a limit

$$e^{-j\sigma t} \mathcal{T}_{-\tau t} f_{j,0}(t) \longrightarrow F_j \quad \text{in every finite } H^m \text{ as } t \rightarrow \infty. \quad (40)$$

For $j = 1$ the decaying part is $O(e^{-2\sigma t})$ after this normalization, so

$$F_1 = f_\epsilon(r)e^{4i\varphi} + \text{complex conjugate}.$$

For $j \geq 2$, put $g_j = e^{-j\sigma t}\mathcal{T}_{-\tau t}f_{j,0}$. Its equation is

$$\partial_t g_j = (\mathcal{L}_{\text{rot}} - j\sigma)g_j - \sum_{j_1+j_2=j} (K * g_{j_1}) \cdot \nabla g_{j_2}.$$

On its finite set of sectors, the homogeneous propagator decays at least as $Ce^{-(j-1)\sigma t}$ by (8). Induction, the coefficient bounds, and dominated convergence in Duhamel's integral prove (40) and the finite homological relations

$$F_j = -(j\sigma - \mathcal{L}_{\text{rot}})^{-1} \sum_{j_1+j_2=j} (K * F_{j_1}) \cdot \nabla F_{j_2}, \quad 2 \leq j \leq M. \quad (41)$$

The inverse is the integral of the exponentially decaying propagator. All these functions are smooth and supported in the fixed collars. Only the finitely many sectors in $4\mathbb{Z}$ with $|m| \leq 4M$ are used. No all-mode Sobolev spectral-growth assertion is needed.

Define the fixed polynomial ancient profile

$$P_M(s) = W + \sum_{j=1}^M \delta^j e^{j\sigma s} \mathcal{T}_{\tau s} F_j, \quad s \leq 0. \quad (42)$$

For the phase $\vartheta_q = -\tau H_q$, (40) shows that every pure-seed term of the cap-shifted approximation converges to the corresponding term in (42). The positive-viscosity terms tend to zero because $\kappa_q(1+H_q)^D \rightarrow 0$, while $e^{\kappa_q(H_q+s)\Delta} W \rightarrow W$. For each fixed $s \leq 0$, (39) gives the asymptotic error $C_M(\delta e^{\sigma s})^{M+1}$, since its viscosity remainder tends to zero by (38).

We justify actual compactness separately. The kernel bound and (39), integrated as in (26), give

$$\int_0^{H_q} \|K * (\omega^q(t) - W_{\kappa_q}(t))\|_\infty dt \leq C_M \delta + o(1).$$

Choose δ_* so this is less than the positive radial gap fixed before the expansion orders. The exponential Kato proof of (28), with this fixed cap and the already fixed radius R , gives

$$\sup_{0 \leq t \leq H_q} \int_{|x| > R} |x|^4 |\omega^q(t, x)| dx \leq C e^{-c/(\kappa_q H_q)}. \quad (43)$$

Initial velocities have uniformly bounded finite energy, since the initial vorticities are compactly supported and have zero circulation. The NS velocity energy inequality and the uniform vorticity H^k bound therefore bound the velocities in H^{k+1} . The equation bounds $\partial_t \omega^q$ uniformly in H^{k-2} on the cap interval.

On each shifted interval $[-J, 0]$, Rellich compactness and interpolation give local compactness in $C(H^{k-\varepsilon})$. The tail bound and the uniform H^k , hence L^∞ , bound give global L^2 tightness. Interpolation applied to a smooth exterior cutoff then gives global $H^{k-\varepsilon}$ tightness; the cutoff is chosen outside the same fixed B_R . Thus compactness holds globally in that norm. The mass tail gives global L^1 tightness, and local convergence plus uniform L^∞ bounds gives compactness in $C(L^1 \cap L^p)$ for every finite p . A diagonal subsequence over J gives an ancient Euler solution: the transport term passes by the stated strong convergence, and $\kappa_q \Delta \omega^q \rightarrow 0$ on every compact

shifted interval. The solution is supported in B_R on $s \leq 0$, since the right side of (43) tends to zero.

By lower semicontinuity of the H^k norm and the strong L^1 convergence, every such limit satisfies, for all $s \leq 0$,

$$\|\Omega(s) - P_M(s)\|_{Z_k} \leq C_M(\delta e^{\sigma s})^{M+1}. \quad (44)$$

One may first take rational times; continuity in the weaker compactness norm and weak H^k continuity extend the bound to every time.

3.3 Uniqueness of the ancient subsequential limit

We use a norm in which no extra derivative of a comparison orbit is required. If Ω_1, Ω_2 are two ancient subsequential limits, their compact supports and uniform H^k bounds imply a uniform bound for $\|\nabla \Omega_j\|_{L^1 \cap L^p}$. Indeed compact support controls the L^1 norm by the L^2 derivative norm, and H^k , $k \geq 3$, controls every finite L^p derivative norm. Their perturbations from W are small in these norms after reducing δ_* , so the stability coefficient is bounded by the fixed constant dominated by E_k above.

For $D = \Omega_1 - \Omega_2$ the exact Euler equation is

$$\partial_s D + (K * \Omega_1) \cdot \nabla D = -(K * D) \cdot \nabla \Omega_2.$$

The same Kato and L^p argument as (21) gives, for $s_0 \leq s_1 \leq 0$,

$$\|D(s_1)\|_{L^1 \cap L^p} \leq e^{E_k(s_1 - s_0)} \|D(s_0)\|_{L^1 \cap L^p}.$$

Both limits have the same polynomial (42). Using (44) at $s_0 = -T$ yields

$$\|D(0)\|_{L^1 \cap L^p} \leq C_M \delta^{M+1} e^{-(M+1)\sigma - E_k T} \longrightarrow 0.$$

The same comparison with any fixed endpoint $s_1 \leq 0$ proves equality throughout the ancient interval. The strict inequality (37) uses one finite M only. Thus every subsequence has the same limit, proving the full convergence in (29). Repeating the argument for any different K_* gives the same limit; only its permissible viscosity expansion order and small- q threshold change.

For clarity, a direct H^k difference stability estimate would require an H^{k+1} bound on the comparison vorticity because it differentiates $(K * D) \cdot \nabla \Omega_2$ k times. The $L^1 \cap L^p$ comparison above deliberately avoids this extra hypothesis. The high Sobolev estimate is used for compactness and for the available first derivatives.

3.4 Forward extension and the whole-moment asymptotics

The ancient limit has a smooth enough, compactly supported initial state at zero. The classical smooth two-dimensional Euler construction extends it uniquely to all $s \geq 0$. We recall the bounds behind global continuation. Its transported vorticity has conserved L^1 and L^∞ norms. The near-far Biot–Savart split gives a time-independent velocity bound. Splitting the gradient kernel at a small radius and using its cancellation gives the logarithmic estimate

$$\|\nabla u\|_\infty \leq C_{\|\omega\|_1, \|\omega\|_\infty} [1 + \log(2 + \|u\|_{H^{k+1}})].$$

The near part uses the Hölder norm of ω supplied by H^k ; the intermediate annuli give the logarithm, and the far part uses L^1 . Choosing the small radius inversely proportional to a fixed

positive power of $2 + \|u\|_{H^{k+1}}$ gives the displayed estimate. The velocity Sobolev energy inequality $\frac{d}{ds} \|u\|_{H^{k+1}} \leq C \|\nabla u\|_\infty \|u\|_{H^{k+1}}$ then prevents finite-time blowup. The local smooth solution is obtained by regularization and these same energy bounds; their difference energy bound gives uniqueness. Finite energy at zero follows from compact support and zero circulation, so these velocity norms are legitimate.

The bounded velocity transports its support into B_{R+Vs} . Consequently

$$\int |x|^4 |\Omega_\delta(s, x)| dx \leq \|\Omega_\delta(0)\|_1 (R + Vs)^4 \quad (s \geq 0),$$

which proves (31). On $s \leq 0$, its support is in the fixed B_R . Pairing (44) with $\bar{z}^4$ is therefore bounded by its L^1 error times R^4 . The first homological coefficient has moment b , rotation multiplies it by $e^{i\tau s}$, and every higher term is $O(\delta^2 e^{2\sigma s})$. This proves (30) and completes the theorem.

Corollary 3.2 (Every fixed source window, including after the cap). *For the isolated family in Theorem 3.1, every fixed $J < \infty$ satisfies*

$$\begin{aligned} \omega^q(H_q + \cdot) &\longrightarrow \Omega_\delta \quad \text{in } C([-J, J]; L^1 \cap L^p), \\ \sup_{|s| \leq J} |B_4(\omega^q(H_q + s)) - B_4(\Omega_\delta(s))| &\longrightarrow 0. \end{aligned}$$

Write $B_q(s) = B_4(\omega^q(H_q + s))$ and $B_\delta(s) = B_4(\Omega_\delta(s))$. For fixed $S, U < \infty$ and $a \in \mathbb{R}$, its source moment therefore satisfies

$$\int_{-S}^U e^{-as} |B_q(s) - B_\delta(s)| ds \longrightarrow 0.$$

Proof. The negative window follows from (29) and (43). On $0 \leq s \leq J$, let $\zeta_q(s) = \omega^q(H_q + s) - \Omega_\delta(s)$. Its exact equation is

$$\partial_s \zeta_q + (K * \omega^q(H_q + s)) \cdot \nabla \zeta_q = \kappa_q \Delta \zeta_q - (K * \zeta_q) \cdot \nabla \Omega_\delta + \kappa_q \Delta \Omega_\delta.$$

The global Euler extension has bounded H^k norm and compact support on this fixed interval. Since $k \geq 3$, its second derivatives belong to H^1 , which embeds in every finite L^p in dimension two; compact support also bounds their L^1 norms. Thus $\|\nabla \Omega_\delta\|_{L^1 \cap L^p} + \|\Delta \Omega_\delta\|_{L^1 \cap L^p} \leq C_J$. Kato's inequality, the L^p inequality and (14) give

$$\sup_{0 \leq s \leq J} \|\zeta_q(s)\|_{L^1 \cap L^p} \leq e^{C_J J} (\|\zeta_q(0)\|_{L^1 \cap L^p} + C_J \kappa_q) \longrightarrow 0.$$

No actual NS derivative bound is used in this comparison.

We remove the moment cutoff separately. Scalar norm contraction gives one uniform post-cap velocity bound V . Choose a fixed radius $R' > R + VJ + d$, also enclosing the Euler support on $[0, J]$, with $d > 0$. The positive post-cap propagator has a forward diffusion with $|X_s - y| \leq VJ + \sqrt{2\kappa_q} |B_s|$. For $|y| \leq R$, its terminal fourth moment outside $B_{R'}$ is at most $C_J e^{-c_J/\kappa_q}$ by Gaussian endpoint moments. For unrestricted y its full fourth moment is at most $C_J(1 + |y|^4)$. The positive domination of the signed solution by the propagation of $|\omega^q(H_q)|$ therefore gives

$$\sup_{0 \leq s \leq J} \int_{|x| > R'} |x|^4 |\omega^q(H_q + s, x)| dx \leq C_J e^{-c_J/\kappa_q} + C_J \int_{|y| > R} (1 + |y|^4) |\omega^q(H_q, y)| dy \longrightarrow 0$$

by (43). A compact radial cutoff of $\bar{z}^4$, the just-proved L^1 convergence, and this tail prove uniform whole-moment convergence. On a fixed weighted time window the weight is bounded, so its final assertion follows by integration. $\square$

Corollary 3.3 (A nonzero bilateral discounted moment). *Define*

$$\mathcal{F}_\delta(z) = \int_{-\infty}^{\infty} e^{-zs} B_4(\Omega_\delta(s)) ds.$$

It is analytic and is not identically zero on $0 < \Re z < \sigma$. More precisely it extends meromorphically to $0 < \Re z < 2\sigma$ as

$$\mathcal{F}_\delta(z) = \frac{\delta b}{\lambda - z} + \mathcal{G}_\delta(z), \quad (45)$$

where $\mathcal{G}_\delta$ is analytic on that larger strip. For every fixed real c , the exceptional set

$$\{\beta > 1 : \mathcal{F}_\delta((\sigma - ic)/\beta) = 0\}$$

is discrete in $(1, \infty)$. In particular one can select arbitrarily large β outside it, including for $c = 1/2$.

Proof. The negative-time estimate (30) and the positive-time polynomial bound give absolute, locally uniform convergence on $0 < \Re z < \sigma$, including after any fixed number of z derivatives. Subtract the leading negative-time exponential. Its integral is

$$\int_{-\infty}^0 \delta b e^{(\lambda - z)s} ds = \frac{\delta b}{\lambda - z}.$$

The remaining negative-time integral is analytic for $\Re z < 2\sigma$, and the positive-time integral is analytic for $\Re z > 0$. This proves (45). Its pole at $z = \lambda$ has nonzero coefficient, so the transform cannot be identically zero on the smaller strip. Its zeros there are isolated. The curve $\beta \mapsto (\sigma - ic)/\beta$ is nonconstant, lies in that strip for $\beta > 1$, and has no interior accumulation of its zeros. Any terminal interval of large β contains values outside the zero set, proving the last assertion. Accumulation as $\beta \rightarrow \infty$ is possible at the boundary $z = 0$ and is not excluded or needed. $\square$

Corollary 3.4 (Changing the canonical cap). *If $0 < \delta' < \delta < \delta_*$ and $s_0 = \sigma^{-1} \log(\delta'/\delta)$, then*

$$\Omega_{\delta'}(s) = \mathcal{T}_{-\tau s_0} \Omega_\delta(s + s_0) \quad (-\infty < s < \infty), \quad \mathcal{F}_{\delta'}(z) = e^{(z - i\tau)s_0} \mathcal{F}_\delta(z).$$

Thus the zero set of the transform is independent of the canonical cap. At $z = (\sigma - ic)/\beta$ its modulus scales by $(\delta'/\delta)^{1/\beta}$.

Proof. The time-translated and rotated orbit has the polynomial (42) with δ replaced by δ' , since $\delta e^{\sigma s_0} = \delta'$. It has the same order- $M + 1$ remainder. The ancient uniqueness argument therefore identifies it with $\Omega_{\delta'}$ on negative times, and forward Euler uniqueness extends the identity to every time. Rotation $\mathcal{T}_\theta$ multiplies B_4 by $e^{i\theta}$. Substituting $v = s + s_0$ into the absolutely convergent bilateral transform gives the stated exponential factor, which never vanishes. Its modulus at the displayed point is $e^{\sigma s_0/\beta}$. $\square$

Remark 3.5 (Scope). The ancient profile, its cap compactness and its nonzero transform have been proved for one isolated fixed smooth annulus. They can supply an isolated source profile and a possible discounted transfer coefficient. This corollary does not identify that coefficient in a coupled receiver, does not estimate the effect of all other colours, and does not construct distinct vanishing-viscosity accumulation points from one fixed datum.

4 The smooth receiver's protected coordinate and transfer coefficient

Throughout this section $W = W_\epsilon$ is the one fixed profile of Section 2, $\lambda = \sigma + i\tau$ its growing fourth-sector eigenvalue, and $f_+ = f_\epsilon$ its normalized radial eigenfunction. All angular conventions are unchanged.

Lemma 4.1 (A global growing-coordinate test). *There is a smooth bounded complex test*

$$T(x) = a(r)e^{-4i\theta}$$

whose derivatives of every fixed order are bounded, such that

$$\mathcal{A}(g) = \int_{\mathbb{R}^2} T(x)g(x) dx, \quad \mathcal{A}(f_+(r)e^{4i\theta}) = 1, \quad \mathcal{A}(\overline{f_+(r)}e^{4i\theta}) = 0,$$

and $\mathcal{A}(\mathcal{L}g) = \lambda\mathcal{A}(g)$ for smooth integrable perturbations for which the pairings are defined. Near the origin T is a constant multiple of $\bar{z}^4$; outside a fixed ball it is a constant multiple of z^{-4} . In particular, ΔT is compactly supported away from the origin.

Proof. On a fixed radial interval containing the derivative collars, the rank-one Riesz projection in the positive fourth sector has form $P_+g = f_+\ell(g)$ with $\ell(f_+) = 1$ and $\ell(\mathcal{L}_4g) = \lambda\ell(g)$. Represent this continuous L^2 functional by a bilinear density $2\pi a(r)r dr$; complex conjugation is included in the representing density rather than in the pairing. Substitution of (7) gives

$$(\lambda + 4i\Omega_\epsilon(s))a(s) = -\frac{i}{2} \int_0^\infty a(r)W'_\epsilon(r)h_4(r, s) dr. \quad (46)$$

On the collars, the radial Green equation and this nonvanishing denominator bootstrap a to smoothness. Extend it to every $s > 0$ by the right side of (46) divided by its denominator. This agrees with the original density where W' is supported, so it still satisfies the equation everywhere. Integration of the Green kernel and Fubini then verify the global eigenfunctional identity.

For s smaller than every derivative collar, the numerator is a constant times s^4 and Ω_ϵ is the constant core angular velocity $-1/5$. Thus $a(s) = Cs^4$ there and $T = C\bar{z}^4$. Beyond the profile support the numerator is a constant times s^{-4} and $\Omega_\epsilon = 0$ by zero circulation. Thus $a(s) = C's^{-4}$ and $T = C'z^{-4}$. These two extensions are harmonic; on the remaining fixed annulus the test is smooth. This proves all the regularity and Laplacian assertions. Normalize by $2\pi \int a f_+ r dr = 1$. Since the stable eigenvalue is $-\bar{\lambda} \neq \lambda$, the eigenfunctional identity annihilates its eigenfunction. The angular test kills every sector other than $+4$. $\square$

Lemma 4.2 (Exact inverse-quintic forcing coefficient). *Let a source be real, compactly supported, quarter-turn symmetric and of zero circulation. At points outside its support, its leading complex velocity is*

$$U(z) = \frac{iB_4}{2\pi\bar{z}^5} + \frac{iB_8}{2\pi\bar{z}^9} + \dots$$

The first term forces the receiver's positive fourth vorticity sector by $B_4F_\epsilon(r)e^{4i\theta}$, where

$$F_\epsilon(r) = -\frac{iW'_\epsilon(r)}{4\pi r^5}.$$

Its growing projection is $\gamma_\epsilon B_4$, with $\gamma_\epsilon = \mathcal{A}(F_\epsilon e^{4i\theta})$. For one fixed sufficiently small smoothing length,

$$\gamma_\epsilon \longrightarrow \gamma_0 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0 \quad (\epsilon \downarrow 0). \quad (47)$$

Proof. In complex notation Biot–Savart is $U(z) = \frac{i}{2\pi} \int (\bar{z} - \bar{y})^{-1} w(y) dy$. The convergent exterior geometric series gives the stated expansion; quarter-turn symmetry eliminates every exponent not divisible by four, and zero circulation eliminates the constant moment. For its first term,

$$u_r = \Re(Ue^{-i\theta}) = \frac{iB_4}{4\pi r^5} e^{4i\theta} + \text{complex conjugate}.$$

The linearized forcing is $-u_r W'$, giving F_ϵ .

We check nonvanishing without assuming an adjoint convergence theorem. The inhomogeneous resolvent Schur reduction in the proof of Lemma 2.2 applies to this forcing. Its natural radial measure has limit

$$F_\epsilon(r) r dr \rightarrow \sum_{i=1}^2 \frac{ic_i}{4\pi r_i^4} \delta_{r_i}.$$

Thus in interface variables its forcing vector tends to $h = (i/(4\pi\rho^6), i/(4\pi))$. The growing dual of M_4 , normalized on $(1, -i\rho)$, is $\ell_0(q) = (q_1 + iq_2/\rho)/2$. Its pairing with h is γ_0 in (47). More explicitly, on a small fixed contour around the growing root the inhomogeneous Schur equations converge uniformly to $(z - M_4)q = h$. Integrating their resolvent in z gives convergence of the rank-one projected forcing coefficient. The eigenfunction normalization agrees with that in Section 2. This proves the limit and nonvanishing for a fixed sufficiently small ϵ . $\square$

The direction of this interaction matters. A smaller source acts on the larger receiver by the inverse-quintic exterior field above. A larger annulus acts in its interior through a cubic harmonic field. They are different coefficients and cannot be interchanged.

For comparison, an exterior annulus's interior harmonic field $U(z) = c_3 \bar{z}^3$ has positive fourth radial coefficient $\bar{c}_3 r^3/2$. Its frozen-base forcing is $-\bar{c}_3 r^3 W'/2$, with growing projection $\gamma_\epsilon^{\text{out}} \bar{c}_3$. The same resolvent limit gives

$$\gamma_\epsilon^{\text{out}} \rightarrow (\rho^2 + i/\rho)/4 \neq 0.$$

Indeed its interface forcing vector is $(\rho^2/2, 1/2)$, paired against the same growing dual. Thus older cubic fields can genuinely seed the growing coordinate; their size must be bounded, rather than discarded by symmetry.

Lemma 4.3 (Duhamel coefficient for a child source). *Let $A_c = \beta A_r$, $R_c = q R_r$, and use receiver time $s = A_r t$. Suppose, in a frame removing all common solid rotation, that the source's normalized fourth moment is $B(u)$, with child time $u = \beta s - s_*$. The receiver core has angular velocity $-A_r/5$; this rotates the child's moment by $e^{4is/5}$ in the receiver frame. The ideal linear growing receiver coefficient obeys*

$$a'(s) = \lambda a(s) + \beta q^6 \gamma_\epsilon e^{4is/5} B(\beta s - s_*).$$

For the response with no independent growing coefficient at $s = -\infty$,

$$e^{-\lambda s} a(s) = q^6 \gamma_\epsilon e^{-\mu_\beta s_*} \int_{-\infty}^{\beta s - s_*} e^{-\mu_\beta u} B(u) du, \quad \mu_\beta = \frac{\sigma + i(\tau - 4/5)}{\beta}. \quad (48)$$

This integral exists whenever $B(u) = O(e^{\sigma u})$ at $-\infty$ and has polynomial growth at $+\infty$, provided $\beta > 1$.

Proof. The dimensional fourth moment is $A_c R_c^6 B$. Dividing the child velocity by $A_r R_r$ leaves the factor βq^6 , so Lemma 4.2 gives the forcing magnitude. Rotation through angle Ωt multiplies B_4 by $e^{-4i\Omega t}$. The parent constant core has vorticity $-2A_r/5$, hence angular velocity $-A_r/5$,

which gives the factor $e^{4is/5}$. Multiplication by the integrating factor $e^{-\lambda s}$ and the substitution $u = \beta s - s_*$ give (48). Its negative tail has exponential exponent $\sigma - \sigma/\beta > 0$; its positive tail has discount $\sigma/\beta > 0$. $\square$

Lemma 4.4 (A nonzero bilateral source transform from an ancient orbit). *Suppose a global Euler source trajectory has a fourth moment $B(u)$ with*

$$B(u) = cb_\epsilon e^{\lambda u} + O(e^{2\sigma u}) \quad (u \rightarrow -\infty), \quad c \neq 0,$$

and $|B(u)| \leq C(1+u)^4$ for $u \geq 0$. Define

$$\mathcal{F}(z) = \int_{\mathbb{R}} e^{-zu} B(u) du, \quad 0 < \Re z < \sigma.$$

Then $\mathcal{F}$ is analytic and not identically zero. For every finite interval $[B_*, B_* + 1] \subset (1, \infty)$ there is a choice of β in that interval such that $\mathcal{F}(\mu_\beta) \neq 0$.

Proof. The two explicit bounds give absolute locally uniform convergence of the integral and of all complex derivatives, proving analyticity. Split at zero and subtract the leading negative-time asymptotic. It gives

$$\mathcal{F}(z) = \frac{cb_\epsilon}{\lambda - z} + \mathcal{G}(z), \quad 0 < \Re z < \sigma,$$

where $\mathcal{G}$ is analytic in $0 < \Re z < 2\sigma$: the remainder on the negative half-line has rate 2σ , and the positive half-line has polynomial growth. Thus its meromorphic extension has a nonremovable simple pole at $z = \lambda$. In particular it cannot vanish identically on the original strip. The analytic nonconstant curve $\beta \mapsto \mu_\beta$ lies inside that strip for real $\beta > 1$. If the transform vanished along any real interval, its zeros would have an interior accumulation point and the identity theorem would force identical vanishing. Its zeros along the displayed compact interval are therefore discrete, and a nonzero choice exists. $\square$

The canonical ancient Euler orbit of Theorem 3.1 has precisely this negative-time estimate. It comes from the high-Sobolev construction and not merely from the isolated X -norm cap theorem. Its positive-time moment bound is elementary once a compactly supported bounded-vorticity orbit exists: L^1, L^p conservation bounds its velocity by (14), its support radius grows at most linearly, and its fourth absolute moment is therefore polynomial of degree at most four. Lemma 4.4 isolates the precise ancient-orbit input needed for the noncancellation assertion.

Lemma 4.5 (Quantitative window and source errors). *Let $\beta > 1$ be fixed and $\mu = \mu_\beta$. Suppose $\mathcal{F}(\mu) \neq 0$ and the source satisfies the bounds of Lemma 4.4. Truncating the source interval to $[-S, U]$ changes its transform by at most*

$$Ce^{-(\sigma - \Re \mu)S} + C_\beta(1+U)^4 e^{-\Re \mu U}.$$

Replacing its moment there by $\tilde{B}$ adds at most

$$\int_{-S}^U e^{-\Re \mu u} |\tilde{B}(u) - B(u)| du.$$

Consequently source windows and errors smaller than $|\mathcal{F}(\mu)|/4$ preserve a nonzero growing transmission. An independent old growing seed a_0 cannot cancel it if

$$|a_0| < \frac{1}{4} q^6 |\gamma_\epsilon| e^{-\sigma s_*/\beta} |\mathcal{F}(\mu)|.$$

If $s_* = \sigma^{-1} \log(\delta/\alpha)$ and old and child seed amplitudes both have size $\alpha = q^{K_*}$, this magnitude condition is algebraically feasible when

$$K_*(1 - 1/\beta) > 6.$$

Fixed logarithmic factors in δ do not change this strict power inequality.

Proof. Integrate the negative exponential bound and the polynomial positive bound against $e^{-\Re \mu u}$. For the positive half-line repeated integration by parts gives a fixed polynomial times $e^{-\Re \mu U}$. The error bound is the triangle inequality. Apply (48); its effective growing coefficient has modulus $q^6 |\gamma_\epsilon| e^{-\sigma s_*/\beta} |\mathcal{F}(\mu)|$. The two strict one-quarter bounds leave a nonzero coefficient. Finally $e^{-\sigma s_*/\beta} = (\alpha/\delta)^{1/\beta}$, so $\alpha/[q^6(\alpha/\delta)^{1/\beta}]$ tends to zero under the stated power inequality, including logarithmically decreasing caps. $\square$

4.1 Exact heat shielding and the remaining coupled terms

The neutral seed (13) has only sectors ± 4 and $B_4 = 0$. Its velocity is zero outside $B_{1+\epsilon}$: in the exterior angular Green formula its coefficient is exactly its fourth moment. Thus its velocity $V_\vartheta = K * \Phi_\vartheta$ is smooth and compactly supported. Heat commutes with Biot–Savart, so a passive physical seed of amplitude $A\alpha$ and radius r has exterior heat velocity bounded, for $|x| \geq R > 2(1 + \epsilon)r$, by

$$|e^{\nu t \Delta} [A\alpha r V_\vartheta(\cdot/r)](x)| \leq C A\alpha r \exp[-cR^2/(\nu t)]. \quad (49)$$

Indeed convolution integrates a bounded function supported in that ball, and the Gaussian mass at distance at least $R/2$ has the displayed bound. No damping estimate of a conserved harmonic moment is invoked. The bound concerns a passive heat evolution; its nonlinear moment generation is an additional term.

For precision, the actual foreign-colour equation yields an exact protected-coordinate identity. Suppose a receiver colour w solves

$$w_t + (K * w + b) \cdot \nabla w = \kappa \Delta w, \quad \operatorname{div} b = 0,$$

and put $f = w - W_\kappa$. Remove any common rigid rotation first. Testing with Lemma 4.1 gives

$$\frac{d}{dt} \mathcal{A}(f) = \lambda \mathcal{A}(f) - \int T b \cdot \nabla W \, dx + \mathcal{E}(t), \quad (50)$$

where the exact error is the sum of

$$\begin{aligned} & \kappa \int \Delta T f \, dx, \quad - \int T [(K * (W_\kappa - W)) \cdot \nabla f + (K * f) \cdot \nabla (W_\kappa - W)] \, dx, \\ & \int f (K * f) \cdot \nabla T \, dx, \quad \int f b \cdot \nabla T \, dx, \quad - \int T b \cdot \nabla (W_\kappa - W) \, dx. \end{aligned}$$

There is no derivative of the actual f after integrating its transport terms by parts. Smooth heat Taylor and (14) give

$$|\mathcal{E}(t)| \leq C[\kappa(1+t)\|f\|_X + \|f\|_X^2] + \int |f| |b| |\nabla T| \, dx + \left| \int T b \cdot \nabla (W_\kappa - W) \, dx \right|. \quad (51)$$

For the term with ∇f , integration by parts bounds it by $\|\nabla T\|_\infty \|K * (W_\kappa - W)\|_\infty \|f\|_1$; the other heat term uses $\|\nabla(W_\kappa - W)\|_1 = O(\kappa t)$. The quadratic self term uses (14). If $b = \Omega(t) Jx$ is a

rigid rotation, its base forcing vanishes and its perturbation contribution is exactly $-4i\Omega(t)\mathcal{A}(f)$; it can be removed by its integrating phase.

Equation (50) identifies quantitative coupled-stage requirements rather than treating the forcing as a free parameter. On the receiver collars one must approximate b by the inverse-quintic child field of Lemma 4.2; its projected error is bounded by the collar sup error times $\|T\nabla W\|_1$. One must also control the two weighted foreign terms in (51) and prove a nonlinear density bound for this actual colour. The canonical source transform must be approximated in the weighted norm of Lemma 4.5, including its post-cap history. Passive shielding (49) and the isolated activation theorem alone do not establish these facts for an infinite interacting stack. No nonlinear receiver transmission theorem is asserted without those estimates.

4.2 A nonlinear receiver theorem for the concrete canonical input

There is an intermediate statement that does close without assuming a foreign field has unspecified desirable properties. It uses the actual moment of the canonical isolated Euler orbit, but realizes its exterior harmonic field through a fixed smooth extension. This extension is an auxiliary prescribed drift, not an assertion that the source and receiver jointly solve the unforced Navier–Stokes system.

Fix an ancient orbit Ω_{δ_c} from Theorem 3.1, with fixed small $\delta_c > 0$, and put $B(u) = B_4(\Omega_{\delta_c}(u))$. Choose $\beta > 1$ for which

$$C_* := \gamma_\epsilon \mathcal{F}(\mu_\beta) \neq 0.$$

Such a choice is possible in every sufficiently large fixed interval by Lemma 4.4. Take a smooth radial cutoff ζ_0 equal to zero near the origin and equal to one on and outside the two receiver derivative collars. For $c \in \mathbb{C}$ set

$$\Psi_c(r, \theta) = -\frac{\zeta_0(r)}{16\pi r^4} (ce^{4i\theta} + \bar{c}e^{-4i\theta}), \quad \mathcal{V}_c = \nabla^\perp \Psi_c.$$

This is a smooth real divergence-free field with every fixed spatial derivative bounded by a constant times $|c|$. On the receiver collars it agrees exactly with the inverse-quintic field of a source moment c .

Theorem 4.6 (Transmission for a prescribed canonical harmonic input). *Fix $K_* > 0$ such that $K_*(1 - 1/\beta) > 6$. For small q put*

$$\alpha = q^{K_*}, \quad \varepsilon_q = q^6, \quad \delta_q = [\log(1/q)]^{-4}, \quad t_0 = \frac{1}{\beta\sigma} \log \frac{\delta_c}{\alpha}, \quad S_q = \frac{1}{\sigma} \log \frac{\delta_q}{\varepsilon_q |C_*|}, \quad H_q = t_0 + S_q.$$

For $0 < \kappa \leq q^{3/2}$ define the prescribed foreign drift

$$b_q(t, x) = \beta \varepsilon_q \mathcal{V}_{e^{4i(t-t_0)/5} B(\beta(t-t_0))}(x), \quad 0 \leq t \leq H_q.$$

Let w^q solve the coloured equation

$$\partial_t w^q + (K * w^q + b_q) \cdot \nabla w^q = \kappa \Delta w^q, \quad w^q(0) = W + \alpha \Phi_0.$$

Then uniformly in $0 < \kappa \leq q^{3/2}$,

$$\sup_{0 \leq t \leq H_q} \|w^q(t) - e^{\kappa t \Delta} W\|_X \leq C \delta_q,$$

$$\mathcal{A}(w^q(H_q)) = \delta_q e^{i(\tau S_q + \arg C_*)} + o(\delta_q),$$

$$B_4(w^q(H_q)) = b_\epsilon \delta_q e^{i(\tau S_q + \arg C_*)} + o(\delta_q).$$

The whole fourth moment is finite. Multiplying the source moment in b_q by a fixed unit complex phase multiplies the leading cap phase by the same phase; all bounds are uniform in that phase.

Proof. For completeness, the positive-viscosity solution with this prescribed drift exists on the whole displayed window. On a short interval its mild equation is a contraction in $C_t(L^1 \cap L^p)$: the heat-gradient estimate is $\|\nabla e^{\kappa t \Delta}\|_{L^r \rightarrow L^r} \leq C(\kappa t)^{-1/2}$, the Biot–Savart map sends X to L^∞ , and b_q is bounded there. Scalar convex-energy estimates give the L^1, L^p contractions and hence a time-uniform velocity bound. Iterating the local construction prevents finite-time loss of the mild solution. Smoothing follows by the same mild equation from the smooth spatial coefficients. This argument uses only bounded drift on the finite window, not the conclusion of the theorem. Use $s = t - t_0$ and write $b_q = \varepsilon_q b_1(s)$. The ancient and future moment bounds give

$$\|b_1(s)\|_{C^j} \leq \begin{cases} C_j e^{\beta \sigma s}, & s \leq 0, \\ C_j (1+s)^4, & s \geq 0. \end{cases}$$

In particular this is bounded by $C_j e^{\sigma s}$ for all s , and its growing-mode weighted time integral is finite, uniformly in the initial time $-t_0$. The first seed-order coefficient solves

$$(\partial_t - \mathcal{L})g_{1,0} = -b_1(t - t_0) \cdot \nabla W, \quad g_{1,0}(0) = (\alpha/\varepsilon_q)\Phi_0.$$

Its growing coordinate is

$$e^{-\lambda s} \mathcal{A}(g_{1,0}(t)) = \frac{\alpha}{\varepsilon_q} e^{\lambda t_0} + \gamma_\epsilon \int_{-t_0}^{\beta s} e^{-\mu_\beta u} B(u) du.$$

The first term tends to zero in modulus because $K_*(1 - 1/\beta) > 6$. The integral tends to $\mathcal{F}(\mu_\beta)$ as $t_0, s \rightarrow \infty$. Thus its growing coefficient tends to C_* . The complementary propagator has exponent $2\eta < \sigma$; integrating its ancient exponential forcing and its future polynomial forcing gives a complementary bound $Ce^{2\eta s}$ for $s \geq 0$. For $s \leq 0$ the whole first coefficient is bounded by $Ce^{\sigma s}$. Indeed its forcing has ancient rate $\beta\sigma > \sigma$ and $(\alpha/\varepsilon_q)e^{\sigma t_0}$ is bounded. All bounds are uniform in q .

Expand in ε_q and κ exactly as in (15), replacing α there by ε_q . In the recursion add $-b_1 \cdot \nabla g_{j-1,\ell}$ for $j \geq 2$ and add $-b_1 \cdot \nabla W_\ell$ for $j = 1$. The initial first coefficient is the one above and the other initial coefficients vanish. The support and finite angular-sector properties are unchanged. Induction gives pure-seed bounds $Ce^{j\sigma s}$ and positive-viscosity bounds $C(1+t)^D e^{j\sigma s}$, uniformly in t_0 : the first coefficient has just been estimated, and for $j \geq 2$ the external forcing is at most $Ce^{\sigma s} e^{(j-1)\sigma s}$, leaving the same nonresonant gap $j\sigma - \sigma$. Positive-viscosity resonances produce only the finite absolute-time polynomial. Consequently the residual estimate (19) holds with

$$a(t) = \varepsilon_q e^{\sigma(t-t_0)}, \quad a(0) = q^{6+K_*/\beta} \delta_c^{-1/\beta}, \quad a(H_q) = \delta_q/|C_*|.$$

The residual contains no forcing-free term: the exact heat base is retained, and $b_1 \cdot \nabla$ of its heat Taylor remainder has the seed factor ε_q . Its bound is $C\kappa^{N+1}(1+t)^D a(t)$ by the displayed input estimate.

The prescribed drift is identical in the actual and approximate equations, so it disappears from the transport part of their Kato comparison. Choose M using the fixed exponent C_1 as before, and choose N with

$$\frac{3}{2}(N+1) > (6 + K_*/\beta)r_1.$$

The exact same X error calculation yields Ca^{M+1} plus a viscosity remainder $o(\delta_q)$ at the cap. It is uniform for $\kappa \leq q^{3/2}$. The higher pure-seed terms contribute $O(\delta_q^2)$ and the positive-viscosity correctors contribute $O(\kappa(1 + H_q)^D \delta_q)$. The complementary first-coefficient response at S_q is $\varepsilon_q e^{2\eta S_q} = o(\delta_q)$. Its growing response gives exactly the asserted protected-coordinate cap. These estimates also give the density bound.

For the whole moment, the integrated self-perturbation velocity is $O(\delta_q)$ as in (26). The additional prescribed drift has integral at most $Cq^6(1 + H_q)^5 = o(\delta_q)$, by its ancient and future bounds. The radial exponential Kato tail argument therefore applies unchanged. All formal coefficients are supported in the fixed collars, and the leading growing coefficient has moment b_ϵ . The complementary, nonlinear and viscosity errors estimated above, followed by the Gaussian tail bound, prove the whole-moment cap. This completes the proof. $\square$

Theorem 4.6 proves nonlinear transmission for this concrete smooth extension of a canonical source. Substituting the actual Biot–Savart field of interacting viscous source colours still requires controlling its omitted higher multipoles, core leakage, and feedback. For a quantitative approximation over a logarithmic stage, keeping only the fourth moment is generally inadequate: its next q^{10} error may be amplified by the generic X exponent. One can instead retain arbitrarily many finite harmonic terms in the prescribed coefficient recursion. Their omitted remainder must be bounded to a sufficiently high power of q to beat that exponent. This identifies an extension of the method, not an estimate already proved for the actual infinite stack.

5 Actual viscous outer cores and the local harmonic field

This section proves a conditional estimate for the actual coloured Navier–Stokes evolution. Its nonradial-density hypothesis is explicit; it is not deduced from material contours or from named unstable coordinates. We identify $x = (x_1, x_2)$ with $z = x_1 + ix_2$ and identify a velocity with $u_1 + iu_2$. Thus

$$K(z) = \frac{i}{2\pi\bar{z}}, \quad Jz = iz.$$

Let P denote circular averaging. For $f \in L^1 \cap L^\infty$, splitting the Biot–Savart integral at distance d and optimizing d gives

$$\|K * f\|_\infty \leq C\sqrt{\|f\|_1 \|f\|_\infty}. \quad (52)$$

Indeed the near integral is at most $Cd\|f\|_\infty$ and the far integral is at most $C\|f\|_1/d$.

Lemma 5.1 (Fourfold growth at the origin). *If $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is invariant under $z \mapsto iz$, then*

$$|K * f(z)| \leq C\|f\|_\infty |z|.$$

Neither zero circulation nor regularity of f is required.

Proof. For $r = |z| > 0$, the integral over $|y| \leq 2r$ is at most $C\|f\|_\infty r$, by integrating $|z - y|^{-1}$ over $B_{3r}(z)$. For $|y| > 2r$ use the absolutely convergent expansion

$$\frac{1}{\bar{z} - \bar{y}} = -\sum_{n \geq 0} \frac{\bar{z}^n}{\bar{y}^{n+1}}.$$

The inverse moment vanishes unless $n + 1$ is a multiple of four, by changing variables under a quarter turn. The surviving degrees are 3, 7, 11, ... Their absolute sum is at most

$$Cr^3 \int_{|y|>2r} \frac{|f(y)|}{|y|^4} dy \leq C \|f\|_\infty r^3 \int_{2r}^\infty s^{-3} ds \leq C \|f\|_\infty r.$$

Termwise integration is justified by the integrable majorant $C|f(y)|/|y|$ on the far region. At $z = 0$ the velocity vanishes by rotational covariance and continuity. $\square$

Lemma 5.2 (Exit controlled only by radial drift). *Let a diffusion satisfy*

$$dX_s = v(s, X_s) ds + \sqrt{2\nu} dB_s, \quad X_0 = x,$$

and let σ be its first exit from B_b , where $|x| < b$. Suppose, before exit,

$$|v(s, y) \cdot y/|y|| \leq V(s) \quad (0 < |y| < b), \quad V \geq 0, \quad D := \int_0^T V(s) ds < b - |x|.$$

Then

$$\mathbb{P}(\sigma \leq T) \leq 2 \exp\left(-\frac{(b - |x| - D)^2}{8\nu T}\right). \quad (53)$$

There is no bound on the tangential component of v .

Proof. For $\lambda > 0$ set $\phi(y) = \cosh(\lambda|y|)$. Its even power series shows that it is smooth at zero. In dimension two,

$$\Delta\phi = \lambda^2 \cosh(\lambda r) + \frac{\lambda}{r} \sinh(\lambda r) \leq 2\lambda^2 \phi, \quad v \cdot \nabla\phi \leq \lambda V(s)\phi,$$

using $\sinh a \leq a \cosh a$ for $a \geq 0$. Therefore Itô's formula shows that

$$e^{-2\nu\lambda^2(s\wedge\sigma)-\lambda\int_0^{s\wedge\sigma} V(\tau) d\tau} \phi(X_{s\wedge\sigma})$$

is a nonnegative supermartingale. Stopping at T gives

$$\mathbb{P}(\sigma \leq T) \leq \frac{\cosh(\lambda|x|)}{\cosh(\lambda b)} e^{2\nu\lambda^2 T + \lambda D} \leq 2e^{-\lambda(b-|x|-D)+2\nu\lambda^2 T}.$$

Choose $\lambda = (b - |x| - D)/(4\nu T)$. The stopped calculation remains valid for bounded drifts smooth for positive times: stop first before their time endpoint and then pass to that endpoint. No approximation of a radial drift bound is used. $\square$

Consider real, fourfold symmetric colours solving

$$\partial_t w_l + u^\nu \cdot \nabla w_l = \nu \Delta w_l, \quad u^\nu = K * \sum_l w_l, \quad (54)$$

with geometric scales

$$A_l = A_1 \beta^{l-1}, \quad R_l = R_1 q^{l-1}, \quad 1 < \beta_{\min} \leq \beta \leq \beta_{\max}, \quad q \leq \frac{1}{4}, \quad \beta q \leq \frac{1}{2}.$$

Assume

$$\|w_l(0)\|_1 \leq C_0 A_l R_l^2, \quad \|w_l(0)\|_\infty \leq C_0 A_l. \quad (55)$$

In the countable case assume that the initial series converges absolutely in $L^1 \cap L^p$, $p > 2$, and use its mild Navier–Stokes solution and the associated linear coloured evolutions. The common velocity is bounded and smooth for positive times. Norm contraction identifies its vorticity with the sum of the colours, and (52) implies uniform convergence of the velocity series because $\sum_l A_l R_l < \infty$.

Fix an index i , a window $T = H/A_i$, and

$$\kappa_i = \frac{\nu}{A_i R_i^2}.$$

Suppose that for some fixed $a_0 > 0$ every $l \leq i$ has an initial constant core,

$$w_l(0) = c_l \quad \text{on } B_{a_0 R_l}, \quad |c_l| \leq C_0 A_l. \quad (56)$$

Write $h_l = (I - P)w_l$ and impose the actual density hypothesis

$$\sup_{0 \leq t \leq T} \|h_l(t)\|_1 \leq \delta A_l R_l^2 \quad (l \leq i), \quad 0 < \delta \leq 1. \quad (57)$$

No nonradial hypothesis is imposed on $l > i$.

The scalar stochastic representation used below is

$$w_l(t, x) = \mathbb{E}[w_l(0, X_t)], \quad dX_s = -u^\nu(t-s, X_s) ds + \sqrt{2\nu} dB_s, \quad X_0 = x. \quad (58)$$

Its sign is verified by applying Itô's formula to $w_l(t-s, X_s)$: the drift is $-\partial_t w_l - u^\nu \cdot \nabla w_l + \nu \Delta w_l = 0$. For completeness, first stop at $t - \varepsilon$, where the coefficients are smooth. Boundedness of the drift gives a continuous limiting process at t . Its transition densities away from starting time are bounded by $C/[\nu s]$. This bound follows by L^1 contraction, the Nash inequality $\|f\|_2^4 \leq C \|f\|_1^2 \|\nabla f\|_2^2$, the L^2 energy identity, and the corresponding adjoint estimate, composed at the midpoint. Thus $w_l(\varepsilon) \rightarrow w_l(0)$ in L^1 suffices to pass to the endpoint in expectation. Approximation of the bounded initial colour in L^1 by continuous functions also justifies evaluation at the limiting endpoint. This verifies (58) for the actual colours without smoothing an asserted radial bound.

Proposition 5.3 (Uniform persistence of actual constant cores). *There is $C \geq 1$, depending only on $C_0, a_0, \beta_{\min}, \beta_{\max}$, with the following property. Set*

$$\mathcal{D} = \sqrt{\delta} + \frac{\delta q^2}{\beta - q^2} + \frac{\beta q}{1 - \beta q}, \quad c_* := \frac{9}{512}, \quad \varepsilon_* := \frac{a_0(\beta - 1)}{16CH}.$$

If

$$CH\mathcal{D} \leq \frac{a_0}{16}, \quad 4C_0 e^{-c_* a_0^2 / (\kappa_i H)} \leq \frac{\varepsilon_*}{2}, \quad (59)$$

then, with $a = a_0/2$, for $l \leq i$ and $0 < t \leq T$,

$$\|w_l(t) - c_l\|_{L^\infty(B_{aR_l})} \leq 4C_0 A_l e^{-c_* a_0^2 R_l^2 / (\nu t)}. \quad (60)$$

All constants are independent of i , viscosity, and the number of colours.

Proof. The maximum principle gives $\|h_l\|_\infty \leq 2C_0 A_l$. The radial average induces a purely tangential velocity, so (52) and (57) imply

$$|(K * w_l) \cdot e_r| \leq \|K * h_l\|_\infty \leq C\sqrt{\delta} A_l R_l \quad (l \leq i).$$

Every unrestricted inner colour has $\|K * w_k\|_\infty \leq C A_k R_k$, whence

$$\sum_{k>l} \|K * w_k\|_\infty \leq C A_l R_l \frac{\beta q}{1 - \beta q}. \quad (61)$$

Temporarily assume that all probe-core defects satisfy

$$\|w_k - c_k\|_{L^\infty(B_{aR_k})} \leq \varepsilon A_k.$$

For $k < l$ and $|z| \leq a_0 R_l$, one has $|z| \leq a R_k/2$. Decompose the older colour into

$$w_k = c_k \mathbf{1}_{B_{aR_k}} + (w_k - c_k) \mathbf{1}_{B_{aR_k}} + w_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}}.$$

The first term gives exactly $(c_k/2)Jz$. The second gives a velocity bounded by $C\varepsilon A_k|z|$ by Lemma 5.1. The radial part of the third has zero velocity inside the disk. Its remaining density is $h_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}}$, whose L^1 norm is at most $\delta A_k R_k^2$. Its exterior series starts at degree three, so

$$|K * (w_k \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_k}})(z)| \leq C \delta A_k |z|^3 / R_k^2. \quad (62)$$

Here and below the density is evaluated at its actual viscous time. Summing the self, inner and older bounds gives

$$|u^\nu(t, z) \cdot e_r| \leq C A_l R_l \left(\mathcal{D} + \frac{\varepsilon}{\beta - 1} \right) \quad (|z| \leq a_0 R_l, l \leq i). \quad (63)$$

We used the geometric sums

$$\sum_{k<l} \frac{A_k}{R_k^2} \leq \frac{A_l}{R_l^2} \frac{q^2}{\beta - q^2}, \quad \sum_{k<l} A_k \leq \frac{A_l}{\beta - 1}.$$

Fix C large enough for (63).

On a bootstrap interval with $\varepsilon \leq \varepsilon_*$, the integrated radial drift in the backward representation is at most $a_0 R_l/8$, because $A_l T \leq H$ and (59) controls both terms. Start at $|x| \leq a_0 R_l/2$ and stop on exiting $B_{a_0 R_l}$. If the backward path does not exit, its endpoint belongs to the initial constant core. The possible discrepancy outside it is at most $2C_0 A_l$. Lemma 5.2 therefore gives

$$|w_l(t, x) - c_l| \leq 4C_0 A_l \exp\left(-\frac{(3a_0 R_l/8)^2}{8\nu t}\right),$$

which is (60). Subtracting c_l is legitimate: a constant is itself a bounded scalar solution, and bounded stochastic representation does not require its integrability.

The maximum normalized probe defect over the finite set $l \leq i$ is continuous for positive time. It tends to zero at time zero, since the probe has a fixed margin inside the initial constant disk and the common drift is bounded. If it first attained ε_* , the preceding estimate, $R_l \geq R_i$, $t \leq H/A_i$, and (59) would instead bound it by $\varepsilon_*/2$. This closes the bootstrap. The finite maximum adds no dependence on i ; every infinite inner sum was already controlled by (61). $\square$

Proposition 5.4 (Harmonic exterior field and actual density leak). *Under Proposition 5.3, set*

$$b_i = \sum_{l \leq i} K * w_l, \quad \Omega_i = \frac{1}{2} \sum_{l \leq i} c_l, \quad H_i = \sum_{l \leq i} K * (w_l \mathbf{1}_{\mathbb{R}^2 \setminus B_{aR_l}}), \quad L_i = \sum_{l \leq i} K * ((w_l - c_l) \mathbf{1}_{B_{aR_l}}).$$

For $|z| \leq aR_i/2$ one has the exact decomposition

$$b_i(t, z) = \Omega_i Jz + H_i(t, z) + L_i(t, z). \quad (64)$$

The field H_i is divergence free and curl free on B_{aR_i} , and

$$|H_i(t, z)| \leq C\delta A_i |z|^3 / R_i^2, \quad |DH_i(t, z)| \leq C\delta A_i |z|^2 / R_i^2, \quad (65)$$

$$|L_i(t, z)| \leq CA_i e^{-c_* a_0^2 / (\kappa_i H)} |z|. \quad (66)$$

Its cubic coefficient is exactly

$$c_{3,i}(t) = -\frac{i}{2\pi} \sum_{l \leq i} \int_{|y| \geq aR_l} \frac{w_l(t, y)}{\bar{y}^4} dy. \quad (67)$$

With $\eta_l(t) = \|h_l(t)\|_1 / (A_l R_l^2)$,

$$|c_{3,i}(t)| \leq C \frac{A_i}{R_i^2} \sum_{m=0}^{i-1} (q^2/\beta)^m \eta_{i-m}(t) \leq C\delta A_i / R_i^2. \quad (68)$$

Furthermore,

$$|H_i(t, z) - c_{3,i}(t) \bar{z}^3| \leq C\delta A_i |z|^7 / R_i^6, \quad (69)$$

$$|D(H_i(t, z) - c_{3,i}(t) \bar{z}^3)| \leq C\delta A_i |z|^6 / R_i^6. \quad (70)$$

The derivative denotes the real spatial Jacobian. No derivative bound on L_i is asserted.

Proof. The constant disks induce $(c_l/2)Jz$ in the specified region, proving (64). Exterior densities induce curl free velocities away from their supports, and all Biot–Savart velocities are divergence free. Radial averaging commutes with radial truncation, and an exterior radial density induces zero velocity inside its truncation radius. Consequently, on this region,

$$H_i(t, z) = -\frac{i}{2\pi} \sum_{l \leq i} \sum_{m \geq 0} \bar{z}^{4m+3} \int_{|y| \geq aR_l} \frac{h_l(t, y)}{\bar{y}^{4m+4}} dy.$$

The inverse radial moment is zero, so this gives exactly (67). The coefficient integral is at most

$$\frac{\|h_l\|_1}{(aR_l)^4}.$$

This proves (68).

The series and its first spatial derivatives converge uniformly on $|z| \leq aR_i/2$. Indeed each series ratio is at most $1/16$, and the derivative majorant adds only the factor $4m+3$. Thus, for each colour, its harmonic field and derivative are bounded by $C\delta A_l |z|^3 / R_l^2$ and $C\delta A_l |z|^2 / R_l^2$, respectively. Removing $m=0$ gives $C\delta A_l |z|^7 / R_l^6$ and $C\delta A_l |z|^6 / R_l^6$. Summing uses

$$\sum_{l \leq i} \frac{A_l}{R_l^d} \leq \frac{A_i / R_i^d}{1 - q^d / \beta} \quad (d = 2, 6).$$

These prove (65), (69), and (70). Finally Lemma 5.1, (60), and $\sum_{l \leq i} A_l \leq A_i \beta / (\beta - 1)$ prove (66). $\square$

Scope for the localized tail problem. If $R_* = \kappa_i^{1/4} R_i \leq aR_i/2$, then throughout B_{R_*} ,

$$|b_i - \Omega_i Jz| \leq C\delta A_i |z|^3 / R_i^2 + CA_i e^{-c_* a_0^2 / (\kappa_i H)} |z|.$$

The integrated linear-leak coefficient is at most $CH e^{-c_* a_0^2 / (\kappa_i H)}$. The actual density leak is therefore quantified; no material hole has been retained by assumption. The harmonic derivative estimate (65) is available for a coupled transfer or energy calculation, while the core leak has only the growth estimate (66). The parameter margins are compatible with $H \asymp \log(1/q)$ and $\kappa_i = q^{3/2}$ if $\sqrt{\delta}H$ is sufficiently small. This compatibility does not prove (57); obtaining that hypothesis and controlling the coupled inner aggregate over its whole receiver window remain separate nonlinear tasks. In particular no fixed-data nonunique vanishing-viscosity theorem is concluded here.

6 A quantitative parabolic localization estimate

The next estimate concerns an actual convection–diffusion equation, and gives a quantitative substitute for compact material support. Its local cubic drift hypothesis must still be verified for a coupled Navier–Stokes cascade.

Lemma 6.1 (Localized tail exit). *Let f solve*

$$\partial_t f + U \cdot \nabla f = \nu \Delta f, \quad U(t, x) = \Omega(t)x^\perp + V(t, x) + b(t, x), \quad \operatorname{div} U = 0,$$

on $0 \leq t \leq T$, with $\|f(0)\|_1 \leq m$ and $\operatorname{supp} f(0) \subset B_r$. Suppose

$$\begin{aligned} \|V(t)\|_\infty &\leq C_0 m (\nu t)^{-1/2}, \\ |b(t, x)| &\leq L|x| \quad \text{for all } x, \\ |b(t, x)| &\leq C_3 |x|^3 + \ell(t)|x| \quad \text{for } |x| \leq R_*. \end{aligned}$$

Assume $\ell \geq 0$, Ω is integrable, and

$$r \leq R_*/4, \quad C_3 R_*^2 T + \int_0^T \ell(t) dt \leq \frac{1}{4}, \quad C_0 \frac{m}{\nu} \sqrt{\nu T} \leq R_*/8.$$

There are absolute $c, C > 0$ such that

$$\sup_{t \leq T} \int_{|x| \geq R_*} |f(t, x)| dx \leq C m \exp\left(-\frac{cR_*^2}{\nu T}\right).$$

For each fixed integer $k \geq 0$ there are constants $c, C_k > 0$ such that

$$\int_{|x| \geq R_*} |x|^k |f(T, x)| dx \leq C_k m e^{C_k L T} R_*^k \exp\left(-\frac{cR_*^2}{\nu T}\right),$$

provided additionally $\sqrt{\nu T} \leq cR_$. The conclusion holds for smooth drifts and for limits for which the parabolic solution and the corresponding diffusion representation are obtained by smooth approximation.*

Proof. Kato's inequality bounds $|f|$ by the positive scalar solution from $|f(0)|$. Normalize this initial mass and represent the positive solution by the diffusion

$$dX_t = U(t, X_t) dt + \sqrt{2\nu} dB_t.$$

A time-dependent rotation removes $\Omega(t)x^\perp$; the rotated Brownian motion remains standard and all norm assumptions are unchanged. Let σ be the first exit from B_{R_*} . Before that exit, deterministic Gronwall applied to the integral stochastic equation gives

$$\sup_{s \leq t \wedge \sigma} |X_s| \leq e^{C_3 R_*^2 t + \int_0^t \ell} \left[r + 2C_0 \frac{m}{\nu} \sqrt{\nu t} + \sqrt{2\nu} \sup_{s \leq t} |B_s| \right].$$

The stated margins imply that $\sigma \leq T$ requires $\sup_{s \leq T} |B_s| \geq cR_*/\sqrt{\nu}$. The coordinate reflection bound for planar Brownian motion yields

$$\mathbb{P}(\sigma \leq T) \leq C \exp \left(-\frac{cR_*^2}{\nu T} \right).$$

An endpoint outside B_{R_*} requires this exit, proving the mass bound.

For the moment assertion, condition at σ . The remaining drift has linear growth bounded by $L|x|$ plus $\|V(t)\|_\infty$. Its integrated bounded contribution is at most $2C_0(m/\nu)\sqrt{\nu T} \leq R_*/4$. Gronwall and Brownian moments therefore give, uniformly in the exit time,

$$\mathbb{E}[|X_T|^k | \mathcal{F}_\sigma] \leq C_k e^{C_k L T} R_*^k \quad \text{on } \{\sigma \leq T\}.$$

Multiply by the indicator of this event and use its probability bound. Finally restore the initial mass, which is at most m . $\square$

For a receiver with $\kappa_i = \nu/(A_i R_i^2)$ and stage duration $T = H/A_i$, the choice $R_* = \kappa_i^{1/4} R_i$ gives

$$\frac{R_*^2}{\nu T} = \frac{1}{\kappa_i^{1/2} H}, \quad C_3 R_*^2 T \leq C x_i \kappa_i^{1/2} H \quad \text{if } C_3 \leq C A_i x_i / R_i^2.$$

A global bound $L \leq C A_i$ costs only $e^{C_k H}$, independent of depth. The exponential exit factor dominates that cost as $\kappa_i \downarrow 0$ for logarithmic stage lengths. This parameter calculation does not verify the required local cubic hypothesis for the actual foreign field.

7 Nonlinear inner-tail suppression through a receiver window

The estimate in this section is conditional on quantitative information about the actual foreign velocity. It closes a longer-time nonlinear subproblem than the isolated-tail and first-diffusion-time estimates. It does not establish the required nonradial-density bound for the active and older outer colours.

Write $J(x_1, x_2) = (-x_2, x_1)$, $z = x_1 + ix_2$, and

$$K(x) = \frac{Jx}{2\pi|x|^2}, \quad B_4(f) = \int_{\mathbb{R}^2} \bar{z}^4 f(z) \, dz.$$

A scalar is fourfold symmetric if $f(Jx) = f(x)$; a vector field has the corresponding covariance if $b(t, Jx) = Jb(t, x)$. Constants denoted by C can change from line to line.

7.1 A Gaussian-energy estimate for a globally weak foreign field

Proposition 7.1 (Weak foreign field through a finite window). *There are universal constants $c, \lambda_0, C > 0$ with the following property. Let $p > 2$, and let $f_0 \in L^1 \cap L^p$ be real, fourfold symmetric, supported in B_1 , and satisfy*

$$\int f_0 = 0, \quad B_4(f_0) = 0, \quad m = \|f_0\|_1.$$

Suppose b is divergence free, has fourfold covariance, and, after removal of a time-dependent solid rotation, satisfies

$$|b(s, x)| \leq \ell(s)|x|, \quad |b(s, x)| \leq d|x|^3, \quad (1+s)\ell(s) \leq \lambda_0.$$

Let f solve

$$\partial_s f + (K * f + b) \cdot \nabla f = \Delta f.$$

For $S \geq 2$, assume the displayed hypotheses on $0 \leq s \leq S-1$ and

$$mS^{1/25}\sqrt{1+\log S} \leq c. \tag{71}$$

Then

$$\sup_{0 \leq s \leq S-1} |B_4(f(s))| \leq C \left[m^2 S^{7/50} (1 + \log S) + mdS^{102/25} \right]. \tag{72}$$

The constants are independent of a decomposition of the datum into smaller scales. The assertion holds for smooth drifts and for the parabolic limits obtained by approximation with the stated bounds.

Proof. Scalar contraction and the divergence-free Nash estimate give

$$\|f(s)\|_1 \leq m, \quad \|f(s)\|_\infty \leq Cm/s, \quad \|K * f(s)\|_\infty \leq Cm/\sqrt{s}.$$

The last estimate follows by splitting Biot–Savart at a radius and optimizing. Take $s_0 = 1/100$. The positive scalar solution from $|f_0|$ dominates $|f|$. Its diffusion representation, after removing solid rotation, obeys

$$|X_s| \leq e^{\lambda_0 s} \left[1 + Cm\sqrt{s} + \sqrt{2} \sup_{a \leq s} |B_a| \right] \quad (0 \leq s \leq s_0).$$

The Gaussian exponential moment of the Brownian supremum is finite at this fixed small time. With $\beta = 3/10$ it follows that

$$\int e^{\beta|x|^2/(1+s_0)} |f(s_0, x)| \, dx \leq Cm.$$

Multiplication by the Nash L^∞ bound gives a weighted L^2 norm $O(m)$. Fixed polynomial absolute moments are $O(m)$ on $[0, s_0]$. The exact harmonic-moment equation and symmetrization of self-advection therefore yield

$$|B_4(f(s_0))| \leq C(m^2 + md). \tag{73}$$

For clarity, the self contribution is bounded by $Cm \int |x|^2 |f|$, because $|\bar{z}^3 - \bar{y}^3|/|z - y| \leq C(|z|^2 + |y|^2)$; the foreign contribution is bounded by $d \int |x|^6 |f|$. These bounds also prove the result directly if the final time is smaller than s_0 .

For $s \geq s_0$ put

$$\tau = \log(1 + s), \quad y = x/\sqrt{1 + s}, \quad w(\tau, y) = (1 + s)f(s, x), \quad E(\tau, y) = \sqrt{1 + s}b(s, \sqrt{1 + s}y).$$

Then

$$w_\tau = \mathcal{L}w - (K * w + E) \cdot \nabla w, \quad \mathcal{L} = \Delta + \frac{1}{2}y \cdot \nabla + 1,$$

with $|E| \leq \lambda(\tau)|y|$, $\lambda \leq \lambda_0$, and $|E| \leq a(\tau)|y|^3$, where $a(\tau) = de^{2\tau}$. Let

$$F_\beta = \left\| e^{\beta|y|^2/2} w \right\|_2, \quad D_\beta = \left\| e^{\beta|y|^2/2} \nabla w \right\|_2^2, \quad \beta = 3/10.$$

Writing $g = e^{\beta|y|^2/2} w$ gives the exact identities

$$\begin{aligned} \langle w, \mathcal{L}w \rangle_\beta &= \frac{1}{2} \|g\|_2^2 - \|\nabla g\|_2^2 - \beta(1/2 - \beta) \| |y|g \|_2^2, \\ D_\beta &= \|\nabla g\|_2^2 + \beta^2 \| |y|g \|_2^2 + 2\beta \|g\|_2^2. \end{aligned}$$

For $a_\beta = \sqrt{\beta(1/2 - \beta)}$, completing squares in $\sum_j \|\partial_j g + a_\beta y_j g\|_2^2$ proves

$$\mathcal{H}_\beta := \|\nabla g\|_2^2 + \beta(1/2 - \beta) \| |y|g \|_2^2 \geq 2a_\beta \|g\|_2^2 = \frac{\sqrt{6}}{5} \|g\|_2^2.$$

Moreover $D_\beta \leq 3\mathcal{H}_\beta$: one can first bound it by $\frac{3}{2}\mathcal{H}_\beta + \frac{3}{5}\|g\|_2^2$ and then use the last inequality. Splitting $\mathcal{H}_\beta$ into $\frac{3}{50}\mathcal{H}_\beta + \frac{47}{50}\mathcal{H}_\beta$ yields

$$\langle w, \mathcal{L}w \rangle_\beta \leq -\frac{1}{50}D_\beta + \frac{1}{25}F_\beta^2. \quad (74)$$

Here $\frac{1}{2} - \frac{47}{50}\frac{\sqrt{6}}{5} < \frac{1}{25}$.

The Hardy–Littlewood–Sobolev estimate and Gaussian weighted Hölder give

$$\|K * w\|_4 \leq C \|w\|_{4/3} \leq C_\beta F_\beta.$$

The two-dimensional Gagliardo–Nirenberg inequality gives

$$\|g\|_4 \leq C \|g\|_2^{1/2} \|\nabla g\|_2^{1/2}.$$

Since $\| |y|g \|_2 + \|\nabla g\|_2 \leq C_\beta \sqrt{D_\beta}$, self-advection contributes at most

$$CF_\beta^{3/2} D_\beta^{3/4} \leq C_\beta F_\beta D_\beta$$

to the weighted energy. The foreign contribution is $\beta \int E \cdot y e^{\beta|y|^2} w^2$, bounded by $C_\beta \lambda D_\beta$. Thus, while F_β is sufficiently small,

$$\frac{1}{2}(F_\beta^2)' \leq -\frac{1}{100}D_\beta + \frac{1}{25}F_\beta^2.$$

The initial weighted norm and a continuity bootstrap using (71) prove

$$F_\beta(\tau) \leq Cme^{\tau/25}, \quad \int_{\log(1+s_0)}^{\log S} D_\beta(\tau) \, d\tau \leq Cm^2 S^{2/25}. \quad (75)$$

Let P be angular averaging, $G = Pw$, and $h = (I - P)w$. Every ordinary moment of h of degree at most three vanishes: angular averaging removes radial moments, and fourfold symmetry removes all other moments in these degrees. In

$$X = L^2(e^{|y|^2/4} \, dy), \quad H = \|h\|_X, \quad D_h = \|\nabla h\|_X^2,$$

one consequently has

$$D_h \geq 3H^2. \quad (76)$$

Indeed $\mathcal{L}$ is self-adjoint in X , and its Hermite basis is $\partial^\alpha \mathcal{G}$, with $\mathcal{G} = (4\pi)^{-1} e^{-|y|^2/4}$ and eigenvalue $-|\alpha|/2$. Orthogonality to degrees at most three is equivalent to the stated ordinary-moment cancellations, because the corresponding dual Hermite polynomials span these polynomial degrees. Expansion in this basis gives $\langle h, \mathcal{L}h \rangle_X \leq -2H^2$; since the left side is $-D_h + H^2$, (76) follows. This elementary Hermite expansion also follows from the tensor one-dimensional harmonic-oscillator factorization.

Angular averaging is orthogonal for all the radial weights used here, and commutes with $\mathcal{L}$. Writing $V_a = K * a$, the angular equation is

$$h_\tau = \mathcal{L}h - (I - P) \operatorname{div}[V_G h + V_h G + V_h h + EG + Eh].$$

The radial field V_G is tangential, so its weighted energy contribution is zero. The h - h and E - h contributions are bounded by CHD_h and $C\lambda D_h$. Put $D_G = \|\nabla G\|_X^2 \leq D_\beta$. Weighted integration by parts, unweighted HLS, and GN give

$$|\langle h, -\operatorname{div}(V_h G) \rangle_X| \leq CHF_\beta^{1/2} D_G^{1/4} D_h^{1/2} \leq \eta D_h + C_\eta H^2 F_\beta D_G^{1/2}.$$

The strict Gaussian margin $\beta - 1/4 = 1/20$ absorbs cubic multiplication:

$$\left\| e^{|y|^2/8} EG \right\|_2 \leq Ca(\tau) F_\beta.$$

Consequently the E - G contribution is bounded by $\eta D_h + C_\eta a(\tau)^2 F_\beta^2$. Take η and the bootstrap constants sufficiently small. Using (76) yields

$$(H^2)' \leq -\frac{19}{5} H^2 + CF_\beta D_G^{1/2} H^2 + Ca(\tau)^2 F_\beta^2.$$

By (75) and Cauchy-Schwarz,

$$\int F_\beta D_G^{1/2} \, d\tau \leq Cm^2 S^{2/25} \sqrt{1 + \log S} \leq Cc^2.$$

Variation of constants therefore gives

$$H(\tau) \leq Cm \left[e^{-19\tau/10} + de^{51\tau/25} \right]. \quad (77)$$

Factors depending on the fixed initial time are absorbed in C .

Finally $B_4(f(s)) = e^{2\tau} B_4(w(\tau))$. Harmonicity cancels diffusion, and differentiating $e^{2\tau}$ cancels the dilation term. Radial self-advection vanishes. HLS and Gaussian weighted Hölder show that

$$\left| \frac{d}{d\tau} B_4(f(e^\tau - 1)) \right| \leq Ce^{2\tau} [F_\beta H + a(\tau) F_\beta].$$

Substitution of (75) and (77), followed by integration, gives a first term $Cm^2 S^{1/25+1/10} (1 + \log S)$ and remaining terms at most $CmdS^{102/25}$, using $m \leq 1$. Together with (73), this proves the proposition.

All energy tests can first be made with truncated weights and smooth data. The short-time exponential moment starts the weighted evolution; the uniform weighted estimates justify removal of the truncations and passage to the parabolic limit. No initial L^∞ norm enters a constant. $\square$

7.2 Comparison with the actual local foreign field

Proposition 7.2 (Local harmonic field and a rough small leak). *Let $\nu, r, T, R_* > 0$, $R_* \geq 4r$. Let f_0 have the properties of the preceding proposition at support radius r , with physical mass $M = \|f_0\|_1$. Let f solve*

$$f_t + (K * f + b) \cdot \nabla f = \nu \Delta f.$$

After removal of a solid rotation, suppose b is divergence free, has fourfold covariance, and admits, on B_{2R_} , the decomposition*

$$\begin{aligned} b &= b_h + e, & |b(t, x)| &\leq L|x| \quad \text{globally,} \\ |b_h(t, x)| &\leq C_3|x|^3, & |Db_h(t, x)| &\leq C_3|x|^2, \\ |e(t, x)| &\leq \epsilon L|x|. \end{aligned} \tag{78}$$

where b_h is divergence free with fourfold covariance and $0 \leq \epsilon \leq 1$. Its streamfunction is defined on B_{2R_} . Assume*

$$C_3 R_*^2 T + \epsilon L T \leq \frac{1}{4}, \quad C(M/\nu) \sqrt{\nu T} \leq R_*/8, \quad \sqrt{\nu T} \leq c R_*. \tag{79}$$

Take a fixed radial cutoff of that streamfunction, equal to one in B_{R_} and zero outside B_{2R_*} , and denote its velocity by $\tilde{b}$. Let g solve the same self-consistent equation as f , with foreign field $\tilde{b}$ and the same initial datum. Set $\ell = C C_3 R_*^2$. If M/ν and $\ell(r^2/\nu + T)$ are sufficiently small, then*

$$\begin{aligned} \sup_{t \leq T} |B_4(f(t)) - B_4(g(t))| &\leq C M R_*^4 e^{CLT} \left[\epsilon L (T + r \sqrt{T/\nu}) \right. \\ &\quad \left. + (1 + L R_* \sqrt{T/\nu}) e^{-c R_*^2/(\nu T)} \right]. \end{aligned} \tag{80}$$

No derivative bound on the leak e is required.

Proof. Choose the streamfunction ψ_h zero at the origin. Integration along radial segments gives $|\psi_h(x)| \leq C C_3 |x|^4$. A streamfunction cutoff gives

$$|\tilde{b}| \leq C C_3 |x|^3, \quad |\tilde{b}| \leq \ell |x|, \quad \|D\tilde{b}\|_\infty \leq \ell.$$

It is divergence free, has fourfold covariance, and equals b_h in B_{R_*} . A second useful bound is $|\tilde{b}(x)| \leq CL|x|$: from $\epsilon \leq 1$ and (78), $|b_h| \leq 2L|x|$ on B_{2R_*} , so $|\psi_h| \leq CL|x|^2$ there, and cutoff derivatives preserve this linear growth. Thus this bound does not require $C_3 R_*^2 \leq L$.

We first verify the velocity-gradient estimate needed for comparison. Normalize $r = \nu = 1$ and write $m = M/\nu$; the cutoff growth and Lipschitz constant are normalized in the same way. Nash gives

$$\|g(t)\|_\infty \leq C m/t, \quad \|u_g(t)\|_\infty \leq C m/\sqrt{t}, \quad u_g = K * g.$$

For $M_k(t) = \int |x|^k |g(t, x)| \, dx$, Kato's inequality gives

$$\begin{aligned} M_2' &\leq 4m + C m t^{-1/2} \sqrt{m M_2} + 2\ell M_2, \\ M_4' &\leq 16M_2 + C m t^{-1/2} m^{1/4} M_4^{3/4} + 4\ell M_4. \end{aligned}$$

Since $M_2(0), M_4(0) \leq m$, a continuity bootstrap gives

$$M_2(t) \leq C m(1+t), \quad M_4(t) \leq C m(1+t)^2$$

when m and $\ell(1+T)$ are small. On these trial bounds the integrated self terms are respectively $O(m^2(1+t))$ and $O(m^2(1+t)^2)$; $\ell T \ll 1$ absorbs the foreign terms. Hence

$$\|xg(t)\|_1 \leq Cm\sqrt{1+t}, \quad \|xg(t)\|_4 \leq Cmt^{-3/4}\sqrt{1+t}.$$

The second follows from $\|xg\|_4^4 \leq \|g\|_\infty^3 M_4$.

With H_t the heat kernel, scaling and the smooth kernels' exterior decay give

$$\begin{aligned} \|\nabla K * H_t\|_\infty &\leq Ct^{-1}, \\ \|\nabla K * \operatorname{div} H_t\|_\infty &\leq Ct^{-3/2}, \quad \|\nabla K * \operatorname{div} H_t\|_1 \leq Ct^{-1/2}, \quad \|\nabla K * \operatorname{div} H_t\|_{4/3} \leq Ct^{-3/4}. \end{aligned}$$

For the nonlinear differentiated kernel the tail is $O(|x|^{-3})$, so the last three norms are finite before scaling. The self flux $u_g g$ has L^1 norm $Cm^2 t^{-1/2}$ and L^∞ norm $Cm^2 t^{-3/2}$. Split the heat-mild time integral at $t/2$, using the kernel's L^∞ norm first and its L^1 norm second; its contribution to ∇u_g is Cm^2/t . The foreign flux has L^1 norm $C\ell m\sqrt{1+t}$ and L^4 norm $C\ell m t^{-3/4}\sqrt{1+t}$. Use the kernel's L^∞ norm first and its $L^{4/3}$ norm second. This contribution is

$$C\ell m t^{-1/2}\sqrt{1+t} \leq Cm/t,$$

because $\ell(1+T)$ is small. The initial heat contribution is Cm/t . Undoing normalization proves

$$\|\nabla u_g(t)\|_\infty \leq C(M/\nu)/t. \quad (81)$$

Put $\delta = f - g$, $v = K * \delta$, and $F = (b - \tilde{b})f$. Then

$$\delta_t - \nu \Delta \delta + (u_f + \tilde{b}) \cdot \nabla \delta + v \cdot \nabla g = -\operatorname{div} F.$$

Let ψ be a streamfunction of δ . Testing in H^{-1} gives

$$\frac{1}{2}(\|v\|_2^2)' + \nu \|\delta\|_2^2 \leq (\|\nabla u_g\|_\infty + \|D\tilde{b}\|_\infty) \|v\|_2^2 + \|v\|_2 \|F\|_2. \quad (82)$$

Here $v \cdot \nabla \psi = 0$. Thus the term $v \cdot \nabla g$ vanishes when paired with ψ , and the same identity replaces $u_f = u_g + v$ by u_g in the other self transport term. Integration by parts bounds that term by $\|\nabla u_g\|_\infty \|v\|_2^2$. The identical calculation applies to $\tilde{b}$. The circulation of δ is zero, and its polynomial moments are finite, so $v \in L^2$ and these tests are justified by approximation.

The true coefficient in (82) is integrable at time zero, although the upper bound (81) is singular there. Indeed $g_0 \in L^p$, $p > 2$, gives from the same mild formula

$$\|\nabla u_g(t)\|_\infty \leq C_0(t^{-1/p} + t^{1/2-1/p}), \quad 0 < t \leq 1.$$

The constant may depend on the actual initial L^p norm and on the bounded cutoff drift. The kernel norms are $Ct^{-1/p}$ and $Ct^{-1/2-1/p}$ in $L^{p'}$; the latter time singularity is integrable, and $(u_g + \tilde{b})g$ has bounded L^p norm near zero by contraction and the initial L^1 - L^p Biot-Savart bound. Therefore the integrating factor in (82) starts from $v(0) = 0$. Only after writing its source integral do we use (81) on $[s, t]$, obtaining

$$\|v(t)\|_2 \leq \int_0^t (t/s)^{CM/\nu} e^{\ell(t-s)} \|F(s)\|_2 \, ds. \quad (83)$$

Common smooth approximations have bounded initial L^p norms, so no arbitrary initial term at a positive time is discarded in this argument.

The localization estimate, applied at each time $s \leq T$, gives

$$\int_{|x| \geq R_*} |x|^2 |f(s, x)| \, dx \leq CM e^{CLs} R_*^2 e^{-cR_*^2/(\nu s)}.$$

The cutoff field and the actual field have global growth $CL|x|$. Combining this tail estimate with Nash yields

$$\|(b - \tilde{b})f \mathbf{1}_{|x| \geq R_*}\|_2 \leq CLM \frac{R_*}{\sqrt{\nu s}} e^{CLs} e^{-cR_*^2/(\nu s)}.$$

Inside B_{R_*} the difference of drifts is exactly e . A weighted Kato estimate gives $\int |x|^2 |f(s)| \leq CM(r^2 + \nu s)e^{CLs}$. Consequently

$$\|ef \mathbf{1}_{|x| < R_*}\|_2 \leq C\epsilon LM \sqrt{\frac{r^2 + \nu s}{\nu s}} e^{CLs}.$$

Substitute these bounds into (83). Since $CM/\nu < 1/4$, integration of $s^{-1/2-CM/\nu}$ is legitimate, and yields, uniformly for $t \leq T$,

$$\|v(t)\|_2 \leq CM e^{CLT} \left[\epsilon L(T + r\sqrt{T/\nu}) + LR_* \sqrt{T/\nu} e^{-cR_*^2/(\nu T)} \right]. \quad (84)$$

The bounded factor $e^{\ell T}$ is absorbed into C .

For a radial cutoff χ_* equal to one in B_{R_*} and zero outside B_{2R_*} ,

$$\|\nabla(\chi_* \bar{z}^4)\|_2 \leq CR_*^4, \quad \left| \int \chi_* \bar{z}^4 \delta \right| \leq CR_*^4 \|v\|_2.$$

Both f and g satisfy the localized exit estimate, with the margins (79). Their exterior absolute fourth moments are at most $CMR_*^4 e^{CLT} e^{-cR_*^2/(\nu T)}$. Combining this with (84) proves (80). $\square$

7.3 Physical normalization, the available power margin, and scope

Set

$$m = M/\nu, \quad d = CC_3 r^4/\nu, \quad S = 2 + \nu T/r^2.$$

The space-time change $x = ry$, $t = r^2 s/\nu$, and $f(t, x) = \nu r^{-2} F(s, y)$ gives these quantities exactly; moreover $B_4(f) = \nu r^4 B_4(F)$. Applying Proposition 7.1 to the cutoff-field solution and Proposition 7.2 to the actual solution gives

$$\begin{aligned} \sup_{t \leq T} |B_4(f(t))| &\leq CM r^4 \left[\frac{M}{\nu} S^{7/50} (1 + \log S) + \frac{C_3 r^4}{\nu} S^{102/25} \right] \\ &\quad + CM R_*^4 e^{CLT} \left[\epsilon L(T + r\sqrt{T/\nu}) + (1 + LR_* \sqrt{T/\nu}) e^{-cR_*^2/(\nu T)} \right]. \end{aligned} \quad (85)$$

This requires (71) and the small normalized linear-growth bound of the cutoff field in addition to the local hypotheses. No cutoff has been applied to the actual nonlinear solution.

For the geometric cascade budget take a receiver (A_i, R_i) and

$$r = qR_i, \quad M \leq C\beta A_i q^2 R_i^2, \quad \kappa = \frac{\nu}{A_i R_i^2} = q^{3/2}, \quad T = H/A_i, \quad R_* = \kappa^{1/4} R_i.$$

Suppose the actual outer-colour estimates furnish

$$L \leq CA_i, \quad C_3 \leq C\delta A_i/R_i^2, \quad \epsilon \leq Ce^{-c/(\kappa H)}.$$

Then $m = O(\beta q^{1/2})$, $d = O(\delta q^{5/2})$, and $S = O(q^{-1/2}H)$. Dividing the first line of (85) by $A_i R_i^6$ gives

$$O\left(\beta^2 q^{643/100} H^{7/50} (1 + \log S) + \beta \delta q^{323/50} H^{102/25}\right). \quad (86)$$

The powers are 6.43 and 6.46. The localization terms are smaller than every q -power for logarithmic H , since

$$\frac{R_*^2}{\nu T} = \frac{1}{\sqrt{\kappa} H}, \quad LT = O(H), \quad C_3 R_*^2 T = O(\delta \sqrt{\kappa} H).$$

The other margins, including $(M/\nu)\sqrt{\nu T} \ll R_*$ and $mS^{1/25}\sqrt{1 + \log S} \ll 1$, hold for sufficiently small q . Both terms in (86) are consequently $o(q^6)$. A fixed prefactor in the support radius changes only the constants.

The actual outer-core argument supplies the required harmonic-plus-leak hypothesis conditionally on the density bound

$$\sup_{t \leq T} \|(I - P)w_l(t)\|_1 \leq \delta A_l R_l^2, \quad l \leq i.$$

Its harmonic exterior multipole series can be differentiated to give $|Db_h(x)| \leq CC_3|x|^2$ throughout B_{2R_*} when this disk lies strictly inside the constant-core probe region; the exponentially small core density defect contributes the rough leak. Thus the present section closes full-window non-linear inner-tail suppression under this explicit outer-density input. It does not prove that input for the active colour and older seeds, a signed outward transfer, or distinct vanishing-viscosity accumulation paths.

Remark 7.3 (Selected-cutoff scope). The displayed $q^{6.43}$ and $q^{6.46}$ budget uses $\kappa_i = q^{3/2}$ at the selected cutoff level. At later, larger receivers the same physical viscosity has a smaller normalized value; their aggregate of smaller colours includes the intended active child. The budget is not an erasure estimate for that entire aggregate at every later stage. After a source leaves its small cap neighbourhood, its constant-core probe on a canonical source window requires a separate comparison; it is not a consequence of the small-density hypothesis over that post-cap window.

8 Fixed seeds: separate initial and forced prefix contributions

The result in this section is a majorant lemma. Its coupling inequality is an explicit hypothesis; no assertion that diffusive colours already satisfy that inequality is made.

Lemma 8.1 (Two prefix norms). *Let $\beta \geq 4$, $A_i = \beta^{i-1}$, and fix a receiving index $j \geq 2$. Suppose nonnegative continuous functions x_i , $1 \leq i < j$, satisfy, on $0 \leq \theta \leq S$,*

$$x_i(\theta) \leq Mbe^{\lambda_0 \theta} + M \int_0^\theta e^{\lambda_0(\theta-s)} \left(\sum_{l < j} K_{il} x_l(s) + f_i(s) \right) ds, \quad \lambda_0 = \frac{\sigma + \eta}{\beta}, \quad (87)$$

where $M, C, \sigma > 0$, $\eta, r_* > 0$, $f_i \geq 0$ are integrable, and the nonnegative matrix K obeys

$$K_{ii} \leq Cr_*/\beta, \quad K_{il} \leq \begin{cases} C\beta^{l-j}q^{2(i-l)}, & l < i, \\ C\beta^{l-j}q^{6(l-i)}, & i < l < j. \end{cases}$$

Assume $\beta q^6 < 1$. Define

$$r_B = \frac{C}{\beta} \left(r_* + \frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right), \quad \lambda_B = \lambda_0 + M r_B.$$

Then $x_i \leq h_i + p_i$, where the initial contribution satisfies

$$\max_{i < j} h_i(\theta) \leq M b e^{\lambda_B \theta}. \quad (88)$$

For $D > \beta$ with $Dq^8 < \beta$, put $w_i = (Dq^8)^{j-i}$ and

$$r_P = \frac{C}{\beta} \left(r_* + \frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right), \quad \lambda_P = \lambda_0 + M r_P.$$

The forced contribution satisfies

$$P(\theta) := \max_{i < j} \frac{p_i(\theta)}{w_i} \leq M \int_0^\theta e^{\lambda_P(\theta-s)} \max_{i < j} \frac{f_i(s)}{w_i} ds. \quad (89)$$

All constants are independent of j . One can choose η, r_* small, then D large and q small, so both λ_B and λ_P are strictly smaller than $\sigma/2$.

Proof. The row sums of K are bounded by r_B : for $l = i + d$ the inner coefficient is at most $C\beta^{i-j}(\beta q^6)^d$, and for $l = i - d$ the outer coefficient is at most $C\beta^{i-j}(q^2/\beta)^d$. Use $\beta^{i-j} \leq 1/\beta$ and sum the geometric series.

Solve the positive comparison system

$$Y' = (\lambda_0 I + MK)Y + Mf, \quad Y(0) = Mb\mathbf{1}.$$

Iteration of (87), or its integral comparison principle, gives $x \leq Y$. The variation-of-constants formula splits $Y = h + p$, with

$$h = e^{(\lambda_0 I + MK)\theta} Mb\mathbf{1}, \quad p = M \int_0^\theta e^{(\lambda_0 I + MK)(\theta-s)} f(s) ds.$$

The unweighted maximum norm gives (88). For the weighted maximum norm, the inner row acquires $(\beta/D)^d$ and the outer row acquires $(Dq^8/\beta)^d$. Their geometric sums give r_P and hence (89). Since λ_0 is close to $\sigma/\beta \leq \sigma/4$, first make the diagonal term small. Increasing D makes the remaining q -independent term in r_P small; subsequently decreasing q makes all other terms small. $\square$

Corollary 8.2 (The fixed-seed baseline is small in its actual field). *Assume the hypotheses of Lemma 8.1, with $\lambda_B \leq \lambda_* < \sigma/2$, in the clock $\theta = A_{jt}$. Suppose the completed cap clocks satisfy*

$$T_{j,N} \leq S/A_N + H \sum_{m=j}^{N-1} A_m^{-1}, \quad N \geq j, \quad S \geq H\beta/(\beta - 1).$$

Let $b = \delta e^{-\sigma S}$. Then through $T_{j,N}$ the initial contribution is bounded uniformly by

$$B = M\delta e^{-(\sigma - \lambda_*)S}.$$

Its normalized inward feedback on the receiver j is at most

$$C \sum_{i < j} \beta^{i-j} q^{2(j-i)} h_i \leq CB \frac{q^2/\beta}{1 - q^2/\beta}. \quad (90)$$

The forced contribution has feedback at most

$$CP \frac{Dq^8/\beta}{1 - Dq^8/\beta}.$$

In particular, with $H = h_0 \log(1/q)$ one can take $S = C_0 \log(1/q)$ so that $B = O(q^6)$ and its feedback is $O(q^8)$, with constants independent of depth.

Proof. Writing $a = \beta^{j-N} \in (0, 1]$ gives

$$A_j T_{j,N} \leq Sa + \frac{H\beta}{\beta - 1}(1 - a) \leq S.$$

Apply (88). The actual inward row in (90) is a geometric series with ratio q^2/β . For $p_i \leq w_i P$ its ratio is Dq^8/β . Choose $C_0 \geq h_0\beta/(\beta - 1)$ and $C_0(\sigma - \lambda_*) > 6$. No weighted norm of the initial seeds is used. If $Dq^6 < 1$, the local chart bound follows from $x_i \leq B + w_i P \leq B + Dq^6 P$ whenever the latter quantity is below r_* . $\square$

The corollary repairs a specific scalar obstruction. To use it for Navier–Stokes one must prove (87), including its spatial q^2 and q^6 rows for diffusive colours, and bound the forced term P . Those remain coupled PDE hypotheses.

Remark 8.3 (A generic smooth-profile self rate). The same positive-matrix proof works with $\lambda_0 = \gamma/\beta$ for any fixed $\gamma > 0$. Choose $\beta > 4\gamma/\sigma$, then choose the foreign-row parameters so Mr_B and Mr_P are less than $\sigma/4 - \gamma/\beta$. For a differential inequality whose diagonal rate is at most $\gamma\beta^{i-j}$, use γ/β as its common upper bound. This permits a generic fixed-profile Kato rate; it does not identify that rate with the sharp unstable eigenvalue. The transform selection in Lemma 4.4 permits choosing a good β in any fixed sufficiently large interval, so this parameter change causes no conflict with the isolated source transform.

9 A fixed passive colour is determined by the total Euler path

The next result allows signed smooth colours as well as indicator colours. It therefore does not require a patch reference profile.

Proposition 9.1 (Recovery of fixed passive colours). *Let u be bounded and divergence free on $[0, T] \times \mathbb{R}^2$, with $u \in L^1([0, T]; W_{\text{loc}}^{1,p})$ for some $p > 1$. For each compactly supported $f_0 \in L^1 \cap L^\infty$ there is at most one solution of*

$$\partial_t f + u \cdot \nabla f = 0, \quad f(0) = f_0,$$

in $L^\infty([0, T]; L^1 \cap L^\infty)$ with its weak initial trace. It is renormalized and belongs to $C([0, T]; L^2)$.

Suppose $u_n \rightarrow u$ uniformly, $\sup_n \|u_n\|_\infty \leq V$, $\nu_n \rightarrow 0$, and $0 \leq \nu_n \leq 1$. Let f_n solve

$$\partial_t f_n + u_n \cdot \nabla f_n = \nu_n \Delta f_n, \quad f_n(0) = f_0,$$

with the L^1 , L^2 , and L^∞ contraction bounds and the weak-form equation. Then their transport limit f is unique and

$$f_n \longrightarrow f \quad \text{in } C([0, T]; L^1 \cap L^2).$$

The same conclusion holds for transported colours with $\nu_n = 0$.

Proof. We give the renormalization argument needed for uniqueness. Let $f_\varepsilon = f * \eta_\varepsilon$ be a spatial mollification. It satisfies

$$\partial_t f_\varepsilon + u \cdot \nabla f_\varepsilon = r_\varepsilon, \quad r_\varepsilon(x) = \int \nabla \eta_\varepsilon(y) \cdot [u(x) - u(x - y)] f(x - y) dy.$$

On a compact spatial set the difference-quotient estimate bounds its L^1 norm by $C \|\nabla u\|_p \|f\|_{p'}$, on a slightly larger set. This follows by writing the velocity difference as the integral of its gradient along the segment and then applying Hölder's inequality. Approximate u in $L^1_t W^{1,p}_{\text{loc}}$ by divergence-free spatial convolutions. For a smooth approximating velocity, divergence freedom allows subtraction of $f(x)$ in the commutator integral. Translation continuity of f then makes the commutator tend to zero in $L^1_{t,x}$. The uniform difference-quotient bound handles the velocity approximation. Thus $r_\varepsilon \rightarrow 0$ locally in L^1 .

Multiplication by a bounded derivative of a C^1 function and passage to the limit shows that bounded renormalizations satisfy the transport equation, including their initial trace. For the difference of two bounded solutions, use a renormalization equal to s^2 on their bounded value range. Integration against expanding cutoffs has a flux bounded by $C \|u\|_\infty R^{-1} \int |f|^2$. Since bounded integrable functions are square integrable, that flux tends to zero. The initial square norm is zero, which proves uniqueness. The same argument preserves the square norm of one solution. Weak time continuity and preservation of that norm give strong L^2 time continuity.

For convergence, the weak-form identity gives, for compact smooth φ ,

$$\left| \frac{d}{dt} \int \varphi f_n \right| \leq \|f_0\|_1 (V \|\nabla \varphi\|_\infty + \|\Delta \varphi\|_\infty).$$

The contraction bounds and this equicontinuity give weak L^2 convergence uniformly in time along subsequences. The Kato inequality and a radial cutoff also give uniform tightness:

$$\sup_{t \leq T} \int_{|x| > 2R} |f_n(t, x)| dx \leq \int_{|x| > R} |f_0(x)| dx + CT \|f_0\|_1 (V/R + 1/R^2).$$

Passing to the distributional equation identifies every limit as a bounded integrable transported solution with the given initial trace. Uniqueness identifies all subsequences.

To prove uniform strong L^2 convergence, suppose otherwise and choose bad times $t_n \rightarrow t$. Weak time equicontinuity gives $f_n(t_n) \rightharpoonup f(t)$ in L^2 , while contraction and renormalization give

$$\limsup_n \|f_n(t_n)\|_2 \leq \|f_0\|_2 = \|f(t)\|_2.$$

Hilbert-space uniform convexity gives strong convergence, a contradiction. Local L^1 convergence follows from L^2 , and tightness gives global uniform L^1 convergence. $\square$

For $p > 2$ the kernel split gives the explicit useful bound

$$\|K * h\|_\infty \leq C_p (\|h\|_1 + \|h\|_p) : \tag{91}$$

the near kernel belongs to $L^{p'}(B_1)$ because $p' < 2$, and the far kernel is bounded. Thus strong convergence of total vorticity paths in $C_t(L^1 \cap L^p)$ implies uniform convergence of their velocities. Whenever the limiting velocities have the Sobolev regularity in Proposition 9.1, the proposition applies to every fixed bounded initial colour. In particular, if two total Euler paths coincide, their recovered fixed colours coincide. No uniqueness of the total nonlinear Euler equation is used.

Let $f_{1,0}, \dots, f_{k,0}$ be finitely many fixed bounded compact colours and choose complex constants c_l . A suitable observable is

$$a_u(t) = \sum_{l=1}^k c_l \int \Phi_l(x) f_l[u](t, x) dx, \quad \Phi_l(x) = \chi_l(|x|) \bar{z}^4, \quad (92)$$

where each χ_l is a fixed smooth radial compact cutoff. The weak-form estimate makes a_u uniformly Lipschitz for a bounded family of drifts and $\nu \leq 1$, and the proposition makes it continuous under the strong total-path convergence just specified. These tests cover one smooth outer colour and also a moment coordinate made from several patch colours. If uncut moments are wanted in an Euler limit, the cutoffs can equal one on a common support ball: bounded velocity gives finite propagation. Indeed test $|f|$ against a nondecreasing smooth function of $|x| - R_0 - Vt$; its material derivative is nonpositive and its initial pairing is zero.

10 Protected first caps and phase shooting

A phase evaluated at an arbitrary cap time need not distinguish trajectories. The following first-cap functional does distinguish them and requires no common cap clock.

Lemma 10.1 (Continuity of a protected first-cap phase). *Fix $\delta, c, e, h, L > 0$, with $2e < T$. Let $a : [0, T] \rightarrow \mathbb{C}$ be L -Lipschitz, let*

$$\tau(a) = \inf\{t \geq 0 : |a(t)| \geq \delta\}, \quad \mathcal{P}(a) = a(\tau(a))/\delta,$$

and suppose

$$\begin{aligned} |a(0)| &\leq \delta - h, & e &\leq \tau(a) \leq T - e, \\ |a(t)| &\leq \delta - h & (0 \leq t \leq \tau(a) - e), \\ |a(t)| - |a(s)| &\geq c(t - s) & (\tau(a) - e \leq s < t \leq \tau(a) + e). \end{aligned}$$

For any continuous b with $\varepsilon = \|a - b\|_\infty < \min(h/2, ce/4)$, its first cap exists,

$$|\tau(a) - \tau(b)| \leq 2\varepsilon/c, \quad |\mathcal{P}(a) - \mathcal{P}(b)| \leq \delta^{-1}(1 + 2L/c)\varepsilon.$$

Families satisfying the displayed conditions with common constants retain a protected first cap under uniform convergence. Hence $\mathcal{P}$ is a continuous functional on that protected class.

Proof. Before $\tau(a) - e$ the strict margin excludes any cap of b . On $[\tau(a) - e, \tau(a)]$ the radial lower-slope inequality gives $|a(t)| \leq \delta - c(\tau(a) - t)$. Thus b cannot cap before $\tau(a) - 2\varepsilon/c$. At $\tau(a) + 2\varepsilon/c$, its modulus is greater than or equal to $\delta + \varepsilon$, so it has capped by that time. This proves the time estimate. Since both first-cap values have modulus δ , the phase estimate follows from

$$|a(\tau(a)) - b(\tau(b))| \leq L|\tau(a) - \tau(b)| + \|a - b\|_\infty.$$

For a uniformly convergent protected sequence, take a convergent subsequence of its cap times. The common Lipschitz bound, the precap margin, and the integrated radial lower-slope inequality pass to the limit. They identify the limiting time as the unique first cap. This argument applies to every subsequence and proves the final assertion. $\square$

Corollary 10.2 (Separation of total inviscid paths). *Consider one fixed total initial datum and fixed bounded initial colours. Suppose two Navier–Stokes subsequences converge strongly in $C_t(L^1 \cap L^p)$, their limits satisfy the velocity regularity of Proposition 9.1, and their observables (92) have uniformly protected first caps. If their limiting first-cap phases are different, the two total Euler paths are different. The cap times may differ.*

Proof. Proposition 9.1 and Lemma 10.1 pass the colours, observables, and phases to the limit. Equal total paths would have equal velocities and recovered colours, hence equal first-cap phases, a contradiction. $\square$

Proposition 10.3 (Conditional degree shooting). *Fix $\nu > 0$ and an active index $N \geq 2$. Suppose there is a continuous initial-data family indexed by $(r, \theta) \in [0, 1] \times \mathbb{S}^1$: the active seed at N has phase θ , and all other seed amplitudes are multiplied by r . Assume the total and passive-colour solution paths at this fixed viscosity depend continuously on these parameters. Assume their outer observables have first caps protected by the common constants of Lemma 10.1.*

If at $r = 0$ all other colours are radial and the active seed family is obtained by spatial rotation through angle $-\theta/4$, then the output phase map at $r = 1$ is surjective onto $\mathbb{S}^1$.

Proof. The first-cap map is continuous by Lemma 10.1. At $r = 0$, rotational covariance and the radial cutoffs in (92) give

$$\mathcal{P}(a_{0,\theta}) = e^{i\theta} \mathcal{P}(a_{0,0}).$$

Its degree is one. The degree is constant along the homotopy in r : a continuous argument lifted on the rectangle $[0, 1] \times [0, 2\pi]$ has endpoint difference $2\pi d(r)$, and the continuous integer $d(r)$ is constant. A circle map omitting one point admits a periodic continuous argument and has degree zero. The map at $r = 1$ therefore cannot omit a point. $\square$

The continuity hypothesis at fixed positive viscosity is routine and does not hide a uniform-in-viscosity estimate. For completeness, in $X = L^1 \cap L^p$, (91) makes $(K * w)w$ locally Lipschitz on bounded X balls. The mild vorticity equation gives

$$d(t) \leq d(0) + \frac{C_p M}{\sqrt{\nu}} \int_0^t (t-s)^{-1/2} d(s) \, ds.$$

Short intervals on which the integral coefficient is at most $1/2$, followed by iteration, prove continuity for each fixed ν . The passive mild equation has the same kernel and an additional source controlled by the uniform drift difference. It gives continuity of colours, including when their initial bounded domains vary continuously. Uniform cap protection across the homotopy, by contrast, is a substantive coupled PDE hypothesis.

Proposition 10.4 (Conditional sparse-phase construction). *Fix $p > 2$ and $\beta q^{2/p} < 1$. Let a family of infinitely seeded, zero-circulation compact initial data be indexed by a phase at every depth. Assume its series converges absolutely in $X = L^1 \cap L^p$, that changing only the phase at depth N changes the datum by at most $\eta_N \rightarrow 0$, uniformly in all phases, and that depths $N \geq 2$ leave the fixed outer initial colours unchanged.*

Suppose $\nu_N \downarrow 0$, and for every depth and every choice of all other phases, Proposition 10.3 applies with common cap-protection constants. Finally suppose that for every datum in this family its Navier–Stokes family is relatively compact in $C_t X$, and every limit satisfies the velocity hypothesis of Proposition 9.1. Then some single datum in this family has two distinct vanishing-viscosity accumulation paths.

Proof. Start with arbitrary default phases at all depths. At step k , choose a new depth $N_k > N_{k-1}$ with $\nu_{N_k} < \nu_{N_{k-1}}/2$ and

$$\eta_{N_k} < 2^{-k} \min(1, r_1, \dots, r_{k-1}).$$

Shoot its phase to give first-cap output $(-1)^k$ at ν_{N_k} , using Proposition 10.3, and call the resulting datum g_k . Fixed-positive-viscosity continuity gives $r_k > 0$ such that an admissible datum within r_k of g_k changes that output by less than $1/4$. All admissible future phase choices already have protected caps by hypothesis; openness of a derivative lower bound in the uniform topology is not asserted.

The summable phase changes give one limiting datum g_∞ ; each coordinate is changed at most once and the absolutely convergent series identifies the limit with that phase sequence. For each k ,

$$\|g_\infty - g_k\|_X \leq \sum_{l>k} 2^{-l} r_k < r_k.$$

Hence its selected-viscosity output lies within $1/4$ of $(-1)^k$. Relative compactness supplies limiting paths along each parity. Their first-cap phases lie in the disjoint closed $1/4$ -neighbourhoods of $+1$ and -1 . Corollary 10.2 proves that the total paths are different. Zero circulation and compact support pass by strong L^1 convergence. $\square$

The proposition is an implication with explicit uniform cap and coupling hypotheses. It is not an unconditional Navier–Stokes counterexample. Every intermediate datum retains the full unbounded infinite background; the proof does not assert arbitrary-small macroscopic controllability around a bounded finite prefix.

11 Actual inactive densities and a large amplitude ratio

Fix the smooth radial reference W and $p > 2$. The estimate below uses the actual coloured Navier–Stokes equation and requires no derivative bound on an inactive colour. Its input is a weighted history bound for the current and deeper packet. That input remains part of the interacting active theorem, and is not proved in this section.

For clarity, the amplitude ratio can be chosen large after W and p are fixed. A large fixed ratio compensates for the ordinary transport-stability constant of the smooth reference; it is not asserted that this constant equals its unstable eigenvalue.

Let P denote angular averaging. Use

$$\begin{aligned} A_l &= A_1 \beta^{l-1}, & R_l &= R_1 q^{l-1}, \\ G_l(t, x) &= A_l \left[e^{\nu t R_l^{-2} \Delta} W \right] (x/R_l), \\ \zeta_l &= w_l - G_l, & h_l &= (I - P)w_l = (I - P)\zeta_l. \end{aligned}$$

For any density v put

$$\|v\|_{X_l} = \frac{\|v\|_1}{A_l R_l^2} + \frac{\|v\|_p}{A_l R_l^{2/p}}, \quad x_l(t) = \|\zeta_l(t)\|_{X_l}.$$

Assume the initial colours are real, fourfold symmetric, have zero circulation, are supported in $B_{c_0 R_l}$, equal a constant in $B_{a_0 R_l}$, and satisfy

$$\|w_l(0)\|_1 \leq C_0 A_l R_l^2, \quad \|w_l(0)\|_\infty \leq C_0 A_l.$$

Here a_0, c_0, C_0 are fixed. For the smooth seeded family these facts follow from the fixed compact derivative collars. The series of colours is absolutely convergent in $L^1 \cap L^p$, and $\beta q < 1$, so its common velocity and coloured evolution are well defined as before.

Choose a fixed broad annulus $\mathcal{A}_l$, containing both derivative collars of $W(x/R_l)$, whose two radii are fixed multiples of R_l .

Proposition 11.1 (Stopped inactive row with explicit parabolic errors). *There are constants $\gamma, C, c > 0$, depending only on the fixed smooth profile, p , and the displayed initial prefactors, with the following property. Fix $j \geq 2$ and*

$$T \leq S/A_j, \quad \kappa_j = \frac{\nu}{A_j R_j^2} \leq q^{3/2}, \quad 1 \leq S \leq C_1 \log(1/q).$$

Stop at the first time any $i < j$ reaches $x_i = \delta$. Assume $\beta \geq 4$, $\beta q \leq 1/2$, q sufficiently small, and

$$CS(\sqrt{\delta} + \beta q) \leq c. \quad (93)$$

In the clock $\theta = A_j t$, every inactive $i < j$ obeys

$$\frac{d^+ x_i}{d\theta} \leq \gamma \beta^{i-j} x_i + C \sum_{l < i} \beta^{l-j} q^{2(i-l)} x_l + C \sum_{i < l < j} \beta^{l-j} q^{6(l-i)} x_l + f_i(\theta) + E_{j-i}, \quad (94)$$

where

$$f_i(\theta) = \frac{1}{A_j} \left\| \mathbf{1}_{\mathcal{A}_i} \left(\sum_{l \geq j} K * h_l \right) \cdot \nabla G_i \right\|_{X_i} \quad \text{at } t = \theta/A_j$$

is the actual current/deep nonradial forcing. With

$$e_m = C e^{CS} \exp \left[-\frac{cq^{-2m}}{\kappa_j S} \right], \quad m \geq 1,$$

one can take

$$E_k = C \left[\beta^{-k} e_k + \sum_{r > k} \beta^{-r} e_r + \sum_{1 \leq r < k} \beta^{-r} q^{6(k-r)} e_r + e_k \sum_{1 \leq r < k} \beta^{-r} q^{k-r} \right]. \quad (95)$$

The sums may be enlarged to infinite sums of nonnegative terms. No small nonradial-density assumption is imposed on colours $l \geq j$.

Proof. The exact radial heat profile is an isolated Navier–Stokes solution. For the actual common velocity u , subtraction gives

$$(\partial_t + u \cdot \nabla - \nu \Delta) \zeta_i = - \left[K * \zeta_i + \sum_{l \neq i} K * w_l \right] \cdot \nabla G_i. \quad (96)$$

Every radial foreign velocity is tangential and annihilates ∇G_i , so the foreign fields on the right can be replaced by $K * h_l$. In particular diffusion produces no separate angular source. If every initial colour is radial, $\zeta_i = h_i = 0$ solves this system.

The scaled Biot–Savart split gives $\|K * \zeta_i\|_\infty \leq C_p A_i R_i x_i$. Heat contraction of the fixed reference gradient in L^1 and L^p , and the scalar Kato inequalities, bound the self term by $\gamma A_i x_i$. For example γ can be chosen as a constant multiple of $C_p(\|\nabla W\|_1 + \|\nabla W\|_p)$. No derivative of ζ_i enters this estimate; the actual nonlinear transport term disappears from the norm inequalities.

We verify the localized foreign-field estimates under the stop. For any fixed index l , the global radial component obeys

$$|u(t, x) \cdot e_r| \leq C A_l |x| + C A_l R_l \beta q. \quad (97)$$

Indeed every radial average has tangential velocity. For $k \leq l$, the fourfold origin-growth bound gives $|K * h_k(x)| \leq C \|h_k\|_\infty |x| \leq C A_k |x|$, and $\sum_{k \leq l} A_k \leq C A_l$. For $k > l$ use $\|K * w_k\|_\infty \leq C A_k R_k$ and $\sum_{k > l} A_k R_k \leq C A_l R_l \beta q$. The same estimate holds for only the foreign radial field. Thus no divergent sum of amplitudes is used, even when κ_j is much smaller than $q^{3/2}$.

Apply the actual outer-core bootstrap to the stopped finite prefix $l \leq j - 1$. Its nonradial-density hypothesis follows from $\|h_l\|_1 \leq 2 A_l R_l^2 x_l \leq 2 \delta A_l R_l^2$. All colours $k \geq j$ enter that bootstrap through the unrestricted, convergent velocity sum $\sum_{k > l} A_k R_k$. The parameter margin (93) and small q therefore give the individual core estimate

$$\|w_l(t) - c_l\|_{L^\infty(B_{a R_l})} \leq C A_l e^{-c R_l^2 / (\nu t)}, \quad a = a_0/2. \quad (98)$$

The bootstrap error threshold is of order S^{-1} ; its improvement follows from $\kappa_j S \ll 1$ just as in the outer-core proof. Retaining the individual radius in that proof, rather than its maximum over l , gives (98).

Fix $c_2 > c_0$. On $B_{c_2 R_l}$ the own nonradial velocity is bounded by $C \sqrt{\delta} A_l R_l$. Older constant cores contribute only rotation; their harmonic remainders contribute $C \delta A_l R_l q^2 / \beta$ and their core leaks are exponentially small. All inner velocities contribute at most $C A_l R_l \beta q$. Choose q so that this ball is strictly inside every older core probe disk. By (93), the integrated radial drift is less than a fixed fraction of $(c_2 - c_0) R_l$. The radial exit estimate for the positive scalar evolution from $|w_l(0)|$ gives

$$\int_{|x| > c_2 R_l} |w_l(t, x)| \, dx \leq C A_l R_l^2 e^{-c R_l^2 / (\nu t)}. \quad (99)$$

For its post-exit fourth moment use $\phi(x) = (R_l^2 + |x|^2)^2$. Equation (97) gives $u \cdot \nabla \phi \leq C A_l \phi$, while $\nu \Delta \phi \leq C \nu R_l^{-2} \phi$. Conditioning at exit from $B_{c_2 R_l}$ consequently gives $\mathbb{E}[\phi(X_t) \mid \mathcal{F}_{\text{exit}}] \leq C R_l^4 e^{C(A_l + \nu/R_l^2)T} \leq C R_l^4 e^{CS}$. Multiplication by the exit probability proves

$$\int_{|x| > c_2 R_l} |x|^4 |w_l(t, x)| \, dx \leq C A_l R_l^6 e^{CS} e^{-c R_l^2 / (\nu t)}. \quad (100)$$

Angular averaging preserves radial integrals. Inside $B_{c_2 R_l}$ use $\|h_l\|_1 \leq 2 A_l R_l^2 x_l$; outside use (100). It follows that

$$\int |x|^4 |h_l(t, x)| \, dx \leq C A_l R_l^6 [x_l(t) + e_{j-l}]. \quad (101)$$

For an inner colour $i < l < j$, take a fixed small $\eta > 0$ such that $B_{\eta R_i}$ is inside the initial constant disk of colour i and $c_0 q < \eta/4$. In this ball the radial velocity estimate based on index i has the same small integrated bound: the own nonradial term is $C \sqrt{\delta} A_i R_i$, older remainders

are controlled by (98), and all inner colours enter by $CA_i R_i \beta q$. Starting the l th positive colour in $B_{c_0 R_l}$, radial exit from $B_{\eta R_i}$ therefore gives

$$\int_{|x| > \eta R_i} |w_l(t, x)| \, dx \leq CA_l R_l^2 e^{-cR_l^2/(\nu t)}. \quad (102)$$

This receiver-radius estimate is essential.

On $\mathcal{A}_i$, for $l < i$, split h_l into its core and exterior. Equation (98) bounds the core density by $CA_l e^{-cR_l^2/(\nu t)}$; the origin-growth estimate bounds its velocity by $CA_l |x| e^{-cR_l^2/(\nu t)}$. The exterior part has mass at most $2A_l R_l^2 x_l$ and a fourfold inverse-moment expansion starting at degree three. Thus

$$\frac{|K * h_l(x) \cdot e_r|}{R_i} \leq CA_l [q^{2(i-l)} x_l + e_{j-l}]. \quad (103)$$

For $i < l < j$, truncate h_l at radius ηR_i . Every radial truncation has zero circulation because $Ph_l = 0$, and fourfold symmetry cancels harmonic moments of degrees one to three. Its exterior field is bounded by $CR_i^{-5} \int |y|^4 |h_l(y)| \, dy$. For the remaining density use (102), $\|h_l\|_\infty \leq CA_l$, and Biot–Savart interpolation. This gives

$$\frac{|K * h_l(x) \cdot e_r|}{R_i} \leq CA_l [q^{6(l-i)} (x_l + e_{j-l}) + q^{l-i} e_{j-i}]. \quad (104)$$

The fixed exponent c in e_m has been reduced to absorb square roots and polynomial factors.

Away from $\mathcal{A}_i$, ∇G_i is the Gaussian convolution of the compactly supported reference gradient. Its scaled L^1 and L^p tails after multiplication by $|x| + R_i$ are bounded by $Ce^{-cR_i^2/(\nu t)}$. The foreign version of (97) consequently bounds the total omitted forcing by $CA_i e_{j-i}$. Combining this with the self estimate, (103), (104), and the explicitly retained f_i proves (94)–(95). $\square$

Corollary 11.2 (Inactive density closure under packet history). *Let $\sigma > 0$ be the growth rate of the fixed smooth reference. Choose β sufficiently large that $\gamma/\beta < \sigma/4$, then a fixed $D > \beta$ sufficiently large, then q sufficiently small. Set $w_i = (Dq^6)^{j-i}$ and $b = \delta e^{-\sigma S}$. Assume $x_i(0) \leq Mb$, and the packet-history bound*

$$\sup_{\theta \leq A_j T} \int_0^\theta e^{\lambda_P(\theta-s)} \max_{i < j} \frac{f_i(s)}{w_i} \, ds \leq C_* \delta, \quad (105)$$

where $\lambda_P < \sigma/2$ is the weighted comparison rate below. Take $S = C_2 \log(1/q)$, with C_2 sufficiently large and fixed, and δ small enough for (93). Then the stopped solution remains strictly below its stopping threshold:

$$x_i(t) \leq B + w_i [CC_* \delta + o(q^K)] < \delta/4, \quad B \leq M\delta e^{-(\sigma - \lambda_B)S},$$

for every $i < j$ and $t \leq T$. Here K is any fixed prescribed power, and the constants are independent of depth. This is an actual inactive-density conclusion conditional on (105); that packet input is not established by the corollary.

Proof. Bound each self coefficient in (94) by $\lambda_0 = \gamma/\beta$. This replaces, rather than identifies with, the earlier sharp spectral rate $(\sigma + \varepsilon)/\beta$. The unweighted off-diagonal row sum is at most

$$r_B = \frac{C}{\beta} \left[\frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right].$$

The row sum in the forced weights w_i is at most

$$r_P = \frac{C}{\beta} \left[\frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right].$$

Thus $\lambda_B = \lambda_0 + r_B$ and $\lambda_P = \lambda_0 + r_P$ are both below $\sigma/2$ in the prescribed parameter order. The positive comparison equation splits exactly into initial, packet, and parabolic-error contributions. Its initial part is bounded by $Mbe^{\lambda_B \theta}$, giving the stated B .

For the explicit error put

$$\eta(q) = \sup_{m \geq 1} \frac{e_m}{(\beta D q^6)^m}.$$

This is smaller than every power of q , uniformly when $\kappa_j \leq q^{3/2}$ and $S = O(\log(1/q))$. Indeed successive ratios in this supremum are less than $1/2$ for small q , and its first term has Gaussian exponent of order $q^{-7/2}/S$. A direct summation of the four terms of (95) gives

$$\sup_{k \geq 1} \frac{E_k}{(Dq^6)^k} \leq C\eta(q) \left[1 + \frac{Dq^6}{1 - Dq^6} + \frac{1}{D - 1} + \frac{\beta q}{1 - \beta q} \right].$$

For example the third sum becomes $D^{-k} \sum_{r < k} \beta^{-r} q^{-6r} e_r \leq \eta(q) D^{-k} \sum_{r < k} D^r$. The fourth uses $\sum_{r < k} \beta^{-r} q^{k-r} \leq \beta^{-k} \beta q / (1 - \beta q)$. Its weighted convolution is therefore at most $C\eta(q) S e^{\lambda_P S}$, still smaller than every q -power.

The positive-matrix variation of constants and (105) yield the asserted bound. Choose C_2 so $C_2(\sigma - \lambda_B) > 6$; then $B \ll \delta q^6$. Finally $\max_{i < j} w_i = Dq^6$, so small q makes the complete right side less than $\delta/4$, contradicting a first stopping time. The absolute leak envelope has not been called a physical angular source; the exact equation (96) is zero on the all-radial family. In the fixed nonzero seeded family, its error envelope is much smaller even than the prescribed seed b . $\square$

Corollary 11.3 (A static packet bound suffices for inactive density). *For the inactive-density conclusion alone, the packet-history hypothesis (105) may be replaced by*

$$\sup_{\theta \leq A_j T} \max_{i < j} \frac{f_i(\theta)}{w_i} \leq C_* \delta.$$

Fix $C_2 > 6/\sigma$. Choose β, D , and then q , so that

$$C_2(\sigma - \lambda_B) > 6, \quad C_2 \lambda_P < 1.$$

With $S = C_2 \log(1/q)$, the actual inactive densities still remain below $\delta/4$. Their normalized inward nonradial velocity on the current receiver's fixed collars is

$$O(\delta S q^{8 - C_2 \lambda_P}) + o(q^K) = o(\delta q^6).$$

The static packet bound is an explicit input to this corollary. Section 13 proves the stronger actual packet estimate used in the construction. This corollary does not supply a signed transfer or preserve the sharper pre-cap histories needed for phase protection.

Proof. The stated parameter choices are possible since λ_B, λ_P can be made arbitrarily small by taking the fixed β, D large and then q small. The packet contribution to the forced weighted norm is at most $CC_*\delta S e^{\lambda_P S}$. Thus

$$x_i \leq B + (Dq^6)^{j-i} \left[CC_*\delta S q^{-C_2\lambda_P} + o(q^K) \right].$$

The baseline $B = o(\delta q^6)$ and the nearest forced bound $C\delta S q^{6-C_2\lambda_P}$ are both below the stopping threshold. The parabolic errors remain smaller than every power of q . For the inward field use the actual older-core multipole estimate:

$$\sum_{i < j} \beta^{i-j} q^{2(j-i)} x_i \leq B \frac{q^2}{\beta - q^2} + CC_*\delta S q^{-C_2\lambda_P} \frac{Dq^8}{\beta - Dq^8} + o(q^K).$$

Since $C_2\lambda_P < 1$, this is $o(\delta q^6)$. The proof controls magnitudes; it asserts no lower bound on a complex unstable-mode impulse. $\square$

12 Compactness for one fixed compact zero-circulation datum

The compactness needed for phase separation does not require a new cascade hypothesis. We prove it for the initial-data class used in the construction. Only standard existence of the planar Navier–Stokes solution, its vorticity contraction bounds, and its finite-energy inequality are inputs. These inputs are obtained by smooth approximation of the initial datum; no Euler uniqueness or viscosity-selection principle is used.

Lemma 12.1 (The velocity estimates used below). *Let $p > 2$, $\alpha = 1 - 2/p$, and $w \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Then*

$$\|K * w\|_\infty + \|K * w\|_p \leq C_p(\|w\|_1 + \|w\|_p), \quad [K * w]_{C^\alpha} \leq C_p\|w\|_p, \quad \|\nabla K * w\|_p \leq C_p\|w\|_p.$$

*If w has compact support and zero integral, then $K * w \in L^2$.*

Proof. For the supremum norm split K at radius one: its near part is in $L^{p'}$ since $p' < 2$, and its far part is bounded. For the L^p norm the near part is in L^1 and the far part is in L^p ; apply Young's inequality. For a velocity difference at points a distance d apart, the two radius- $2d$ disks contribute at most $C_p\|w\|_p d^\alpha$. On their complement use $|K(x-z) - K(y-z)| \leq Cd|x-z|^{-2}$ and Hölder's inequality; the same power d^α follows.

We recall a direct proof of the derivative bound to specify the analytic input. Each component T of $\nabla K *$ has a Fourier multiplier bounded in absolute value, and hence is bounded on L^2 by Plancherel. Its off-origin kernel satisfies $|t(x)| \leq C|x|^{-2}$, $|\nabla t(x)| \leq C|x|^{-3}$ and has zero angular average; a possible multiple of the identity is harmless. For $f \in L^1 \cap L^2$ decompose at height λ into $g + \sum b_Q$ using maximal dyadic squares with average of $|f|$ larger than λ . The squares are disjoint, and

$$\|g\|_\infty \leq 4\lambda, \quad \|g\|_1 \leq \|f\|_1, \quad \sum |Q| \leq \|f\|_1/\lambda, \quad \int b_Q = 0, \quad \sum \|b_Q\|_1 \leq 2\|f\|_1.$$

The L^2 bound controls the level set of Tg by $C\|f\|_1/\lambda$. Cancellation and the kernel derivative estimate give

$$\int_{\mathbb{R}^2 \setminus 5Q} |Tb_Q(x)| dx \leq \int_Q |b_Q(y)| \int_{\mathbb{R}^2 \setminus 5Q} |t(x-y) - t(x-c_Q)| dx dy \leq C\|b_Q\|_1.$$

The last inner integral is bounded by $C|y - c_Q| \int_{C\ell(Q)}^\infty r^{-3} r \, dr$. This proves weak type $(1, 1)$, first for finite sums and then by their L^2 convergence. Splitting f above and below a height proportional to λ consequently gives

$$|\{|Tf| > \lambda\}| \leq C \left[\lambda^{-1} \int_{|f| > c\lambda} |f| + \lambda^{-2} \int_{|f| \leq c\lambda} |f|^2 \right].$$

Integration against $s\lambda^{s-1} d\lambda$, with $1 < s < 2$, and Fubini bound the L^s norm: the two integrals produce finite factors $(s-1)^{-1}$ and $(2-s)^{-1}$. The adjoint has the same bounds, so duality with $s = p'$ proves the required L^p estimate. Density identifies the result with the distributional derivative of $K * w$.

Finally, compact support gives a finite first absolute moment. At low frequencies zero circulation implies $|\widehat{w}(\xi)| \leq |\xi| \int |x| |w(x)| \, dx$. At high frequencies use $w \in L^2$, which follows by interpolation between L^1 and L^p . The Fourier multiplier $i\xi^\perp/|\xi|^2$ and Plancherel prove $K * w \in L^2$. $\square$

Theorem 12.2 (Strong vanishing-viscosity precompactness). *Let $2 < p < \infty$ and let one fixed real initial vorticity satisfy*

$$\omega_0 \in L^1 \cap L^p, \quad \text{supp } \omega_0 \subset B_{R_0}, \quad \int \omega_0 = 0.$$

For every sequence $\nu_n \downarrow 0$, the unforced planar Navier–Stokes vorticities with this same initial datum have a subsequence converging strongly in $C([0, T]; L^1 \cap L^p)$ for each fixed T . Every limit is a renormalized Euler solution with initial vorticity ω_0 . Its velocity is bounded, belongs to $L_t^\infty W^{1,p}$, and its vorticity preserves the L^1 and L^p norms and circulation at every time. In particular the Sobolev-velocity and compactness hypotheses needed for fixed-colour first-cap separation hold for this class.

Proof. Write $m = \|\omega_0\|_1$, $P = \|\omega_0\|_p$ and $E = \|K * \omega_0\|_2$. The last quantity is finite by Lemma 12.1. Contraction and the kinetic-energy inequality give uniformly, for $0 < \nu_n \leq 1$,

$$\|\omega_n(t)\|_1 \leq m, \quad \|\omega_n(t)\|_p \leq P, \quad \|u_n(t)\|_2 \leq E, \quad \|u_n(t)\|_\infty \leq V, \quad V = C_p(m + P). \quad (106)$$

The velocity lemma also bounds $[u_n(t)]_{C^\alpha}$ and $\|\nabla u_n(t)\|_p$, where $\alpha = 1 - 2/p$.

Local compactness of velocity. Apply the Fourier projection onto divergence-free fields to the velocity equation. It is bounded on every H^s by its bounded matrix multiplier. Since $\|u_n \otimes u_n\|_2 \leq VE$, the equation gives

$$\|\partial_t u_n\|_{H^{-2}} \leq CE(V + 1).$$

For a spatial mollifier η_h , $0 < h \leq 1$, $\|\eta_h\|_{H^2} \leq Ch^{-3}$ in two dimensions. Hence

$$\|u_n(t) - u_n(s)\|_\infty \leq Ch^\alpha + Ch^{-3}|t - s|.$$

Here the first term comes from the spatial Hölder bound and the second from duality of H^{-2} and H^2 after convolution. Taking $h = |t - s|^{1/(3+\alpha)}$ proves uniform time equicontinuity. Arzelà–Ascoli on compact cylinders and a diagonal extraction give

$$u_n \longrightarrow u \quad \text{locally uniformly on } [0, T] \times \mathbb{R}^2.$$

The uniform gradient bound passes weakly to $\nabla u \in L_t^\infty L^p$. Also u is bounded and divergence free. The L^p velocity bound in Lemma 12.1 passes by lower semicontinuity, so $u \in L_t^\infty W^{1,p}$.

Weak vorticity compactness and its tightness. For compact smooth φ the scalar equation gives

$$\left| \frac{d}{dt} \int \varphi \omega_n \right| \leq m(V \|\nabla \varphi\|_\infty + \|\Delta \varphi\|_\infty).$$

A countable dense set of tests in $L^{p'}$, the uniform L^p bound, and Arzelà–Ascoli provide a weakly continuous L^p path ω such that

$$\sup_{t \leq T} \left| \int \varphi(\omega_n(t) - \omega(t)) \right| \rightarrow 0 \quad \text{for every } \varphi \in L^{p'}. \quad (107)$$

Its initial value is ω_0 . Each weak value also has L^1 norm at most m , by testing on finite balls and then enlarging them.

Let χ_R be zero on B_R , one outside B_{2R} , with first and second derivatives bounded by C/R and C/R^2 . Kato's inequality and (106) give

$$\sup_{t \leq T} \int_{|x| > 2R} |\omega_n(t, x)| dx \leq \int_{|x| > R} |\omega_0(x)| dx + CTm(V/R + 1/R^2). \quad (108)$$

The same bound passes to $\omega(t)$ by local lower semicontinuity and monotone enlargement of the outer integration region.

The local uniform convergence of u_n and weak vorticity convergence pass the nonlinear drift product in compactly supported tests. The diffusion term is bounded by $\nu_n P \|\Delta \varphi\|_{p'}$ and vanishes. Thus

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \omega(0) = \omega_0.$$

We identify $u = K * \omega$ after establishing strong convergence below; this avoids requiring a preliminary global compactness argument for velocities.

Renormalization and traces at every time. Spatial mollification gives

$$\partial_t \omega_\varepsilon + u \cdot \nabla \omega_\varepsilon = r_\varepsilon,$$

where, using divergence freedom to subtract $\omega(x)$,

$$r_\varepsilon(x) = \int \nabla \eta_\varepsilon(y) \cdot [u(x) - u(x - y)] [\omega(x - y) - \omega(x)] dy.$$

The Sobolev difference estimate and Hölder's inequality yield

$$\|r_\varepsilon(t)\|_{p/2} \leq C \|\nabla u(t)\|_p \sup_{|y| \leq \varepsilon} \|\omega(t, \cdot - y) - \omega(t)\|_p.$$

Translation continuity and dominated convergence imply $r_\varepsilon \rightarrow 0$ in $L_t^1 L_x^{p/2}$. Since $p/2 > 1$, this gives local space-time L^1 convergence. For every C^1 function β with bounded derivative, the chain rule therefore passes to the renormalized transport equation.

There is no ambiguity about endpoint traces in this argument. For fixed ε the mollified equation is absolutely continuous in time in local smooth spatial norms. Integrate its renormalized equation between *any fixed* times s, t . At each of these fixed times, $\omega_\varepsilon(s) = \eta_\varepsilon * \omega(s) \rightarrow \omega(s)$ and $\omega_\varepsilon(t) = \eta_\varepsilon * \omega(t) \rightarrow \omega(t)$ strongly in L^p , by the spatial approximate-identity property. The Lipschitz function β passes these endpoint terms in local L^1 , while the commutator error vanishes. Consequently the renormalized identity has traces $\beta(\omega(s))$ and $\beta(\omega(t))$ for every s, t , including zero.

For $\beta(0) = 0$, integrate against cutoffs equal to one on B_R and vanishing outside B_{2R} . The flux error between any two times is at most $CTVm\|\beta'\|_\infty/R$. Letting $R \rightarrow \infty$ proves

$$\int \beta(\omega(t)) = \int \beta(\omega_0) \quad (0 \leq t \leq T).$$

To recover the L^p norm, use the convex tangent truncations

$$\beta_M(s) = \begin{cases} |s|^p, & |s| \leq M, \\ pM^{p-1}|s| - (p-1)M^p, & |s| > M. \end{cases}$$

They are C^1 with bounded derivative, nonnegative, and increase to $|s|^p$. For $|s| > M$ their derivative with respect to M is $p(p-1)M^{p-2}(|s| - M) \geq 0$. Monotone convergence therefore proves

$$\|\omega(t)\|_p = \|\omega_0\|_p \quad \text{at every } t. \quad (109)$$

Using $\sqrt{s^2 + \varepsilon^2} - \varepsilon$ and then $\varepsilon \downarrow 0$ proves conservation of the L^1 norm. Using $\beta(s) = s$ preserves circulation.

Strong convergence, uniformly in time. For $p \geq 2$ the pointwise Clarkson inequality gives

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{\|f\|_p^p + \|g\|_p^p}{2}.$$

Weak time continuity and the constant norm (109) imply $\omega \in C_t L^p$ by this inequality and lower semicontinuity. If $\omega_n \rightarrow \omega$ were not uniform strongly in L^p , choose bad times $t_n \rightarrow t$. Equation (107) gives $\omega_n(t_n) \rightharpoonup \omega(t)$, while contraction gives

$$\limsup_n \|\omega_n(t_n)\|_p \leq P = \|\omega(t)\|_p.$$

The same inequality proves strong convergence to $\omega(t)$; continuity of ω contradicts the bad times. Thus convergence is strong in $C_t L^p$. It is strong in $C_t L^1$ as well: on a fixed ball use Hölder's inequality, and outside it use (108) for both paths.

Finally Lemma 12.1 gives uniform global convergence $u_n = K * \omega_n \rightarrow K * \omega$. Their already obtained local limit is therefore exactly $u = K * \omega$. This proves the Euler equation and all claimed regularity and compactness statements. If a common subsequence on all finite windows is desired, repeat the extraction on $T = 1, 2, \dots$ and take the diagonal subsequence. $\square$

Corollary 12.3. *For the compact zero-circulation initial-data class of Theorem 12.2, the relative compactness as $\nu \downarrow 0$ and Sobolev-velocity assumptions in Proposition 10.4 are automatic. The substantive remaining hypotheses there are uniform protected caps and phase shooting for the actual coupled system.*

13 An actual packet bound closes the inactive density stop

The following estimate removes the packet input from the inactive-density conclusion. It does not give the signed history needed by an active receiver. All colours in this section are the actual scalar colours of the common unforced Navier–Stokes velocity. Use the notation and fixed profile hypotheses of Proposition 11.1. In addition choose q so small that $\beta q^{2/p} \leq 1/2$.

Let $\mathcal{S}(t, s)$ be the positive scalar transport–diffusion propagator for that common velocity and put

$$P_j(t) = \mathcal{S}(t, 0) \left(\sum_{l \geq j} |w_l(0)| \right), \quad v_j = \sum_{l \geq j} w_l.$$

The symbol P_j here denotes a positive envelope, not angular averaging. The initial sum is in $L^1 \cap L^p$ and is supported in $B_{c_0 R_j}$. Positivity and monotone convergence imply

$$\sum_{l \geq j} |w_l(t)| \leq P_j(t), \quad \|P_j\|_1 \leq C A_j R_j^2, \quad \|P_j\|_p \leq C A_j R_j^{2/p}.$$

No finite L^∞ norm of this positive aggregate is asserted.

Lemma 13.1 (Actual packet moments). *For every fixed integer $n \geq 2$, with $\theta = A_j t$ and $\kappa_j = \nu / (A_j R_j^2)$, one has*

$$\int (R_j^2 + |x|^2)^{n/2} P_j(t, x) \, dx \leq C_n A_j R_j^{n+2} (1 + \theta + \sqrt{\kappa_j \theta})^n e^{C_n \theta / \beta}. \quad (110)$$

Constants are independent of j , depth, viscosity and all seed phases. No stopping or small-density assumption is required for this lemma.

Proof. Write $u = u_{<j} + u_{\geq j}$. Individual colour contractions and Biot–Savart interpolation give

$$\|u_{\geq j}\|_\infty \leq C \sum_{l \geq j} A_l R_l \leq C A_j R_j.$$

The fourfold origin-growth inequality applied to each older colour yields $|u_{<j}(x)| \leq C A_j |x| / \beta$ for $\beta \geq 4$. Thus the actual radial drift is bounded by $L|x| + V$, with $L = C A_j / \beta$ and $V = C A_j R_j$. Tangential rotation does not affect radial weights.

Let $M = \int P_j$ and $Q_n = \int (R_j^2 + |x|^2)^{n/2} P_j$. Differentiation of the weight, positivity, and Hölder with respect to $P_j \, dx$ give

$$Q'_n \leq C_n L Q_n + C_n V M^{1/n} Q_n^{(n-1)/n} + C_n \nu M^{2/n} Q_n^{(n-2)/n}.$$

If $M = 0$ there is nothing to prove. Otherwise $Y = (Q_n / M)^{1/n}$ satisfies $Y \geq R_j$ and

$$Y' \leq C_n L Y + C_n V + C_n \nu / Y.$$

A supersolution is $e^{C_n L t} (C_n R_j + C_n V t + C_n \sqrt{\nu t})$, after enlarging C_n : the derivative of the square-root term controls $C_n \nu / Y$ for $t > 0$. The initial support bounds $Y(0)$ by $C_n R_j$, proving the claim. To justify the moment calculation without assuming moments in advance, start with a finite truncation of the positive initial aggregate. The common velocity is bounded on the physical window, so its positive diffusion representation gives finite moments. Apply the displayed uniform calculation and pass monotonically to the full positive datum. Equivalently one may use truncated radial weights. The constants in the refined estimate use only L, V above, not the crude global drift bound. $\square$

Proposition 13.2 (Actual force on a stopped inactive prefix). *Under the stopped hypotheses and parameter margins of Proposition 11.1, set*

$$\mathcal{M}(S) = (1 + S + \sqrt{\kappa_j S})^4 e^{C_4 S / \beta}.$$

For every $i < j$, $k = j - i$, and $0 \leq \theta \leq A_j T \leq S$, the actual packet force in that proposition obeys

$$f_i(\theta) \leq Cq^{6k} \mathcal{M}(S) + Cq^k \exp \left[-\frac{c_p q^{-2k}}{\kappa_j S} \right]. \quad (111)$$

In particular, for the weights $w_i = (Dq^6)^{j-i}$ with fixed $D > 1$,

$$\sup_{\theta \leq A_j T} \max_{i < j} \frac{f_i(\theta)}{w_i} \leq C\mathcal{M}(S) + \epsilon(q),$$

where $\epsilon(q)$ is smaller than every fixed power of q , uniformly for $\kappa_j \leq q^{3/2}$ and $S = O(\log(1/q))$.

Proof. Let $h_{\geq j} = (I - P)v_j$, where P again denotes angular averaging. Its fourth absolute moment is bounded by twice that of P_j . Choose a fixed $\eta > 0$ so that $B_{\eta R_i}$ is inside the constant core probe of colour i , and is separated from A_i . On this ball the own radial field is bounded by $C\sqrt{\delta}A_i R_i$, the older fields by their actual core/harmonic bounds, and all inner fields by $CA_i R_i \beta q$. The stop and its radial margin therefore give a displacement strictly smaller than $\eta R_i/2$ up to T . Every positive packet constituent starts in $B_{c_0 R_j} \subset B_{\eta R_i/4}$. The radial exit estimate consequently gives

$$\int_{|x| > \eta R_i} P_j(t, x) dx \leq CA_j R_j^2 e^{-cR_i^2/(\nu t)}.$$

This estimate uses the unrestricted convergent inner velocity sum, not a small-density assumption on the packet.

Radial truncation of $h_{\geq j}$ has zero circulation and vanishing harmonic moments of degrees one to three. Its field on A_i is therefore bounded by $CR_i^{-5} \int |x|^4 |h_{\geq j}|$. For the exterior leakage use the last exit estimate and the L^p contraction of P_j . The elementary kernel split gives, for $p > 2$,

$$\|K * f\|_\infty \leq C_p \|f\|_1^{\frac{p-2}{2(p-1)}} \|f\|_p^{\frac{p}{2(p-1)}}.$$

Indeed the near integral is at most $C_p d^{1-2/p} \|f\|_p$, and the far integral at most $C \|f\|_1 / d$; optimize d . Angular averaging preserves radial tail masses and contracts L^p . Thus the leaked nonradial packet has velocity at most $CA_j R_j e^{-c_p R_i^2/(\nu t)}$. Combining this with the fourth moment and the uniform scaled heat-gradient bound on G_i proves (111): the force normalization is velocity divided by $A_j R_i$.

After division by w_i , the first term is at most $CD^{-k} \mathcal{M}(S)$. The second is at most

$$CD^{-k} q^{-5k} e^{-c_p q^{-2k}/(\kappa_j S)}.$$

For small q successive such terms decrease, and the first is smaller than every q -power because $\kappa_j S \leq q^{3/2} C \log(1/q)$. This proves the weighted assertion. $\square$

Corollary 13.3 (Unconditional stopped inactive closure). *Fix $K > 6$ and any fixed $C_2 > 6/\sigma$. Suppose $x_i(0) \leq Mq^K$ for every $i < j$. Choose β sufficiently large, then D sufficiently large, and then q sufficiently small so that*

$$C_2 \lambda_B < K - 6, \quad C_2(\lambda_P + C_4/\beta) < 1, \quad \beta q^{2/p} \leq 1/2.$$

Set $S = C_2 \log(1/q)$ and $\delta = (\log(1/q))^{-4}$. For every $T \leq S/A_j$ with $\kappa_j \leq q^{3/2}$ the actual older densities satisfy

$$x_i(t) \leq Mq^{K-C_2 \lambda_B} + C(Dq^6)^{j-i} S(1+S)^4 e^{(\lambda_P + C_4/\beta)S} + o(q^K) < \delta/4. \quad (112)$$

The core persistence conclusion and the harmonic older-field estimates therefore hold on this window without a packet-force assumption. On the current colour's fixed collars, the normalized older inward nonradial field is

$$O(q^{K+2-C_2\lambda_B}) + O\left(S(1+S)^4 q^{8-C_2(\lambda_P+C_4/\beta)}\right) + o(q^K).$$

These statements control magnitudes; they give no nonzero signed impulse.

Proof. Use the positive comparison matrix from the proof of Corollary 11.2. Its initial unweighted part is bounded by $Mq^K e^{\lambda_B S}$. Its packet part is bounded in the forced weights by $CSM(S)e^{\lambda_P S} + o(q^K)$, using Proposition 13.2. The parabolic error sum is the same superpolynomially small sum as in that corollary. Since $\kappa_j S \ll 1$, this proves (112). The two conditions involving λ_B, λ_P ensure that both terms are strictly below $\delta/4$ for small q . The radial margin holds since $S\sqrt{\delta} = O((\log(1/q))^{-1})$ and $S\beta q \rightarrow 0$. A first stopping time is thus impossible. Finally sum the actual older core harmonic row $\sum_{i < j} \beta^{i-j} q^{2(j-i)} x_i$; the initial and packet parts give the two displayed field powers. The choices of β, D are possible because $\lambda_B, \lambda_P, C_4/\beta$ can be made arbitrarily small in this parameter order. No active-stage or signed-history conclusion has been used in this density bootstrap. $\square$

14 A global weighted foreign-field estimate for the adjoint error

The following estimate controls an exact foreign pairing in the scalar growing-coordinate equation. It does not assume that the foreign velocity is uniformly small throughout the plane in the current colour's units.

Let T be the smooth growing adjoint test from Section 4. Only its already proved bounds

$$|\nabla T(y)| \leq C \min\{|y|^3, |y|^{-5}\}$$

are used here, with the usual bounded extension across $|y| = 1$. Normalize a current colour to $(A_j, R_j) = (1, 1)$. Let b be the actual sum of all foreign fields, after subtracting the constant rotation $\Omega_{<j} J y$ of the older cores.

Lemma 14.1 (Global weighted foreign field). *Assume the actual older core/harmonic conclusions of Propositions 5.3 and 5.4, with older nonradial density parameter $\delta \leq 1$ and normalized core defect at most ε . Assume $\beta \geq 4$, $\beta q \leq 1/2$, and q is small enough that the current unit ball is inside every older probe core. Then*

$$\sup_y |b(y)| |\nabla T(y)| \leq C \left(\beta q + \frac{\delta q^2 + q^4 + \varepsilon}{\beta} \right). \quad (113)$$

Consequently, for every $f \in L^1$,

$$\int |f| |b| |\nabla T| \leq C \left(\beta q + \frac{\delta q^2 + q^4 + \varepsilon}{\beta} \right) \|f\|_1.$$

All constants are independent of the current index and packet depth.

Proof. The younger fields have global norm at most $C \sum_{l > j} (A_l/A_j)(R_l/R_j) \leq C\beta q$ by the individual colour contractions. Multiplication by the bounded ∇T gives their term.

On $|y| \leq a/(2q)$, the actual older harmonic/core-leak decomposition, with the rigid rotation removed, gives

$$|b_{<j}(y)| \leq C \frac{\delta q^2}{\beta} |y|^3 + C \frac{\varepsilon}{\beta} |y|.$$

The geometric sums are $q^2/(\beta - q^2)$ for the cubic coefficient and $1/(\beta - 1)$ for the core leak. For $|y| \leq 1$, multiplying this bound by $C|y|^3$ gives at most $C(\delta q^2 + \varepsilon)/\beta$. For $1 \leq |y| \leq a/(2q)$ use $C|y|^{-5}$, giving the same bound.

On the complementary region, the individual fourfold origin-growth bound gives $|\sum_{l < j} K * w_l(y)| \leq C|y|/\beta$ in current units. The subtracted constant rotation has the same bound, since its coefficient is at most $C \sum_{l < j} A_l/A_j \leq C/\beta$. Thus $|b_{<j}||\nabla T| \leq C|y|^{-4}/\beta \leq Cq^4/\beta$ there. This uses the sum of older amplitudes, which is finite and uniformly bounded in current units; the sum of older velocity maxima in current units would not give a depth-uniform estimate and is not used. Combining the regions proves the result. $\square$

Corollary 14.2 (A harmless linear scalar loss). *Suppose on an actual stage of length $H \leq C_0 \log(1/q)$ its perturbation satisfies $\|f(t)\|_X \leq Ca(t)$, where $a(t) = a(0)e^{\sigma t}$, and the preceding lemma holds uniformly. Write*

$$\chi_q = C \left(\beta q + \frac{\delta q^2 + q^4 + \varepsilon}{\beta} \right).$$

Then the exact foreign cross-advection pairing in the growing-coordinate identity contributes at most $C\chi_q H$ to the error divided by the growing envelope at any time in the stage. If β is fixed, $\delta \leq 1$ and ε is superpolynomially small, this loss tends to zero uniformly in depth.

Proof. The scalar identity and variation of constants weight the pairing at time t by $e^{\sigma(s-t)}$. The lemma bounds it by $C\chi_q a(t)$. Division by $a(s)$ cancels the exponential envelope, leaving $C\chi_q s \leq C\chi_q H$. $\square$

This is a linear loss, not a quadratic error in f . The foreign radial-base forcing and its heat-profile correction must still be estimated separately on the derivative collars, with their actual retained coefficients. The corollary proves neither a signed lower bound nor the density envelope assumed in its statement. It removes the need for an unjustified global foreign-field bound of order $a(t)$ in this one scalar term.

15 Actual postcap source moments under a stopped-parent core

The following proposition is a conditional estimate for an actual scalar colour in the common Navier–Stokes velocity. It requires the displayed actual parent-core and positive cap-envelope hypotheses. It does not assume that the signed colour has already been identified with a canonical Euler profile. In particular, it supplies a uniformly integrable future history, not convergence or noncancellation of that history.

Use dimensionless child coordinates, and let f solve

$$\partial_u f + v(u, y) \cdot \nabla f = \kappa \Delta f, \quad 0 \leq u \leq U,$$

where v is the actual common velocity expressed in these coordinates. Let P be its positive scalar evolution with $P(0) = P_0 \geq |f(0)|$. Then positivity gives $|f(u)| \leq P(u)$. A rigid rotation of the coordinates may be made at any time; all assumptions and conclusions below concern radial quantities and are unaffected by it.

Proposition 15.1 (Discounted actual future-tail estimate). *Fix $\beta, \sigma, C_0, C_a, C_d > 0$. Suppose $R \geq 8$, $H_0 \geq 1$, $0 < \kappa \leq 1$, $M = \int P_0 > 0$, and*

$$\int (1 + |y|)^4 P_0(y) dy \leq C_0 M, \quad (114)$$

$$\int_{|y| > R/4} (1 + |y|)^4 P_0(y) dy \leq C_0 M \exp \left[-\frac{c_0 R^2}{\kappa H_0} \right] \quad (115)$$

for a fixed $c_0 > 0$. Assume the actual radial drift satisfies

$$|v(u, y) \cdot e_r| \leq a|y| + D \quad (|y| < R), \quad (116)$$

$$|v(u, y) \cdot e_r| \leq a_0|y| + D \quad (y \in \mathbb{R}^2), \quad (117)$$

where

$$0 \leq a \leq C_a \delta / \beta, \quad 0 \leq a_0 \leq C_a / \beta, \quad 0 \leq D \leq C_d.$$

There is a fixed constant C_* depending only on the displayed constants such that, if

$$e^{aU} (R/4 + DU) \leq R/2, \quad C_* \delta \leq \sigma/4, \quad (118)$$

then, for every $0 \leq u \leq U$,

$$\int |y|^4 P(u, y) dy \leq CM(1 + u + \sqrt{\kappa u})^4 e^{C_* \delta u / \beta} + \mathcal{E}, \quad (119)$$

where one can take

$$\mathcal{E} = CM e^{C_* U / \beta} (1 + R + U + \sqrt{\kappa(H_0 + U)})^4 \exp \left[-\frac{cR^2}{\kappa(H_0 + U)} \right]. \quad (120)$$

The constants $C, c > 0$ depend only on the fixed constants and c_0 . For every $0 \leq J \leq U$ and every real measurable phase $\vartheta(u)$,

$$\int_J^U e^{-\sigma u / \beta} \left| e^{i\vartheta(u)} B_4(f(u)) \right| du \leq C_{\beta, \sigma} M (1 + J)^4 e^{-\sigma J / (2\beta)} + \frac{\beta}{\sigma} \mathcal{E}. \quad (121)$$

In particular the same bound controls any complex moment integral with weight $e^{-\mu u}$ and $\text{Re } \mu = \sigma / \beta$.

Proof. All statements can first be proved for the actual smooth positive-time Navier–Stokes velocity and then by approximation. Its positive scalar representation is the diffusion with drift v and covariance $2\kappa I$. For radial estimates a predictable rotation eliminates the tangential component of its drift: if $Y_u = O_u X_u$, then the martingale $\int O_u dB_u$ is again a planar Brownian motion and $|Y_u| = |X_u|$. Equivalently one may use the radial exit estimate proved above. The local drift bound and Gronwall give, up to the first exit time τ_R from B_R ,

$$\sup_{s \leq u \wedge \tau_R} |X_s| \leq e^{au} \left[|X_0| + Du + \sqrt{2\kappa} \sup_{s \leq u} |B_s| \right].$$

The stopped process is kept at its exit point in this inequality. For $|X_0| \leq R/4$, (118) and the Brownian supremum tail therefore imply

$$\Pr_{X_0}(\tau_R \leq U) \leq C \exp \left[-\frac{cR^2}{\kappa U} \right]. \quad (122)$$

For $U = 0$ this means zero exit probability. The predictable rotation can be regularized at the origin; for a smooth fourfold Navier–Stokes drift $v(0) = 0$ and its tangential angular speed is locally bounded, so removal of that regularization causes no difficulty. No angular drift bound enters the constants.

Split P_0 into its restrictions to $B_{R/4}$ and its complement. On unexited trajectories of the first part, the path bound and the fourth moment of the Brownian supremum give

$$\mathbb{E}_{X_0}[(1 + |X_u|)^4 \mathbf{1}_{\{\tau_R > u\}}] \leq C e^{4au} \left[(1 + |X_0|)^4 + (Du)^4 + (\kappa u)^2 \right].$$

Integration against the initial positive density and (114) bound this contribution by the first term of (119), after enlarging C_* .

The global radial bound also gives, conditionally from any starting time s and point x ,

$$\mathbb{E}[(1 + |X_u|)^4 \mid X_s = x] \leq C e^{C_*(u-s)/\beta} \left[(1 + |x|)^4 + (D(u-s))^4 + (\kappa(u-s))^2 \right].$$

This follows from the same radial rotation and Gronwall argument, now using a_0 . Conditional application at τ_R , where $|X_{\tau_R}| = R$, and (122) bound all exited trajectories of the first initial part by

$$C M e^{C_* U/\beta} (1 + R + U + \sqrt{\kappa U})^4 e^{-cR^2/(\kappa U)}.$$

For the second initial part, the global conditional bound and (115) give at most

$$C M e^{C_* U/\beta} (1 + U + \sqrt{\kappa U})^4 e^{-c_0 R^2/(\kappa H_0)}.$$

Reducing c and combining these estimates gives (120), and hence (119). This proves in particular that the entire fourth moment of f is well defined.

Since $|f| \leq P$, one has $|B_4(f)| \leq \int |y|^4 P$. The cap-smallness condition in (118) makes its discounted first term bounded by

$$C M (1 + u)^4 e^{-3\sigma u/(4\beta)},$$

because $\kappa \leq 1$. Integrating from J to infinity yields $C_{\beta,\sigma} M (1 + J)^4 e^{-\sigma J/(2\beta)}$. The constant-error term integrates to at most $(\beta/\sigma)\mathcal{E}$. Unit complex phases do not change any absolute value, proving (121). $\square$

Corollary 15.2 (Uniform scale margin for an actual source). *Fix all profile constants and a fixed finite β . Let $q \downarrow 0$ and assume*

$$R = c_R/q, \quad U \leq C_U \beta \log(1/q), \quad H_0 \leq C_H \log(1/q), \quad \kappa \leq q^{3/2}, \quad \delta = [\log(1/q)]^{-4},$$

with the preceding proposition's explicit actual drift and cap-envelope hypotheses. For small q , the margins in that proposition hold, and $\mathcal{E}/M$ is smaller than every fixed power of q , uniformly in viscosity, depth and seed phases. The actual discounted source histories then have uniformly integrable future tails:

$$\int_J^U e^{-\sigma u/\beta} |B_4(f(u))| du \leq C_{\beta,\sigma} M (1 + J)^4 e^{-\sigma J/(2\beta)} + M o(q^K)$$

for any fixed prescribed $K > 0$.

Proof. Here $aU = O(\delta \log(1/q)) \rightarrow 0$ and $DU/R = O(q \log(1/q)) \rightarrow 0$, so the displacement margin holds. The exponential in (120) has an exponent bounded above by

$$-\frac{cq^{-7/2}}{\log(1/q)},$$

whereas its remaining factors are bounded by a fixed power of q^{-1} times a fixed power of $\log(1/q)$. This proves the superpolynomial error estimate, uniformly for every smaller κ . The discounted estimate follows from the proposition. $\square$

Remark 15.3 (How the actual drift hypotheses arise). For a child colour at scale (A_j, R_j) , its own field and all descendant fields have normalized velocity bounded by a fixed D , by individual L^1, L^∞ contractions and $\sum_{l \geq j} A_l R_l \leq C A_j R_j$. The global older fourfold fields obey a radial bound $C|y|/\beta$. Inside a fixed fraction of the parent's protected constant core, the stopped actual older densities and their harmonic/core-leak estimate give instead a radial bound $C\delta|y|/\beta$, after the rigid rotation has been removed and exponentially small leaks have been absorbed into that coefficient. These facts supply (116)–(117) when that stopped core estimate is available. The positive cap-envelope moment and parent-radius leak (114)–(115) remain explicit inputs here. They must be obtained from the actual preparation-stage localization; they are not consequences of a signed cap-coordinate estimate alone.

16 The actual deepest selected source

We retain the fixed smooth profile W , its unstable eigenvalue $\lambda = \sigma + i\tau$, its normalized growing coordinate $\mathcal{A}$, and its neutral seeds Φ_ϑ . All colours below solve the scalar equation with the same actual unforced Navier–Stokes velocity.

Lemma 16.1 (Weighted off-diagonal Nash estimate). *Let $\mathcal{S}(t, s)$ be the positive scalar propagator for the actual divergence-free Navier–Stokes velocity u , or its rigidly rotating frame, and diffusion coefficient $\kappa > 0$ in two dimensions. Let ϕ be a fixed bounded smooth function satisfying $|\nabla\phi| \leq 1$, $|\Delta\phi| \leq C_\phi$. Suppose $|u \cdot \nabla\phi| \leq V(t)$, where $V \geq 0$, and put $D(t, s) = \int_s^t V$. For $\ell \geq 0$ one has*

$$\|e^{\ell\phi}\mathcal{S}(t, s)e^{-\ell\phi}\|_{L^1 \rightarrow L^\infty} \leq \frac{C}{\kappa(t-s)} \exp(\ell D(t, s) + \kappa\ell^2(t-s) + C_\phi\kappa\ell(t-s)). \quad (123)$$

If a positive datum of mass M is supported where $\phi = 0$, while $\phi = d_0 > 0$ on a set E , and $D(t, 0) < d_0/4$, then, provided κt is sufficiently small,

$$\|\mathbf{1}_E \mathcal{S}(t, 0) f_0\|_\infty \leq \frac{CM}{\kappa t} e^{-c/(\kappa t)}, \quad \|\mathbf{1}_E \mathcal{S}(t, 0) f_0\|_1 \leq CM e^{-c/(\kappa t)}.$$

The constants do not depend on a bound for the tangential velocity or on derivatives of u .

Proof. For $F = e^{\ell\phi}f$, direct conjugation gives the drift $u + 2\kappa\ell\nabla\phi$ and zeroth-order coefficient

$$\ell u \cdot \nabla\phi + \kappa\ell^2|\nabla\phi|^2 - \kappa\ell\Delta\phi.$$

Integration of its positive equation bounds its L^1 growth by $\Lambda_1 = \ell V + \kappa\ell^2 + C_\phi\kappa\ell$. In the L^2 energy identity half the drift divergence cancels the Laplacian term, so its growth coefficient is at most $\Lambda_2 = \ell V + \kappa\ell^2$. Apply the planar Nash inequality to $e^{-\int_s^t \Lambda_1} F$: its L^1 norm does

not increase, and its L^2 dissipation is at least $\kappa\|\nabla F\|_2^2$ after the same scalar integrating factor. Solving the resulting scalar differential inequality gives the $L^1 \rightarrow L^2$ bound $C[\kappa(t-s)]^{-1/2}e^{\int_s^t \Lambda_1}$. The adjoint calculation has the same bounds, with the signs of u and ℓ reversed. Composition at the midpoint proves (123). The L^1 bound before Nash is $M \exp(\ell D + \kappa \ell^2 t + C_\phi \kappa \ell t)$. Multiply both bounds by $e^{-\ell d_0}$ and optimize $\ell > 0$. Since $D < d_0/4$ and κt is small, this proves the last assertions. The energy calculations can first start at positive time and use smooth data. For the actual initial $L^1 \cap L^p$ vorticity, $p > 2$, the physical velocity is bounded by the Biot–Savart split; its rotating-frame addition is tangential to radial cutoff boundaries. Thus the integrations by parts are justified by cutoff limits. Their estimates use only V , not that possibly large velocity bound. Truncating the positive initial datum and passing monotonically gives the L^1 version, with the same constants. The vorticity generating u may have either sign: divergence freedom is the only property of that velocity used in these energy identities. For any signed transported aggregate g , scalar positivity gives $|g| \leq \mathcal{S}(t, 0)|g(0)|$, or domination by the sum of the absolute initial colours. That latter positive datum need only be in L^1 ; no summed initial L^∞ norm appears in any step. $\square$

Choose once and for all

$$6 < K < \frac{643}{100}, \quad C_2 > K/\sigma + 1.$$

Use the actual packet estimate to choose β, D so large that

$$e_B = C_2 \lambda_B < K - 6, \quad e_P = C_2(\lambda_P + C_4/\beta) < \min\{1, 8 - K\}. \quad (124)$$

Here the comparison rates are those of Corollary 13.3; they tend to zero in its stated parameter order. Choose q sufficiently small after these fixed choices. In particular $\beta q^{2/p} < 1/2$ and $\beta q < 1/2$.

Theorem 16.2 (Actual selected-source activation and protected first cap). *Let $A_l = A_1 \beta^{l-1}$, $R_l = R_1 q^{l-1}$, and*

$$w_l(0, x) = A_l[W(x/R_l) + q^K \eta_l \Phi_{\vartheta_l}(x/R_l)], \quad 0 \leq \eta_l \leq 1.$$

Fix any $j \geq 1$, impose $\eta_j = 1$, and set $\nu = q^{3/2} A_j R_j^2$. The phases and all other amplitudes are arbitrary. Let w_l be the actual scalar colours of the common unforced Navier–Stokes solution. Normalize the j th source to radius and amplitude one, and remove the rigid rotation

$$\Omega_{<j} = \frac{W(0)}{2} \sum_{l < j} \frac{A_l}{A_j}.$$

Denote its normalized, rotated vorticity by $\tilde{w}_j(t)$ and put $f_j = \tilde{w}_j - e^{\kappa t \Delta} W$, $\kappa = q^{3/2}$, $\alpha = q^K$, $a(t) = \alpha e^{\sigma t}$. Fix one sufficiently large finite expansion order M , with $M > C_W/\sigma$ if a reference-state reset is required below. There are constants $B, \delta_ > 0$, allowed to depend on this finite order and independent of j and all phases, such that through $H_* = \sigma^{-1} \log(\delta_*/\alpha)$,*

$$\|f_j(t)\|_X \leq B a(t), \quad \int_0^{H_*} \|K * f_j(t)\|_\infty dt \leq C B \delta_*. \quad (125)$$

The first hitting time t_j of $|\mathcal{A}(f_j)| = \delta_q$, $\delta_q = (\log(1/q))^{-4}$, is well defined and protected: in a fixed neighbourhood of that crossing,

$$\frac{d}{dt} |\mathcal{A}(f_j(t))| \geq \sigma \delta_q / 2.$$

It obeys

$$t_j = \sigma^{-1} \log(\delta_q/\alpha) + O(\delta_q + q^\gamma(1 + \log(1/q))^{D_0})$$

for some $\gamma > 0$ and finite D_0 .

For that fixed finite M , let the matching exact-cap state on the canonical ancient orbit have the same complex coordinate $\mathcal{A}$ as $f_j(t_j)$. After this actual phase matching, the two states differ in X by

$$C_M \delta_q^{M+1} + C_M q^\gamma (1 + \log(1/q))^{D_M} \delta_q + C \kappa t_j. \quad (126)$$

The last, radial term is the explicit difference between the exact Navier–Stokes heat base and the Euler base. It is not a seeded error. One can choose $M > C_W/\sigma$, where C_W is the one-sided X comparison constant of a fixed small ancient segment. All fixed positive-envelope moments at t_j are bounded uniformly; outside one fixed ball they are smaller than every q -power. This theorem concerns the deepest source selected by diffusion. It does not assert inductive transmission to all larger receiving annuli.

Proof. The initial series converges absolutely in $L^1 \cap L^p$. Individual scalar contractions give $\|w_l\|_1 \leq C A_l R_l^2$, $\|w_l\|_\infty \leq C A_l$ and hence $\sum_{l>j} \|K * w_l\|_\infty \leq C A_j R_j \beta q$. These are actual-colour bounds and require no hypothesis on the deep colours' nonradial densities.

Work in the normalized rotating frame. Let C_Φ bound the fixed autonomous first coefficient in X divided by $e^{\sigma t}$, and fix $B > 4(C_\Phi + 1)$ now. All later field estimates use this fixed B ; their constants may depend on it. Stop provisionally when $\|f_j(t)\|_X = B a(t)$. The terminal cap δ_* is a provisional small number, to be fixed after the finite expansion orders, before the final choice of q . By Corollary 13.3, all larger colours already obey their unconditional inactive-density estimate through $S = C_2 \log(1/q)$; their cores and harmonic estimates hold there. Their nonrotational field on any fixed source collar is, in every fixed spatial derivative norm,

$$O(q^{K+2-e_B}) + O(q^{8-e_P}(1+S)^5) + o(q^P) \quad (127)$$

for every fixed P . Derivatives here are taken only on the harmonic part; the core leak has no asserted derivative bound and is included in the last velocity remainder.

The stopped source core persists directly, without a source-core hypothesis in the bootstrap. Write its own velocity exactly as $K * e^{\kappa t \Delta} W + K * f_j$; the first term is tangential everywhere and the second is bounded by $C B a(t)$. Larger cores have already been proved independently by the inactive-prefix result. Their harmonic and leaked nonrotational contributions on the fixed source-core disk have integral tending to zero by (127). Every smaller velocity contributes at most $C \beta q$. Thus the integral of the absolute radial drift on that disk is at most

$$C B \delta_*/\sigma + C \beta q S + o(1).$$

Choose δ_* and then q so this is less than $a_0/8$. In the exact backward representation the sign of the drift is reversed, but the same absolute radial bound holds. Starting in $B_{a_0/2}$ and stopping on exit from the initial constant core B_{a_0} gives

$$\|\tilde{w}_j(t) - W(0)\|_{L^\infty(B_{a_0/2})} \leq C e^{-c/(\kappa t)}$$

by the radial exit lemma and the individual colour maximum principle. Consequently, in the source core all foreign fields acting on the aggregate $g = \sum_{l>j} \tilde{w}_l$ are harmonic plus a rough exponentially small leak; after rotation their cubic constant is at most $C B \delta_*$. Their global growth is $C|x|$, uniformly in j .

The aggregate g has mass norm $M_g \leq C\beta q^2$, initial support in B_{Cq} , zero circulation and zero initial B_4 . Apply the already proved full nonlinear cold-tail estimate with $r = Cq$, $R_* = \kappa^{1/4}$ and window S . Its margins hold since $M_g/\kappa = O(\beta q^{1/2})$, $\kappa S \rightarrow 0$, and $CB\delta_*\sqrt{\kappa}S \rightarrow 0$. It yields

$$\sup_{t \leq H_*} |B_4(g(t))| \leq Cq^{643/100}(1+S)^{D_1}. \quad (128)$$

Every constant in this stopped calculation is independent of j .

For completeness, higher moments require no signed cancellation. Nash for the positive passive envelope gives $\|K * g(t)\|_\infty \leq CM_g/\sqrt{\kappa t}$. Inside B_{2R_*} , foreign radial drift is bounded by $(CB\delta_*R_*^2 + \epsilon(t))|x|$. Its integrated linear-growth coefficient tends to zero. The elementary radial moment calculation, or the integrating-factor diffusion comparison, therefore gives for each fixed integer $n \geq 2$, before exit,

$$\int |x|^n |g(t, x)| dx \leq C_n M_g (q + \sqrt{\kappa S} + M_g \sqrt{S/\kappa})^n.$$

The last displacement is smaller than $\sqrt{\kappa S}$ since $M_g/\kappa \rightarrow 0$. Radial exit from B_{R_*} has probability $Ce^{-cR_*^2/(\kappa S)}$, because both integrated self displacement and the cubic displacement are small compared with R_* . After exit the global foreign linear growth and integrable self displacement amplify a fixed moment by at most $C_n e^{C_n S}$, absorbed by that probability. Thus the whole-moment bound is

$$\sup_{t \leq H_*} \int |x|^n |g(t, x)| dx \leq C_n q^{2+3n/4} (1+S)^{n/2} + o(q^P). \quad (129)$$

In particular $B_8(g) = O(q^8(1+S)^4)$. Using only the absolute fourth moment instead of (128) would fail, since its available power is merely q^5 .

There is also an exponentially small actual density leak on the source collars. Choose in Lemma 16.1 a fixed radial ϕ which rises from zero to d_0 entirely inside the protected source core, and is constant thereafter. On the support of its derivative the integrated radial drift is at most $CB\delta_* + C\beta qS + o(1) < d_0/4$. Apply that lemma to the actual positive envelope of g , not to a new Navier–Stokes solution. It bounds its mass and L^∞ norm beyond the rise by $CM_g e^{-c/(\kappa t)}$ and $CM_g(\kappa t)^{-1} e^{-c/(\kappa t)}$. Kernel interpolation makes the corresponding leaked velocity smaller than every q -power through S .

Take finitely many ACTUAL coefficients of the old interior harmonic series and of the cold exterior multipole series, retaining degrees 3, 7, $\dots$ and 4, 8, $\dots$, respectively. Extend their stream-functions by fixed smooth radial cutoffs, vanishing near zero and outside one fixed ball, so the fields agree on a broad annulus containing both derivative collars. The result b_{app} is real, smooth, divergence free and has a fixed finite set of sectors. For every finite derivative order required below,

$$\|b_{\text{app}}(t)\|_{C^k} \leq \varepsilon_q \alpha e^{\sigma t}, \quad \varepsilon_q \leq Cq^{\gamma_0} (1+S)^{D_2}, \quad (130)$$

$$\|b - b_{\text{app}}\|_{L^\infty(\text{collars})} \leq Cq^P, \quad (131)$$

where $\gamma_0 > 0$. Indeed one may take any number smaller than

$$\min\{643/100 - K, 8 - K - e_P, 2 - e_B\} > 0.$$

The fourth cold coefficient uses (128); the higher coefficients use (129); the old coefficients use (127). Retaining enough moments makes the remainder exponent P arbitrarily large: the next

cold moment has power $2 + 3n/4$, while old Taylor remainders gain arbitrarily many factors of q . All density/core leaks are superalgebraic. This step is valid for each fixed finite P, k and does not assert a derivative bound for a rough leak.

Expand the receiving/source colour in α, κ as in the isolated proof, adding the ACTUAL coefficient functions $b_1 = b_{\text{app}}/\alpha$ to the triangular recursion. Explicitly, in addition to the right-hand side of (16), add

$$-\mathbf{1}_{m=1}b_1(t) \cdot \nabla W_\ell(t) - \mathbf{1}_{m \geq 2}b_1(t) \cdot \nabla g_{m-1,\ell}(t), \quad W_\ell(t) = t^\ell \Delta^\ell W / \ell!.$$

Here $g_{1,0}(0) = \Phi_{\vartheta_j}$ and all other initial coefficients vanish. Their dependence on this actual run does not require analyticity in either parameter: $b_1(t)$ is one fixed coefficient function for that run, and the recursion cancels the actual equation algebraically.

We make the finite derivative hierarchy explicit. Fix an integer $r \geq 4$ and estimate coefficient (m, ℓ) at Sobolev order

$$d_{m,\ell} = r + 2(M - m) + 2(N - \ell), \quad 1 \leq m \leq M, \quad 0 \leq \ell \leq N.$$

The Laplacian input $(m, \ell - 1)$ has two more assigned derivatives. Every heat coupling $(m, \ell - s)$, $s \geq 1$, has at least two more; each nonlinear input has strictly smaller seed order and therefore at least two more assigned derivatives, sufficient for its one differentiated factor. The external differentiated input $(m - 1, \ell)$ also has two more. Use the already established C^k bounds for b_1 up to the finite maximum assigned order, and smoothness of W up to that order plus the finitely many heat Laplacians. The finite-sector propagator acts at each of these assigned orders without radial derivative loss: base transport is angular multiplication on each sector. In two dimensions H^d , $d \geq 4$, is an algebra; on the fixed coefficient support, zero circulation and the Fourier low-frequency estimate give $\|K * g\|_{H^d} \leq C_d \|g\|_{H^d}$. These facts prove all coefficient estimates by induction in $m + \ell$. There is no infinite derivative iteration and no regularity assumption on a rough actual core leak. Coefficients remain smooth, supported in the fixed collars, and of zero circulation, since every forcing is a Laplacian or the divergence of a compactly supported flux. If the retained input stream frequencies are at most $4N_h$, coefficient (m, ℓ) has frequencies at most $4mN_h$; thus only one fixed finite frequency set is used throughout this hierarchy.

The first coefficient differs from the isolated coefficient by $C\varepsilon_q(1+t)e^{\sigma t}$, by the sharp finite-sector semigroup. For orders $m \geq 2$, the forcing has exponent $m\sigma$ and the gap $m\sigma - \sigma > 0$; induction bounds the coefficient differences by $C_m\varepsilon_q(1+S)e^{m\sigma t}$. The finite coefficient bounds are uniform since $\varepsilon_q S \rightarrow 0$. Positive-viscosity terms have only finite time polynomials. The source factor is retained in the heat Taylor remainder.

Use the ACTUAL foreign drift in the transport part of the comparison between the actual source and this expansion. It cancels from the Kato norm calculation. The additional residual is $(b - b_{\text{app}}) \cdot \nabla \omega_{\text{app}}$. All nonheat coefficient gradients have compact collar support, so (131) controls them. Away from a broad collar annulus the exact heat gradient is Gaussian; its product with the foreign radial field is uniform because $|b \cdot e_r| \leq C(|x| + \beta q)$ globally, by origin growth for old colours and the unrestricted deep velocity sum. Thus the residual obeys

$$\|R\|_X \leq Ca^{M+1} + C\kappa^{N+1}(1+t)^D a + Cq^P + o(q^{P'}),$$

where P' can also be arbitrarily large. The Kato coefficient is the same fixed $C_1 = C_p(\|\nabla W\|_X + 1)$ as before, after a small terminal cap. For clarity the parameter choices and bootstrap improvement can be made in the following order. Fix C_1 before the expansion orders. Choose $(M+1)\sigma > C_1$, then N with

$$\frac{3}{2}(N+1) > K(C_1/\sigma - 1)_+ + 1, \quad P > K \max\{1, C_1/\sigma\} + 1,$$

then choose the finite harmonic truncation and derivative hierarchy above. Use the $B > 4(C_\Phi + 1)$ fixed at the start of the density bootstrap. The finite pure-coefficient bounds are uniform when $\varepsilon_q(1 + S)$ is small. Choose δ_* sufficiently small that the summed higher pure coefficients are at most $Ba/32$, the pure gradient sum is at most $1/2$, the pure residual comparison is at most $Ba/32$, and the stopped source core has its fixed radial margin. All remaining constants are finite and fixed. Now choose q small so the forced first-coefficient correction, positive-viscosity sums and comparisons, spatial mismatch comparison, and remaining gradients are each within their respective $Ba/32$ or gradient- $1/2$ budgets. In particular $\|\nabla\omega_{\text{app}}\|_X \leq \|\nabla W\|_X + 1$, so the coefficient is precisely the previously fixed C_1 . The explicit comparison error is bounded by

$$Ca^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)t} + Cq^P(1+t)e^{C_1t} + o(q^{P'}e^{C_1t}),$$

uniformly through H_* . Dividing by $a(t)$, the second term is bounded by $Cq^{\frac{3}{2}(N+1)-K(C_1/\sigma-1)+}(1+S)^{D+1}$; the third by $Cq^{P-K\max\{1,C_1/\sigma\}}(1+S)$. Both tend to zero, with explicit positive powers. Choose P' large enough for the last superalgebraic term. The approximate density and comparison error sum to less than $Ba/2$. Continuity of $\|f_j\|_X/a$ therefore rules out its first stop and proves (125). The core and cold-tail estimates held on that sole provisional density stop and were obtained before its improvement. Therefore no conditional source-density input remains.

The autonomous pure coefficients approach their universal ancient homological coefficients at an exponential rate. To check the rate, use the rotating generator $\mathcal{L}_{\text{rot}}$ of the ancient-profile construction and normalize coefficient m by its factor $e^{m\sigma t}$ and the common angular rotation. The first normalized coefficient differs from its growing limit by $O(e^{-2\sigma t})$. For $m \geq 2$ its equation is

$$F'_m(t) = (\mathcal{L}_{\text{rot}} - m\sigma)F_m(t) - \sum_{m_1+m_2=m} (K * F_{m_1}(t)) \cdot \nabla F_{m_2}(t).$$

The finite-sector propagator of the first operator decays at least as $Ce^{-(m-1)\sigma t}$. Its limiting constant is the unique homological resolvent coefficient. Subtract that constant; the differentiated-input hierarchy above controls the forcing difference. Induction gives a bound C_me^{-ct} , with any fixed $0 < c < \sigma$, in all the finitely required orders. Taking $c = \sigma/2$ turns its cap-time remainder into a positive q -power times a finite logarithmic loss. The above actual-input coefficient difference is $O(\varepsilon_q S)$, so at a cap with linear amplitude δ_q the actual source is the universal ancient polynomial up to

$$C_M\delta_q^{M+1} + C_Mq^\gamma(1+S)^{D_M}\delta_q.$$

This displayed bound concerns f_j relative to the radial heat base. For the full vorticity state used in (126), one must additionally use the exact heat identity

$$\|e^{\kappa t_j \Delta} W - W\|_X \leq \kappa t_j \|\Delta W\|_X.$$

It gives the explicit unseeded radial term $C\kappa t_j$ in that equation. The radial heat term has zero protected coordinate, so it does not alter the first-cap phase or the coordinate crossing estimate below. The protected scalar identity has quadratic self error $O(a^2)$, heat error $O(\kappa(1+t)a)$ and projected base forcing $O(\varepsilon_q a)$. Its other foreign error is small: on a fixed ball the old field is (127), the cold global velocity is $O(\beta q)$, and outside that ball the source's positive envelope has Gaussian tails. It follows that near $a = \delta_q$ its radial crossing derivative is at least $\sigma\delta_q/2$ for small q . More explicitly the same expansion and comparison, using that the protected test kills the stable first coefficient, give on preparation

$$|\mathcal{A}(f_j(t)) - \alpha e^{i\vartheta_j} e^{\lambda t}| \leq Ca(t)[a(t) + \varepsilon_q(1+S) + q^\gamma(1+S)^D].$$

For $a(t) \leq 2\delta_q$ this estimate excludes any cap before the indicated small neighbourhood of the linear crossing. Within that neighbourhood the positive radial derivative gives one crossing. This proves protection and the time estimate. Matching the canonical exact coordinate removes its amplitude and phase bias; local invertibility of the canonical radial coordinate proves (126).

Finally, the integrated radial drift on any fixed ball surrounding the initial source support is at most $CB\delta_* + C\beta qS + o(1)$, small compared with a fixed boundary gap. Forward radial exit of its actual positive passive envelope is therefore bounded by $Ce^{-c/(\kappa t)}$. After exit, the uniform global radial growth amplifies each fixed moment by at most $C_n e^{C_n S}$, absorbed by that probability. This proves the cap-time positive-envelope assertions and completes the theorem. $\square$

Remark 16.3 (All finite jets describe the same ancient orbit). The high-order polynomial used here is a finite jet of the already identified canonical orbit, not a new ancient solution for each order. Indeed fix any larger finite order and repeat the isolated finite-expansion construction at a sufficiently small ancient amplitude, using that order. It is the same isolated seeded Navier–Stokes family. Its subsequential limit must coincide with the previously identified ancient limit by the one-sided X uniqueness argument at the earlier sufficient order. Passing the new finite expansion to this same limit gives the higher-order jet. Time and phase translation then give that jet on the reference orbit at sufficiently negative times. The constants and the range of small amplitudes may depend on the finite order; no convergence of an infinite formal series is used or claimed.

Corollary 16.4 (Actual reference-state reset and fixed source windows). *Choose the order in Theorem 16.2 with $M > C_W/\sigma$, and normalize one sufficiently small canonical reference state in time and phase so that*

$$\mathcal{A}(\Omega_{\text{ref}}(0)) = \delta_{\text{ref}} > 0, \quad \mathcal{A}(\Omega_{\text{ref}}(s)) = c_{\text{ref}} e^{\lambda s} + O(e^{2\sigma s}) \quad (s \rightarrow -\infty).$$

Write $a_{\text{ref}} = |c_{\text{ref}}|$; then $a_{\text{ref}} = \delta_{\text{ref}} + O(\delta_{\text{ref}}^2)$, but the two parameters are not identified. Let $s_q < 0$ be the exact canonical hitting time of $|\mathcal{A}| = \delta_q$ on Ω_{ref} , and remove the actual source's constant phase so that its first-cap coordinate equals $\mathcal{A}(\Omega_{\text{ref}}(s_q))$. Put $L_q = -s_q$ and $t_{\text{ref}} = t_j + L_q$. For every fixed $J < \infty$, after the same phase and rigid-rotation removal,

$$\tilde{w}_j(t_{\text{ref}} + \cdot) \longrightarrow \Omega_{\text{ref}} \quad \text{in } C([-J, J]; L^1 \cap L^p),$$

uniformly in cascade depth and all other seed phases. Whole fourth moments converge uniformly on each such window. At t_{ref} the actual source's positive passive envelope, and the positive envelope of its entire source/deep packet, have all fixed moments bounded uniformly and have superalgebraic moment tails outside one fixed ball. In particular no power of $\log \log(1/q)$ is lost in this reference-time moment reset.

Proof. The canonical radial coordinate is strictly increasing in its small ancient segment. Consequently $s_q = \sigma^{-1} \log(\delta_q/a_{\text{ref}}) + O(\delta_q)$ and $L_q = O(\log \log(1/q))$. Compare the actual cap state with the canonical exact-cap state using (126). On the segment from s_q to zero the canonical gradient and Laplacian have fixed X bounds and one fixed compact support. The one-sided Kato inequality is

$$(\|\zeta\|_X)' \leq C_W \|\zeta\|_X + \|b \cdot \nabla \Omega\|_X + \kappa \|\Delta \Omega\|_X.$$

On that support the actual larger field after rotation is $O(q^{K+2-\epsilon_B} + q^{8-\epsilon_P}(1+S)^5)$; the entire smaller field is at most $C\beta q$. Core leaks are superalgebraic. Thus its last two terms are $O(q)$, uniformly in depth. At time L_q the initial error is amplified only by $C\delta_q^{-C_W/\sigma}$. The first term of

(126) therefore tends to zero by the choice of M , and every q -power term still tends to zero after that fixed logarithmic amplification. The explicit radial heat-base term is amplified by at most $C\kappa t_j \delta_q^{-Cw/\sigma} \leq Cq^{3/2}(1+S)\delta_q^{-Cw/\sigma}$, which also tends to zero. This proves convergence at the reference state and on every fixed earlier window. On a fixed later window the canonical Euler derivatives are bounded by a constant depending on J , and the same comparison applies. All these windows lie inside $C_2 \log(1/q)$ for small q , after choosing the reference amplitude smaller than the terminal cap threshold if necessary.

Moment reset can be proved directly from (125), which remains valid through a fixed ancient amplitude: its own integrated nonradial radial displacement is $O(\delta_{\text{ref}})$. Alternatively, in the preceding comparison the canonical negative-segment perturbation has integrated velocity $O(\delta_{\text{ref}})$, and the integrated comparison error tends to zero. The smaller field contributes $C\beta q L_q = o(1)$ and the larger nonrotational field contributes $o(1)$. Hence the actual positive source envelope has small integrated radial drift from its original compact support through the reference state, not a drift of order L_q . The radial exit calculation then gives Gaussian tails with time horizon $t_j + L_q = O(\log(1/q))$; fixed moment amplification is absorbed by that Gaussian. This gives a fixed-radius reset at the reference state. The identical radial calculation applies to the positive envelope of the entire source/deep packet: its initial mass and support are bounded by the same fixed constants, and it uses the same actual common drift. It requires no summed initial L^∞ norm of that packet. On any subsequent fixed window the source's own velocity is uniformly bounded by its scalar L^1, L^∞ contractions, and the entire smaller velocity remains $O(\beta q)$. Taking a sufficiently larger fixed ball gives the same Gaussian tightness. Compactly cut off fourth moments now converge by strong L^1 convergence, while their whole-moment tails tend to zero uniformly. This proves the claimed whole moment convergence. $\square$

Lemma 16.5 (Gaussian handoff at the parent radius). *Fix any $j \geq 2$ and any viscosity with normalized value $0 < \kappa \leq q^{3/2}$ at this child. The child may have been activated by its incoming packet; it need not be the deepest source selected by diffusion. Use only the common geometric initial colour bounds and constant cores, and assume every actual larger nonradial density is at most $\delta_q A_l R_l^2$ through a normalized horizon $H \leq C_H \log(1/q)$. Let $F_j(t)$ be the actual positive scalar envelope transported from the absolute value of the normalized source datum. The same conclusion holds for the positive envelope of the entire source/deep packet, transported from the sum of its initial absolute colours. At every prepared cap or reference handoff time $0 < t \leq H$, there is a fixed $r_0 > 0$ such that, for every fixed integer $n \geq 0$,*

$$\int_{|x| > r_0/q} (1 + |x|^n) F_j(t, x) dx \leq C_n \exp \left[-\frac{c_n q^{-2}}{\kappa H} \right]. \quad (132)$$

This estimates positive-envelope leakage at the parent radius independently of any fixed-radius cap reset.

Proof. Take r_0/q strictly inside the parent's protected core, for example $r_0 = a_0/4$. The actual larger cores follow from the stipulated density stop by applying the outer-core proposition at the parent index. Its horizon in parent units is $H/\beta = O(\log(1/q))$, its normalized viscosity is $\beta q^2 \kappa \leq \beta q^{7/2}$, and its radial margin tends to zero because $(H/\beta)\sqrt{\delta_q} = O(1/\log(1/q))$ and $qH \rightarrow 0$. Thus no selected-source cold-tail theorem or child small-density assumption is being invoked in this handoff lemma. The actual old core and harmonic decomposition gives on this disk a radial bound $C\delta_q|x|/\beta + \epsilon(t)|x|$, with superalgebraic leak integral. The own source and all smaller velocities together are at most C , by the convergent individual scalar L^1, L^∞ bounds.

Consequently the radial displacement bound for exit from this disk is at most

$$CH + (C\delta_q H/\beta + \text{superalgebraic error})r_0/q.$$

Since $H \ll q^{-1}$ and $\delta_q H = o(1)$, this is less than a fixed fraction of its radius. The initial source support has fixed radius. The radial exit lemma bounds its exit probability by $C \exp[-cq^{-2}/(\kappa H)]$. Globally all older colours satisfy the origin-growth bound $|u_{<j}(x)| \leq C|x|/\beta$, while the source/deep packet velocity is bounded by C . Conditioning at exit therefore bounds its later fixed moment by $C_n q^{-n} e^{C_n H/\beta}$. Both this polynomial prefactor and the fixed q-power from $H = O(\log(1/q))$ are absorbed by reducing the constant in the exit exponential. Positivity of the actual envelope and its fixed initial mass complete (132). $\square$

Proposition 16.6 (Actual positive source history before the parent cap). *Start the source history at the fixed reference reset of Corollary 16.4. The same statement applies to any other actual source colour having the same scalar contractions and uniform fixed-reference positive-envelope moment reset; it does not use how that source was activated. On a following interval $0 \leq u \leq U_q \leq C_0 \beta \log(1/q)$ assume that every larger colour is still in its actual nonradial stop $\|(I - P)w_l\|_1 \leq \delta A_l R_l^2$, and that its actual constant core persists on the appropriate probe disk with the Gaussian defect estimate. These are explicit hypotheses for this following interval; the preceding selected-source theorem alone does not establish the whole parent interval. For sufficiently small $\delta > 0$ one has*

$$|B_4(\tilde{w}_j(t_{\text{ref}} + u))| \leq C(1 + u)^4 e^{C\delta u/\beta} + o(q^P),$$

for every fixed P , uniformly in depth. Choose δ additionally so $C\delta < \sigma/2$. Then this is an integrable uniform positive-history bound under the receiving discount $e^{-\sigma u/\beta}$.

Proof. On the disk $|z| \leq a_0/(4q)$ the actual old harmonic series gives $|b_{\text{old}} \cdot e_r| \leq C\delta|z|/\beta + \epsilon(t)|z|$. Indeed the series is bounded by $C\delta \sum_{k \geq 1} \beta^{-k} q^{2k} |z|^3$, and $q^2|z|^2$ is bounded on this disk. The core leak has superalgebraic time integral. At every time the source's own velocity is bounded by C , while the sum of all smaller velocities is bounded by $C\beta q$, without any nonradial stop. Thus actual radial drift there is at most $C(1 + \beta q) + (C\delta/\beta + \epsilon(t))|z|$. For its actual positive passive envelope use the weight $(1 + |z|^2)^2$. Its diffusion contribution is at most $C\kappa(1 + |z|^2)$, its additive-drift contribution at most $C(1 + |z|^2)^{3/2}$, and its linear-drift contribution at most $C\delta(1 + |z|^2)^2/\beta$. Hölder and scalar Gronwall for the fourth root give the displayed polynomial times weak exponential bound before exit.

For exit from the probe disk use the integrating factor associated with the linear radial drift. Its deterministic radial displacement is at most $C(1 + U_q)e^{C\delta U_q/\beta}$, much smaller than q^{-1} if $C\delta C_0 < 1/4$. The radial Brownian exit probability is bounded by

$$C \exp \left[-\frac{cq^{-2+2C\delta C_0}}{\kappa(1 + U_q)} \right].$$

The reset's initial moment tails have the same superalgebraic status. After exit the actual global radial growth bound is $C|z| + C\beta q$; it amplifies a fixed moment by at most e^{CU_q} , which is a fixed q-power and is absorbed by the exit exponential. This proves the whole-moment bound. Its last integrability assertion follows because $\sigma - C\delta > \sigma/2$. $\square$

Remark 16.7 (The core part of a stopped parent environment). If every larger colour is stopped at the sharper density level $\delta_q = (\log(1/q))^{-4}$, the core hypothesis in Proposition 16.6 follows from the already proved outer-core proposition, applied at the parent index. Its total time horizon

in parent units is $O(\log(1/q))$, its normalized viscosity is $\beta q^2 \kappa \leq \beta q^{7/2}$, and its radial margin is $O(\log(1/q)\sqrt{\delta_q} + \beta q \log(1/q)) = o(1)$. Thus only the actual parent/larger density stop is a substantive input in this usual logarithmic window; a separate post-cap small-density assumption on the source is unnecessary.

Corollary 16.8 (The actual whole-history source transform). *Use the fixed reference gauge of Corollary 16.4 and the actual post-reference environment of Proposition 16.6. Assume also $U_q \rightarrow \infty$. Extend the actual source moment by zero before its physical initial time and after U_q . Then, with $\mu_\beta = [\sigma + i(\tau - 4/5)]/\beta$,*

$$\int_{\mathbb{R}} e^{-\mu_\beta u} B_{\text{actual,ref}}(u) du \longrightarrow \int_{\mathbb{R}} e^{-\mu_\beta u} B_4(\Omega_{\text{ref}}(u)) du,$$

uniformly in depth and all other seed phases. The same conclusion holds for the weighted absolute difference of the two moment functions. This controls the whole source history, including its nonlinear future. If the canonical transform is nonzero at the chosen β , the actual one is nonzero for all sufficiently small q , with a uniform lower bound in this fixed reference gauge.

Proof. On the preparation interval, the actual source norm envelope and fixed-radius Gaussian tails imply $|B_4(\tilde{w}_j(t))| \leq C\alpha e^{\sigma t}$. The exact reference shift satisfies $\alpha e^{\sigma t_{\text{ref}}} = a_{\text{ref}}(1 + o(1))$. Thus the actual reference moment is bounded by $Ce^{\sigma u}$ for $u < 0$, including the part between the shrinking cap and the reference state. Its tail outside the fixed source radius is superalgebraic and smaller than α , so it is included in that same envelope. The post-reference bound gives $C(1+u)^4 e^{C\delta u/\beta}$ for $u > 0$ up to its endpoint, with superalgebraic errors uniformly on a logarithmic horizon. The canonical moment has the corresponding negative exponential and positive polynomial bounds. Since $\beta > 1$ and $C\delta < \sigma/2$, both are integrable under $e^{-\sigma u/\beta}$. The extra superalgebraic errors times the whole logarithmic horizon and its possible fixed q -power amplification still tend to zero. The fixed-window whole-moment convergence now proves the weighted absolute-difference assertion by truncating at $[-J, J]$, first choosing J and then q . This gives the transform assertion and its nonzero conclusion. $\square$

Remark 16.9 (Why the fixed reference gauge matters). If the transform is instead centred at the shrinking exact cap, the time and phase shift multiplies both actual and canonical transforms by the same factor whose modulus is comparable to $(\delta_q/a_{\text{ref}})^{1/\beta}$. The absolute lower bound then shrinks by that factor, while the relative transform error remains small. Using the fixed reference state as the receiving force origin avoids that artificially shrinking coefficient. Merely comparing a fixed window around the shrinking cap would omit the later nonlinear source pulse.

17 Finite actual forcing: density closure and exact-cap reset

We prove the finite approximation statement used by an actual receiving stage. The field retained from the actual foreign colours is frozen at its actual time values. No derivative with respect to viscosity, scale, seed phase, or time is taken of those values. The hypotheses below are field and history bounds; an actual density envelope, an ancient-profile reset, and the existence of a protected first cap are conclusions, not hypotheses. A nonzero growing coefficient and instantaneous cap-window forcing bounds are stated separately where they are necessary.

17.1 Spaces, operators and the derivative reserve

Let $\mathcal{C}$ be the union of the two closed derivative collars of the fixed smooth radial profile W . Choose a closed radial interval $I \Subset (0, \infty)$ containing these collars in its interior. For an integer $J \geq 1$, let E_j^r denote the real finite angular-sector space of functions supported in $\mathcal{C}$ whose sectors are $m \in 4\mathbb{Z}$, $|m| \leq 4Jj$, with radial coefficients in $H^r(I)$. The norm is the sum of their radial Sobolev norms. On these supported finite-sector functions it is equivalent to the full-plane H^r norm, with constants depending on I, J, j, r . Initially and in every recursion below the functions are smooth, flat at the collar endpoints, and have zero integral. The coefficient equations therefore use a common closed supported subspace on each finite Sobolev level; there is no assertion of an unrestricted all-angular-mode spectral estimate.

Write

$$\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W, \quad Q(f, g) = -(K * f) \cdot \nabla g.$$

The inherited finite-sector spectral lemma gives

$$e^{t\mathcal{L}} = e^{\lambda t} P_+ + e^{\bar{\lambda} t} P_- + \mathcal{R}(t), \quad \|\mathcal{R}(t)\|_{H^r \rightarrow H^r} \leq C_r e^{2\eta t}, \quad 2\eta < \sigma, \quad \lambda = \sigma + i\tau. \quad (133)$$

This is used only in one of the finitely many spaces E_j^r . The growing eigenfunction F_+ lies in sector $+4$ and is normalized by $\mathcal{A}(F_+) = 1$; its real conjugate lies in sector -4 . The global test defining $\mathcal{A}$ selects sector $+4$. For the pattern rotation set

$$(\mathcal{T}_\theta f)(r, \varphi) = f(r, \varphi + \theta/4), \quad \mathcal{L}_{\text{rot}} = \mathcal{L} - (\tau/4)\partial_\varphi.$$

For vector fields its definition is the pushforward $\mathcal{T}_\theta V(x) = R_{-\theta/4} V(R_{\theta/4} x)$, where R is the planar rotation matrix. Equivalently its polar components are shifted by $\theta/4$. This gives $K * (\mathcal{T}_\theta f) = \mathcal{T}_\theta(K * f)$ and $\mathcal{T}_\theta[V \cdot \nabla f] = (\mathcal{T}_\theta V) \cdot \nabla(\mathcal{T}_\theta f)$. This rotation is applied to the entire function, separately in every angular sector. It is not replaced by the scalar factor $e^{-ij\tau s}$ on an order- j coefficient. The operators commute, and rotations are isometries in radial Sobolev norms. Thus $e^{t\mathcal{L}_{\text{rot}}} \leq C e^{\sigma t}$ and the growing real space has eigenvalue σ .

Fix truncation orders $M, N \geq 1$ and a desired final Sobolev order k . Set

$$k_* := \max\{4, k\}, \quad r_{j\ell} = k_* + 2(M - j) + 2(N - \ell), \quad R = k_* + 2M + 2N + 3. \quad (134)$$

The reserve is explicit. A diffusion source uses $g_{j,\ell-1}$, whose order exceeds $r_{j\ell}$ by two. A quadratic or foreign-advection source uses a coefficient with strictly smaller seed order, whose order exceeds the requested order by at least two. A heat-interaction source uses $g_{j,\ell-r}$, again with at least two additional derivatives. All profile derivatives used below are fixed derivatives of W ; W is smooth in every finite order.

For zero-integral functions supported in one fixed ball,

$$\|K * f\|_{H^{r+1}(\mathbb{R}^2)} \leq C_r \|f\|_{H^r}, \quad \|Q(f, g)\|_{H^r} \leq C_r \|f\|_{H^r} \|g\|_{H^{r+1}}, \quad r \geq 2. \quad (135)$$

To check the first low-frequency estimate, compact support and zero mass give $|\hat{f}(\xi)| \leq C|\xi| \|f\|_1$ for $|\xi| \leq 1$. The Biot–Savart multiplier has order -1 , so it is bounded there by this cancellation and is of order -1 at high frequency. Plancherel proves the first estimate. For the second, $H^r(\mathbb{R}^2)$ is an algebra for $r > 1$, and $\nabla g \in H^r$; multiplication and the first estimate prove the assertion. On the supported spaces H^{k_*} also controls all the L^1, L^p product norms and the two extra derivatives in residual Laplacians. These elementary bounds justify, rather than hide, every transport derivative in the finite induction.

17.2 The finite recursion and its exact residual

Put $X = L^1 \cap L^p$, $p > 2$, with its sum norm. Splitting Biot–Savart at radius one gives $\|K * f\|_\infty \leq C_p \|f\|_X$. Let $H < \infty$, $t_0 \geq 0$, $\varepsilon > 0$, $0 < \kappa \leq 1$, and

$$s = t - t_0, \quad a(t) = \varepsilon e^{\sigma(t-t_0)}, \quad W_\kappa(t) = e^{\kappa t \Delta} W, \quad W_r(t) = \frac{t^r}{r!} \Delta^r W.$$

Let $B(t, x)$ be real and divergence free, spatially C^∞ for almost every time, and locally integrable in time with values in every fixed finite C^m norm. Only its C^R norm is used in the quantitative bounds. This qualitative regularity is automatic for the finite smooth stream basis with actual moment coefficients. Its polar velocity components have finitely many angular sectors $m \in 4\mathbb{Z}$, $|m| \leq 4J$. The retained field is $b_J = \varepsilon B$. Assume

$$\frac{\alpha}{\varepsilon} e^{\sigma t_0} \|\Phi\|_{H^{r_{10}}} \leq A_0, \quad D(t) := e^{-\sigma(t-t_0)} \|B(t)\|_{C^R}, \quad \int_0^H D(t) dt \leq B_0, \quad (136)$$

where $\Phi \in E_1^{r_{10}}$ is a real smooth zero-integral seed. The bounded constants A_0, B_0 are the input ball; no pointwise upper bound on D is imposed.

Set $g_{10}(0) = (\alpha/\varepsilon)\Phi$, with all other initial coefficients zero. For $1 \leq j \leq M$, $0 \leq \ell \leq N$, solve

$$\begin{aligned} (\partial_t - \mathcal{L})g_{j\ell} &= \mathbf{1}_{\ell \geq 1} \Delta g_{j, \ell-1} - \sum_{r=1}^{\ell} [(K * W_r) \cdot \nabla g_{j, \ell-r} + (K * g_{j, \ell-r}) \cdot \nabla W_r] \\ &+ \sum_{\substack{j_1+j_2=j, \\ \ell_1+\ell_2=\ell}} Q(g_{j_1 \ell_1}, g_{j_2 \ell_2}) - \mathbf{1}_{j=1} B \cdot \nabla W_\ell - \mathbf{1}_{j \geq 2} B \cdot \nabla g_{j-1, \ell}. \end{aligned} \quad (137)$$

Every coefficient on the right precedes (j, ℓ) in the ordering by $j + \ell$. The angular sectors are at most $4Jj$. All outputs are supported in $\mathcal{C}$: Laplacians preserve the support of smooth flat functions; $\Delta^r W$ with $r \geq 1$ has this support; and every advective term differentiates a supported function. Each forcing has zero integral by divergence freedom or the Laplacian identity. The bounded finite-sector generator and Duhamel therefore construct a unique smooth coefficient, absolutely continuous in time in every required lower Sobolev order. Measurable time coefficients are sufficient.

Define

$$F(t) = \sum_{j=1}^M \sum_{\ell=0}^N \varepsilon^j \kappa^\ell g_{j\ell}(t), \quad w_{\text{app}} = W_\kappa + F.$$

The recursion is an algebraic identity at the given run's B, κ . Dependence of B on the actual solution and on κ does not change that identity.

Lemma 17.1 (Coefficients and finite residual). *Under (136), with the reserve (134),*

$$\|g_{j0}(t)\|_{H^{r_{j0}}} \leq C e^{j\sigma s}, \quad \|g_{j\ell}(t)\|_{H^{r_{j\ell}}} \leq C(1+t)^{d_{j\ell}} e^{j\sigma s}, \quad d_{j\ell} = 2\ell(j+\ell), \quad \ell \geq 1. \quad (138)$$

The constants depend on the fixed finite orders and input ball, not on $t_0, H, \varepsilon, \kappa$ or the particular waveform. One may take $d_R = 4N(M+N) + N + 3$. If $a \leq 1$ and $\kappa(1+H)^{d_R} \leq c$, the residual in the retained field satisfies

$$\|\partial_t w_{\text{app}} + (K * w_{\text{app}} + \varepsilon B) \cdot \nabla w_{\text{app}} - \kappa \Delta w_{\text{app}}\|_X \leq C[a^{M+1} + \kappa^{N+1}(1+t)^{d_R a}](1+D(t)). \quad (139)$$

Proof. For g_{10} , (133) and the weighted initial bound show that its homogeneous contribution is at most $Ce^{\sigma s}$. Its forcing convolution is at most

$$C \int_0^t e^{\sigma(t-r)} \|B(r)\|_{C^R} dr = Ce^{\sigma s} \int_0^t D(r) dr.$$

The reserve supplies its requested derivatives. Suppose preceding coefficients have the claimed bounds. A quadratic pure-seed source has size $Ce^{j\sigma(r-t_0)}$ in the requested Sobolev order, by (135). For $j \geq 2$ its convolution has the gap $(j-1)\sigma$, so its bound is $Ce^{j\sigma s}$ with no polynomial. A pure external-advection source has size $Ce^{j\sigma(r-t_0)}D(r)$; dividing its convolution by $e^{j\sigma s}$ leaves $C \int_0^t e^{-(j-1)\sigma(t-r)} D(r) dr \leq CB_0$. This proves the pure-seed estimate.

For positive viscosity order, the heat terms carry $(1+r)^r$ from W_r , the diffusion term has the preceding polynomial, and the quadratic terms add their two polynomial degrees. A convolution at $j = 1$ can add at most one degree; for $j \geq 2$ the strict gap adds none. The choice $d_{j\ell} = 2\ell(j+\ell)$ dominates all these degrees:

$$d_{j\ell} \geq d_{j,\ell-1} + 1, \quad d_{j\ell} \geq r + d_{j,\ell-r} + 1, \quad d_{j\ell} \geq d_{j-1,\ell}, \quad d_{j\ell} \geq d_{j_1\ell_1} + d_{j_2\ell_2}.$$

The second inequality is used for $r \geq 1$ and $\ell \geq r$; the third for $j \geq 2$; the last for the displayed quadratic indices. For the direct $B\nabla W_\ell$ term its degree ℓ is smaller. The weighted D convolution adds no degree beyond the preceding factor. This proves (138) by the finite induction, using precisely the reserve already specified.

For completeness we give the residual identity. Put $G_N = W_\kappa - \sum_{r=0}^N \kappa^r W_r$ and $Q_r h = -[(K * W_r)\nabla h + (K * h)\nabla W_r]$. Write $Q_G h$ for the same expression with G_N replacing W_r . After cancellation of the retained monomials the residual is exactly

$$\begin{aligned} R^J = & - \sum_{j=1}^M \varepsilon^j \kappa^{N+1} \Delta g_{jN} - \sum_{\substack{1 \leq j \leq M, \ 0 \leq \ell \leq N, \ 1 \leq r \leq N \\ \ell+r > N}} \varepsilon^j \kappa^{\ell+r} Q_r g_{j\ell} \\ & - \sum_{\substack{1 \leq j_1, j_2 \leq M, \ 0 \leq \ell_1, \ell_2 \leq N \\ j_1+j_2 > M \text{ or } \ell_1+\ell_2 > N}} \varepsilon^{j_1+j_2} \kappa^{\ell_1+\ell_2} Q(g_{j_1\ell_1}, g_{j_2\ell_2}) \\ & + \varepsilon^{M+1} \sum_{\ell=0}^N \kappa^\ell B \cdot \nabla g_{M\ell} + \varepsilon B \cdot \nabla G_N - Q_G F. \end{aligned} \tag{140}$$

Here juxtaposition with a gradient means a dot product. The exact radial heat base has zero residual on its own. The integral heat Taylor formula gives

$$\|G_N\|_X + \|\nabla G_N\|_X \leq C\kappa^{N+1}t^{N+1}.$$

Use the Biot–Savart X bound and (138) to estimate $Q_G F$. Every remaining factor in (140), other than the heat remainder, is supported on the fixed collars. Pure terms with seed order exceeding M are Ca^{M+1} ; their pure coefficient bounds have no polynomial. A positive viscosity power multiplied by any of the finitely many polynomial factors is absorbed by increasing d_R and requiring $\kappa(1+H)^{d_R} \leq c$. Terms with viscosity order exceeding N are $C\kappa^{N+1}(1+t)^{d_R}a$, since $a^j \leq a$. The external top-seed term uses $\|B\| = e^{\sigma s}D$ and is $Ca^{M+1}D$. The external heat remainder is $C\kappa^{N+1}(1+t)^{d_R}aD$. This proves (139), including all viscosity terms and without assuming a pointwise bound on D . $\square$

17.3 Comparison with the actual foreign velocity

Suppose the actual colour solves

$$\partial_t w + (K * w + b) \cdot \nabla w = \kappa \Delta w, \quad w(0) = W + \alpha \Phi.$$

The actual b is divergence free; it may depend on the entire interacting solution. Assume the usual positive-time smoothness and $C_t X$ initial trace, sufficient for the scalar Kato inequalities. A removed rigid rotation is allowed. Let $\mathcal{U}$ be a fixed annular neighbourhood of $\mathcal{C}$, separated from its complement by a positive radial heat-gradient gap. Assume

$$\sup_{x \in \mathcal{U}} |b - b_J| \leq m_q, \quad |b(t, x) \cdot e_r| \leq C_b(1 + |x|), \quad (141)$$

with fixed C_b . The second is used only against the exact radial heat gradient outside $\mathcal{U}$. No derivative of b is assumed. The stream extensions giving B agree with the retained harmonics on $\mathcal{U}$. These are exactly the collar omission and actual global radial-growth inputs supplied by the packet/core estimates.

Lemma 17.2 (Actual density comparison). *Fix $C_1 = C_p(\|\nabla W\|_X + 1)$ before choosing M, N, J . Choose $(M + 1)\sigma > C_1$. There is a later threshold $a_* > 0$ such that if $a(t) \leq a_*$ and the finite viscosity smallness conditions hold, then $\|\nabla w_{\text{app}}\|_X$ gives the fixed exponent C_1 and*

$$\begin{aligned} \|w - w_{\text{app}}\|_X &\leq Ca^{M+1} + C\kappa^{N+1}(1+t)^{d_R+1}ae^{(C_1-\sigma)+t} \\ &\quad + Cm_q(1+t)e^{C_1 t} + C(1+H)e^{C_1 H}(1+\varepsilon e^{\sigma H})e^{-c/(\kappa H)}. \end{aligned} \quad (142)$$

The last Gaussian term is understood as zero at $H = 0$. Consequently in a scale family

$$\varepsilon = q^r, \quad \sigma t_0 \leq \theta L, \quad H \leq C_H L, \quad \kappa \leq q^\gamma, \quad m_q \leq q^P L^{d_m}, \quad r, \gamma > 0,$$

if

$$\gamma(N+1) > C_H(C_1 - \sigma)_+ + \chi, \quad P > r + \theta + C_H(C_1 - \sigma)_+ + \chi, \quad (143)$$

then

$$\|w - w_{\text{app}}\|_X \leq Ca^{M+1} + Cq^\chi L^{d_*} a, \quad \|w - W_\kappa\|_X \leq Ca. \quad (144)$$

These estimates require no nonzero transmission coefficient.

Proof. The finite coefficient bounds make the pure perturbation gradient at most $1/2$ after choosing a_* sufficiently small. Choose the later viscosity threshold so the remaining finite gradients are at most $1/2$. Heat contracts $\|\nabla W\|_X$. This yields the already fixed C_1 ; the coefficient constants change thresholds, not that exponent. For $z = w - w_{\text{app}}$ the exact equation is

$$(\partial_t + (K * w + b) \cdot \nabla - \kappa \Delta)z = -(K * z) \cdot \nabla w_{\text{app}} - R^J - (b - b_J) \cdot \nabla w_{\text{app}}.$$

The actual foreign field is part of divergence-free transport and cancels from the L^1, L^p norm inequalities. The Biot–Savart X bound gives

$$(\|z\|_X)' \leq C_1 \|z\|_X + \|R^J\|_X + \|(b - b_J) \cdot \nabla w_{\text{app}}\|_X.$$

The initial error is zero. Smooth norm regularizations and compact cutoffs justify the integrated inequality from time zero.

On $\mathcal{U}$ the mismatch is at most $Cm_q(1+a)$. Outside it, $\nabla F = 0$ and the weighted radial heat-gradient norm is $Ce^{-c/(\kappa t)}$, after decreasing c to absorb fixed polynomial heat factors. The actual radial-growth bound and $\|b_J\|_\infty \leq aD$ therefore bound the omitted force by $C(1+aD)e^{-c/(\kappa t)}$. Convolve with $e^{C_1(t-v)}$. For the top pure residual, division by $a(t)^{M+1}$ leaves

$$\int_0^t e^{-[(M+1)\sigma - C_1](t-v)}(1 + D(v)) dv \leq [(M+1)\sigma - C_1]^{-1} + B_0.$$

For the viscous residual, division by $a(t)$ leaves at most $C(1+t)^{d_R+1}e^{(C_1-\sigma)t}(1+B_0)$. The constant collar mismatch gives $Cm_q(1+t)e^{C_1 t}$. The Gaussian force uses $\int D \leq B_0$ and monotonicity of its exponential, proving (142) after enlarging its harmless prefactor. In the scale family, $a(0) \geq q^{r+\theta}$ and (143) makes both non-Gaussian errors relative to $a(t)$ positive powers of q times fixed logarithms. The Gaussian dominates all these fixed power losses uniformly for smaller viscosities. Finally the finite coefficient sum has norm Ca and the error has the same bound, proving (144). $\square$

17.4 A normalized finite polynomial loses the forcing waveform

Extend B by zero after H , and put $S = H - t_0$. Suppose the decomposition holds on $-t_0 \leq s \leq S$, with the following future bound for $s \geq 0$:

$$\begin{aligned} B(t_0 + s) &= B^{\text{main}}(s) + E(s), \\ \|B^{\text{main}}(s)\|_{C^R} &\leq C_F(1+s)^{d_F}e^{hs}, \quad 0 \leq h \leq h_* < \sigma, \\ \int_{-t_0}^S e^{-\sigma s} \|E(s)\|_{C^R} ds &\leq \eta_q. \end{aligned} \tag{145}$$

The main and remainder are finite-sector fields of the same class as B and are zero-extended after S . Before $s = 0$ only the total weighted bound (136) is used; no past convergence hypothesis is needed for the coefficient reset. Define the *full* first-order growing amplitude

$$C_q = \frac{\alpha}{\varepsilon} e^{\lambda t_0} \mathcal{A}(\Phi) + \int_0^H e^{-\lambda(t-t_0)} \mathcal{A}(-B(t) \cdot \nabla W) dt. \tag{146}$$

It includes every retained input and the initial seed. It equals the infinite-future growing amplitude of the zero-extended first coefficient. It is not replaced by a canonical target coefficient.

Let $F_1(C) = CF_+ + \bar{C}\bar{F}_+$, and, for $j \geq 2$, define

$$F_j(C) = -(j\sigma - \mathcal{L}_{\text{rot}})^{-1} \sum_{a+b=j} (K * F_a(C)) \cdot \nabla F_b(C). \tag{147}$$

The inverse is the convergent integral of the finite-sector semigroup $e^{t(\mathcal{L}_{\text{rot}} - j\sigma)}$. It is not a formal inverse. All coefficients are smooth, supported in $\mathcal{C}$, of zero mass, and belong to sectors $4n$ with $|n| \leq j$ and $n \equiv j$ modulo two. Induction proves that they are real polynomials of total degree j in $C, \bar{C}$, and that $F_j(C) = |C|^j \mathcal{T}_{\arg C} F_j(1)$ for $C \neq 0$.

Lemma 17.3 (Finite coefficient memory loss). *Under (136) and (145), for any fixed $0 < d < \min(\sigma - 2\eta, \sigma - h_*)$, the pure coefficients obey*

$$\left\| e^{-j\sigma s} \mathcal{T}_{-\tau s} g_{j0}(t_0 + s) - F_j(C_q) \right\|_{H^{r_{j0}}} \leq C_j(1+s)^{d_j} e^{-ds} + C_j \eta_q, \quad 0 \leq s \leq S. \tag{148}$$

The constants depend only on the fixed finite orders, input ball, future envelope, and the strict rate gap. They do not depend on the waveform, preparation length, or an incoming canonical-profile error. No lower bound on $|C_q|$ is required for this coefficient statement.

Proof. The coefficient bounds imply that every normalized coefficient at $s = 0$ is uniformly bounded in its reserved order. Write $h_j(s) = e^{-j\sigma s} \mathcal{T}_{-\tau s} g_{j0}(t_0 + s)$. For $j = 1$ the growing component is $F_1(C(s))$, where the partial integral in (146) defines $C(s)$. Its missing main future tail has norm at most $C(1+s)^{d_F} e^{-(\sigma-h)s}$; its missing remainder has norm at most $C\eta_q$. Its complementary component has propagator decay $e^{-(\sigma-2\eta)s}$ after normalization. Its main forcing has bound $C(1+s)^{d_F} e^{-(\sigma-h)s}$ in this frame. Duhamel convolution of these two decays is bounded by $C(1+s)^{d_1} e^{-ds}$ for the stated strictly smaller d . The remainder convolution is bounded by $C \int_0^s e^{-\sigma v} \|E(v)\|_{C^R} dv \leq C\eta_q$. This proves the first case.

For $j \geq 2$ the normalized equation is exactly

$$h'_j = (\mathcal{L}_{\text{rot}} - j\sigma)h_j - \sum_{a+b=j} (K * h_a) \cdot \nabla h_b - e^{-\sigma s} (\mathcal{T}_{-\tau s} B(t_0 + s)) \cdot \nabla h_{j-1}.$$

Its homogeneous propagator decays as $Ce^{-(j-1)\sigma s}$ by (133). Subtract (147). Each quadratic difference has at least one lower difference, and its other factor is uniformly bounded in the needed reserved order. The reserve supplies one extra derivative in that lower factor, so (135) applies without loss. By induction these terms are bounded by $C(1+s)^d e^{-ds} + C\eta_q$ in the target order. The direct main external term is $C(1+s)^{d_F} e^{-(\sigma-h)s}$; its remainder has weighted time norm at most $C\eta_q$. Duhamel and $(j-1)\sigma > d$ yield the same bound for the normalized difference, including the decaying bounded initial value at zero. Increasing a finite polynomial degree handles a possible convolution polynomial. This proves the finite induction and (148). $\square$

17.5 Exact protected-coordinate matching

Define the real Banach-valued polynomial

$$\mathcal{P}_M(z) = W + \sum_{j=1}^M F_j(z), \quad z \in \mathbb{C}.$$

Its protected coordinate is $\Psi_M(z) = \mathcal{A}(\mathcal{P}_M(z))$. It is a real polynomial in $z, \bar{z}$, is rotation equivariant, $\Psi_M(e^{i\theta}z) = e^{i\theta}\Psi_M(z)$, and $D\Psi_M(0)$ is the identity on $\mathbb{R}^2$. In particular it has the form $z[1 + O(|z|)]$; the parity actually eliminates its quadratic term, but that stronger fact is not needed.

Lemma 17.4 (Exact polynomial and ancient cap gauge). *There is a fixed $\rho > 0$ such that Ψ_M has a unique inverse on $|\zeta| < \rho$, with inverse derivative uniformly bounded. Suppose $\|v - \mathcal{P}_M(z)\|_X \leq e$, $|z| < \rho$, and $\mathcal{A}(v) = \delta e^{i\theta}$ with $0 < \delta \ll \rho$. Then for $z_* = \Psi_M^{-1}(\delta e^{i\theta})$,*

$$|z - z_*| \leq Ce, \quad \|v - \mathcal{P}_M(z_*)\|_X \leq Ce. \quad (149)$$

If the inherited canonical ancient orbit has every fixed finite jet

$$\left\| \Omega_{\text{ref}}(u) - \mathcal{P}_M(c_{\text{ref}} e^{\lambda u}) \right\|_X \leq C_M |c_{\text{ref}} e^{\lambda u}|^{M+1}, \quad u \ll 0, \quad c_{\text{ref}} \neq 0,$$

and u_δ is its small exact cap, then the rotation ϕ matching its protected coordinate exactly to $\delta e^{i\theta}$ satisfies

$$\|v - \mathcal{T}_\phi \Omega_{\text{ref}}(u_\delta)\|_X \leq C[e + \delta^{M+1}]. \quad (150)$$

Proof. Choose 2ρ so small that $\sup_{|z| \leq 2\rho} \|D\Psi_M(z) - I\| \leq 1/4$. The integral mean-value formula then gives the quantitative inverse estimate $|\Psi_M(z) - \Psi_M(z')| \geq (3/4)|z - z'|$ in that ball. The inverse function theorem, or its contraction proof applied to $z = \zeta - [\Psi_M(z) - z]$, supplies the inverse on a smaller ball. Since $\mathcal{A}$ is bounded on X , the state error bounds the coordinate error by Ce , and the inverse estimate proves the first bound in (149). The derivative of $\mathcal{P}_M$ is bounded on this fixed ball, so the second follows.

At the canonical exact cap, its jet remainder is $O(\delta^{M+1})$ because $|c_{\text{ref}} e^{\lambda u \delta}| \asymp \delta$. The same inverse estimate changes its linear asymptotic parameter to the unique parameter matching its *full* protected coordinate, with error $O(\delta^{M+1})$. Rotation equivariance makes that parameter, after the exact coordinate-matching rotation, precisely z_* . The polynomial and ancient-state errors therefore add to the bound (150). Matching only the linear parameter would instead leave a lower-order error; it is not used here. $\square$

17.6 State reset and the separate instantaneous cap condition

Combine the lemmas in a scale family with $\eta_q \leq q^{\chi_E} L^{d_E}$, $\chi_E > 0$, and $0 < c_* \leq |C_q| \leq C^*$. Define

$$S_q = \sigma^{-1} \log \frac{\delta}{\varepsilon |C_q|}, \quad \delta = L^{-4},$$

and assume the prospective interval contains $t_0 + [S_q - L_0, S_q + L_0]$, where L_0 is fixed. At a time in that window put $z(t) = \varepsilon e^{\lambda(t-t_0)} C_q$. The coefficient memory estimate, the positive-viscosity correctors, the comparison bound, and $\|W_\kappa - W\|_X \leq C\kappa H$ give

$$\|w(t) - \mathcal{P}_M(z(t))\|_X \leq C\delta^{M+1} + q^{\chi_*} L^{d_*}, \quad (151)$$

for some fixed $\chi_* > 0$. Indeed $S_q = (r/\sigma)L + O(\log L)$, so the intrinsic memory transient is a positive power of q times fixed logarithms. The other errors have the strict powers already stated. The exponent χ_* is independent of an incoming canonical state error: that error may change the waveform and C_q , but is not used in the memory estimate. Exact coordinate matching by Lemma 17.4 then gives the same order of ancient-state reset at any exact cap in this window.

A weighted time norm alone does not supply a crossing slope. For a protected first-cap conclusion, impose the following *separate pointwise actual-field bounds* on the late window whenever $3\delta/4 \leq |\mathcal{A}(w)| \leq 5\delta/4$, writing $f = w - W_\kappa$:

$$\left| \int Tb \cdot \nabla W \right| + \int |f| |b| |\nabla T| + \left| \int Tb \cdot \nabla (W_\kappa - W) \right| \leq \rho_q \delta, \quad \rho_q \rightarrow 0. \quad (152)$$

These are instantaneous hypotheses, not consequences of (136). In the actual receiver application they follow, respectively, from the absolute packet moments and older collar bounds, the global weighted adjoint lemma, and the weighted smooth heat-profile estimate with the global radial-growth bound. Thus they require no new unproved estimate beyond those actual-field lemmas. A source with only a small weighted norm but an arbitrarily high time spike would not satisfy (152).

Corollary 17.5 (Protected actual first cap). *Assume the preceding scale-family bounds, the nonzero coefficient bound, (152), and a sufficiently wide prospective interval. Then for small q there is a unique first crossing $|\mathcal{A}(w(t_q))| = \delta$. It lies in the fixed late window and satisfies*

$$t_q = t_0 + S_q + o(1), \quad \frac{d}{dt} |\mathcal{A}(w)| \geq \sigma\delta/2 \quad \text{for } 3\delta/4 \leq |\mathcal{A}(w)| \leq 5\delta/4 \text{ in that window.}$$

The upper slope is $C\delta$, and its exact-cap state satisfies

$$\left\| w(t_q) - \mathcal{T}_{\phi_q} \Omega_{\text{ref}}(u_\delta) \right\|_X \leq C\delta^{M+1} + q^{\chi_*} L^{d_*},$$

where the canonical coordinate and actual coordinate are matched exactly. Any continuous family of actual solutions satisfying these uniform bounds has continuous cap times and phases.

Proof. The all-time density bound and boundedness of $\mathcal{A}$ give $|\mathcal{A}(w)| \leq Ca(t)$. Choose L_0 so large that $Ce^{-\sigma L_0}/c_* < 1/4$. Then no cap occurs for $t - t_0 \leq S_q - L_0$. In the fixed late window (151) and the polynomial coordinate yield $\mathcal{A}(w) = z(t) + O(\delta^2) + o(\delta)$ uniformly. The moduli at the window endpoints straddle δ , so continuity gives a crossing. The adjoint identity proved in Section 4, with (144) and (152), gives

$$\mathcal{A}' = \lambda \mathcal{A} + E_A, \quad |E_A| \leq C\delta^2 + \rho_q \delta + C\kappa(1 + H)\delta.$$

The last term tends to zero relative to δ . The derivative of the modulus is $\sigma|\mathcal{A}| + \text{Re}(\overline{\mathcal{A}}E_A)/|\mathcal{A}|$. On the stated level interval it is at least $3\sigma\delta/4 - C\delta^2 - o(\delta) \geq \sigma\delta/2$. The upper bound follows from the same identity. Let C_{speed} be its fixed upper-speed constant. For all sufficiently small q , the crossing lies in the middle half of the late window. Choose

$$d_{\text{cap}} = \min\{L_0/4, (8C_{\text{speed}})^{-1}\}.$$

On $[t_q - d_{\text{cap}}, t_q + d_{\text{cap}}]$ the radius stays in the level band: before a first exit its change is at most $C_{\text{speed}}\delta d_{\text{cap}} \leq \delta/8$, whereas exit requires a change of $\delta/4$. The lower slope integrates on this entire interval. At earlier times the envelope margin applies or the radius is below $3\delta/4$; a previous return from a larger radius would require a downward band crossing. Thus for each fixed $d \in (0, d_{\text{cap}}]$ there is the explicit uniform precap margin

$$\sup_{0 \leq t \leq t_q - d} |\mathcal{A}(w(t))| \leq \delta - \min\{\delta/4, \sigma\delta d/2\}.$$

A second crossing within the protected window would require a downward passage through this level interval, contradicted by the strictly positive slope. The endpoint estimate also gives $t_q - (t_0 + S_q) = o(1)$. Apply the exact-gauge lemma to (151) at this crossing. Finally a uniform early margin, common crossing slope, and uniform convergence of the continuous coordinate under solution parameters prove continuity of the first hit; its nonzero coordinate makes its circle-valued phase continuous. $\square$

The nonzero C_q is a necessary transmission input to the last corollary: no finite-forcing lemma can exclude cancellation for arbitrary signed histories. The actual source-history theorem supplies it. The finite approximation, derivative reserve, residual integration, and normalized reset above do not assume that conclusion. Likewise the last corollary makes the instantaneous slope input explicit; it is never inferred from a weighted L^1 time norm.

18 An actual receiver and the loss of source memory

This section gives a receiving-stage implication for the actual coloured Navier–Stokes evolution. Its inductive hypothesis is the preceding child’s preparation and exact-cap state; it does not prescribe the foreign velocity. The selected deepest stage must be supplied separately. The global induction and phase shooting are also separate statements.

Write $L = \log(1/q)$, $\delta = L^{-4}$, $\alpha = q^K$, where $K > 6$, and fix the smooth reference W , its protected coordinate $\mathcal{A}$, its growing eigenvalue $\lambda = \sigma + i\tau$, and its canonical ancient orbit $\Omega_{\delta_{\text{ref}}}$. The reference amplitude $\delta_{\text{ref}} > 0$ is fixed, sufficiently small, independently of q . After one fixed time translation and rotation, normalize the canonical orbit so $\mathcal{A}(\Omega_{\delta_{\text{ref}}}(0)) = \delta_{\text{ref}} > 0$. The bilateral transform is multiplied by a nonzero factor by this normalization; its nonvanishing selection is therefore unchanged. Choose a fixed β in a sufficiently large interval so that

$$C_* := \gamma_\epsilon \int_{\mathbb{R}} e^{-\mu_\beta u} B_4(\Omega_{\delta_{\text{ref}}}(u)) du \neq 0, \quad \mu_\beta = [\sigma + i(\tau - 4/5)]/\beta.$$

All choices below are made before q .

Proposition 18.1 (Actual receiving-stage implication). *Consider actual colours w_l at scales $A_l = A_1 \beta^{l-1}$, $R_l = R_1 q^{l-1}$, in the common unforced Navier–Stokes velocity. Their initial conditions are the fixed smooth seeded profiles constructed in Section 2, with all seed amplitudes at most α ; every colour is real, fourfold symmetric and has zero circulation. Fix a receiving index i and its immediate child $c = i + 1$. Work in receiver units and remove the common solid rotation of all older cores. Assume $\kappa_c \leq q^{3/2}$ and hence $\kappa_i = \beta q^2 \kappa_c \leq \beta q^{7/2}$.*

The inductive child hypothesis is as follows. In child units it has an exact first cap τ_c , with $|\mathcal{A}(w_c(\tau_c))| = \delta$,

$$c_0 L \leq \tau_c \leq C_0 L, \quad \left\| w_c(t) - e^{\kappa_c t \Delta} W \right\|_X \leq C \delta e^{\sigma(t-\tau_c)} \quad (0 \leq t \leq \tau_c), \quad (153)$$

$$\inf_{\phi} \|w_c(\tau_c) - \mathcal{T}_\phi \Omega_{\delta_{\text{ref}}}(s_q)\|_X \leq C \delta^{M+1} + q^\chi L^D, \quad (154)$$

where $s_q < 0$ is the canonical exact-cap time $|\mathcal{A}(\Omega_{\delta_{\text{ref}}}(s_q))| = \delta$; the phase in (154) is chosen in the same exact protected-coordinate gauge. Here $\chi > 0$, and M is large enough that the right side of (154), multiplied by the fixed forward comparison factor over $-s_q = O(\log L)$ and by $\delta^{-1/\beta}(1 + |s_q|)$, tends to zero. This slightly stronger finite-order requirement also controls the weighted history on the advance. The positive scalar envelope transported from $|w_c(0)|$ has at τ_c uniformly bounded moments of each fixed order and Gaussian tails outside a fixed ball, as supplied by the preparation localization lemma. All these constants are uniform in depth and in the allowed seed homotopies.

Choose fixed C_0, C_2 with $C_2 > C_0 > \max(K/\sigma, 6/\sigma)$, then β, D sufficiently large so that $C_0 > 6\beta/[\sigma(\beta - 1)]$, the unconditional inactive closure holds and

$$\epsilon_0 := C_2 \sigma / \beta < K - 6, \quad C_2 \lambda_B < K - 6, \quad C_2(\lambda_P + C_4/\beta) < 1, \quad \epsilon_0 < 1/2. \quad (155)$$

For small q , the actual receiver has a unique protected first cap $|\mathcal{A}(w_i(t_i))| = \delta$, with

$$t_i = t_0 + \sigma^{-1} \log \frac{\delta}{q^6 |C_*|} + o(1),$$

where t_0 is the child's fixed-reference time in receiver units. The reference phase ϕ_c used below is measured in the receiver's frame at t_0 . If θ_c is its phase in the child's frame with all older rotations removed, then $\phi_c = \theta_c + 4t_0/5$ modulo 2π : the parent core rotation is $-1/5$ in receiver time. Thus the source field has factor $e^{4i(t-t_0)/5} e^{i\phi_c}$, rather than counting the reference rotation a second time. It obeys an exponential preparation envelope

$$\left\| w_i(t) - e^{\kappa_i t \Delta} W \right\|_X \leq C q^6 e^{\sigma(t-t_0)} \quad (0 \leq t \leq t_i). \quad (156)$$

In its exact-cap phase gauge its state satisfies

$$\|w_i(t_i) - \mathcal{T}_{\phi_i} \Omega_{\delta_{\text{ref}}}(s_q)\|_X \leq C\delta^{M+1} + q^{\chi'} L^{D'}, \quad (157)$$

for some fixed $\chi' > 0, D' < \infty$. The exponent χ' depends only on the fixed stage parameters, and may be chosen independently of the incoming reset exponent χ . Advancing this cap to the fixed reference preserves an upper clock bound $C_0 L$, for small q . At the cap, and throughout the level interval $[3\delta/4, 5\delta/4]$,

$$\frac{d}{dt} |\mathcal{A}(w_i(t))| \geq \sigma\delta/2.$$

The corresponding upper derivative bound is $C\delta$. The positive receiver envelope has uniformly bounded fixed moments and Gaussian tails at the cap. The cap time and phase are continuous under all allowed seed homotopies. If the child's reference phase is ϕ_c , then the receiver cap phase is $\phi_c + \arg C_* + \tau S_q + o(1)$, uniformly, where $S_q = \sigma^{-1} \log[\delta/(q^6 |C_*|)]$.

This proposition is an induction step: it does not assert that the child hypotheses have already been established for every index.

Proof. We order the stopping argument so no signed lower bound is used to establish the receiver density. The unconditional inactive theorem protects all colours older than the current receiver. Introduce the additional current receiver stop

$$\|w_i - e^{\kappa_i t} W\|_X = C_{\text{stop}} \delta,$$

where C_{stop} is a fixed constant chosen below. On this interval, $H\sqrt{C_{\text{stop}}}\delta \rightarrow 0$ for every $H \leq C_2 L$. The actual core bootstrap therefore protects the receiver's core too. The child's inductive cap and its advance to the fixed reference occur before this stop: while the child is current, the unconditional older inactive theorem bounds the receiver independently of the child's nonradial density. The child cap comparison, over $O(\log L)$, then supplies its fixed-reference state. This gives

$$t_0 \leq C_0 L/\beta + O(\log L),$$

with its phase ϕ_c fixed by that comparison. The reference may be taken at the actual protected crossing of the fixed level δ_{ref} : the short comparison and the canonical positive crossing slope shift its matching time and phase only by $o(1)$. The positive reference envelope has uniform fixed moments too. Indeed the source's nonradial velocity on the advance has integral $O(\delta_{\text{ref}}) + o(1)$, by the canonical negative-segment envelope and its Kato error. The descendant global velocity adds $O(\beta q \log L)$; older harmonic radial fields and leaks are smaller. Choosing δ_{ref} small fixes one support radius for the positive envelope through the reference, with Gaussian outside tails. This supplies a reference moment reset, rather than bounding its radius by $O(\log L)$ from a crude velocity estimate.

The actual child history. After reference recentering, its own whole fourth moment satisfies $|B_c(u)| \leq Ce^{\sigma u}$ for $u \leq 0$ by (153), the cap localization, and the short forward comparison. On every fixed window it converges uniformly to $e^{i\phi_c} B_4(\Omega_{\delta_{\text{ref}}}(u))$. The actual postcap moment lemma gives, before the parent receiver cap,

$$|B_c(u)| \leq C(1+u)^4 e^{C\delta u/\beta} + o(q^P), \quad u \geq 0,$$

for every fixed P . Its hypotheses hold because the parent core is protected on the stop, the source's own velocity and all descendant velocities have a uniformly bounded sum in source units, and its

positive reference envelope has uniformly bounded moments. Since $u \leq C\beta L$ and $\delta L \rightarrow 0$, the exponential is uniformly bounded. Splitting into a fixed reference window and its two tails gives

$$\int e^{-\mu_\beta u} B_c(u) du = e^{i\phi_c} \mathcal{F}_{\text{ref}}(\mu_\beta) + o(1), \quad (158)$$

uniformly in depth on the full prospective interval reached below; the same argument gives convergence in the weighted absolute history norm. In particular the absolute main forcing norm is bounded by a fixed reference-history norm plus one, for small q . To make the constant closure explicit, the portion before the child's shrinking cap is at most $C_{\text{past}}\delta^{1-1/\beta}$ times a fixed reference factor. On the advance to the reference, the strengthened finite-order cap error is small in the weighted norm, and the canonical history has fixed norm. At and after reference the positive envelope has one fixed localization radius and uniformly bounded moments. Hence the main input bound may be fixed before choosing C_{past} large enough for the output density envelope; afterwards q is decreased so its contribution $C_{\text{past}}\delta^{1-1/\beta}$ is small. This bound does not multiply a child envelope constant from one stage to the next. The localization radius is fixed before q , since the preparation's own nonradial displacement is $O(C_{\text{past}}\delta)$ and its reference advance adds only $O(\delta_{\text{ref}}) + o(1)$. The moment is extended by zero outside its physical history and the receiving interval. The two endpoints tend respectively to $-\infty$ and $+\infty$; the negative exponential and positive polynomial-discount bounds justify this extension.

All other actual harmonic coefficients. In receiver units the inner packet has total mass $O(\beta q^2)$, L^p norm $O(\beta q^{2/p})$, and global velocity at most $C\beta q$. Inside the protected receiving core the older/current radial remainder has bound $C\delta r^3$ plus Gaussian leakage. Radial moment comparison and exit at a fixed core radius give, for every fixed n ,

$$\int |x|^{4n} |\text{inner packet}(t, x)| dx \leq C_n \beta q^{4n+2} (1 + H)^{4n} + o(q^P) \quad (0 \leq t \leq H \leq C_2 L). \quad (159)$$

Indeed its radial moment root is bounded, before exit, by $Ce^{C_n \delta t} [q + \beta q t + \sqrt{\kappa_i t}]$; $\kappa_i \leq \beta q^{7/2}$ makes the square-root term smaller than q up to logarithms. At exit the probability is $e^{-c/(\kappa_i H)}$, and the global radial growth amplifies each fixed moment by only $e^{C_n H}$, which is absorbed by this Gaussian. This calculation uses the positive aggregate, not its nonexistent summed L^∞ bound.

The descendants below the immediate child have mass $O(\beta q^2)$ in child units. The child's preparation envelope traps their positive aggregate in a fixed child ball up to its reference time; afterwards their moment is bounded by $C\beta q^2(1 + u)^4 e^{C\delta u/\beta}$, with Gaussian errors. Thus their fourth moment adds only

$$O(q^{2-\epsilon_0} L^D)$$

to the weighted child-reference transform. The factor $q^{-\epsilon_0}$ comes from the finite negative-history endpoint, whose source clock is at most $C_2 L + O(\log L)$; its receiving discount is σ/β . This argument controls the actual entire descendant packet and does not assert a separate small-density bound for its colours.

Retain exterior harmonics $4, 8, \dots, 4J$ of the inner packet and interior harmonics $3, 7, \dots, 4J - 1$ of the older colours. Use their *actual* time coefficients, and fixed smooth stream cutoffs agreeing on the derivative collars; call the resulting global field b_J . The leading child fourth harmonic gives the first-order coordinate forcing

$$\beta q^6 \gamma_\epsilon e^{4i(t-t_0)/5} B_c(\beta(t-t_0)).$$

For each finite k , all retained higher inner harmonics have weighted C^k norm, divided by q^6 , at most $C_J q^{4-\epsilon_0} L^D$ by (159). The unconditional inactive theorem bounds the actual older harmonics

on these collars by

$$O(q^{K+2-C_2\lambda_B}) + O(q^{8-C_2(\lambda_P+C_4/\beta)}L^D).$$

After source recentering, their weighted norm divided by q^6 is $o(1)$ by (155). The own initial seed contributes only $O(q^{K-6-\epsilon_0}L^D)$ to the first growing coefficient. All these strict powers are uniform in depth.

For the collar density leak, the inner positive aggregate has Gaussian L^1 tail outside a fixed protected radius and conserved finite L^p norm. Biot–Savart interpolation therefore gives a Gaussian velocity leak, despite its infinite sum of initial L^∞ amplitudes. Older leaks have the individual core estimates. The omitted harmonic remainder is $O(q^{4J+6}L^D)$ for the inner packet, and gains arbitrarily many q -powers for the older Taylor series. Choose J so large that, writing $P = 4J + 6$,

$$P > 6 + \epsilon_0 + C_2(C_1 - \sigma)_+ + \chi_1$$

for a fixed $\chi_1 > 0$; older omission orders are chosen to meet the same inequality. This is a relative envelope requirement, not only an absolute endpoint error bound: the first envelope is $a(0) = q^6 e^{-\sigma t_0} \gtrsim q^{6+\epsilon_0} L^{-D}$, and a q^P forcing error divided by $a(t)$ amplifies by at most $e^{(C_1-\sigma)+C_2L}$. The fixed-profile Kato exponent C_1 is selected before J .

The actual nonlinear density bootstrap. Set $\varepsilon = q^6$, $s = t - t_0$ and $a(t) = \varepsilon e^{\sigma s}$. For every spatial derivative order needed in a finite recursion,

$$\int_0^H \varepsilon^{-1} e^{-\sigma(t-t_0)} \|b_J(t)\|_{C^k} dt \leq C_k. \quad (160)$$

Before excluding a possible short density stop, only an upper input norm is used here. The main harmonic satisfies this by its negative exponential bound, the controlled cap-to-reference advance, and its actual future polynomial bound. No assertion about a truncated short interval having endpoints at both infinities is made at this stage. The preceding strict power estimates give it for all other harmonics. No derivative of their actual time coefficients is used.

Apply Lemmas 17.1 and 17.2 to this actual run. In their notation take $\varepsilon = q^6$, $r = 6$, $\theta = \epsilon_0$, $C_H = C_2$ and $\gamma = 3$: $\kappa_i \leq \beta q^{7/2} \leq q^3$ for small q . For each retained coefficient use the explicit reserve $r_{j\ell} = k_* + 2(M - j) + 2(N - \ell)$ and $R = k_* + 2M + 2N + 3$, with $k_* = 4$ for the X conclusion. The finite stream field has polar sectors at most $4J$; its order- j coefficients therefore have sectors at most $4Jj$. The seed bound in (136) follows from $q^{K-6} e^{\sigma t_0} \|\Phi\|_{H^{r_{10}}} = o(1)$. The other input bound is exactly (160) for this finite derivative order R . Actual harmonics agree on the fixed broad annulus $\mathcal{U}$, with omission $m_q \leq q^P L^D$; Gaussian density leaks can be absorbed in that bound. The actual global radial bound is $|b \cdot e_r| \leq C(|x| + \beta q)$, uniformly in depth. Choose $(M + 1)\sigma > C_1$ and a finite N so that $3(N + 1) > C_2(C_1 - \sigma)_+ + \chi_2$. The harmonic order was chosen to satisfy $P > 6 + \epsilon_0 + C_2(C_1 - \sigma)_+ + \chi_2$. These are precisely the relative-error inequalities (143). The coefficient recursion including every viscosity term is (137); its exact cancelled residual is (140), not a pointwise assumed forcing remainder. The actual foreign field stays in the divergence-free error transport. Consequently (144) gives

$$\|w_i - w_{\text{app}}\|_X \leq Ca(t)^{M+1} + q^{\chi_2} L^D a(t), \quad \|w_i - W_{\kappa_i}\|_X \leq Ca(t). \quad (161)$$

No signed growing-coefficient lower bound is used in this application.

Take the deterministic prospective end $H_+ = t_0 + S_q + L_0$, where L_0 is fixed and sufficiently large that even the half-width $L_0/2$ meets the early-margin requirement of Corollary 17.5. For small q , $H_+ \leq C_2 L$ by the parameter choice. The density bound up to this time is $C\delta e^{\sigma L_0}/|C_*|$.

Choose C_{stop} larger than twice this constant. A first density stop before H_+ is impossible. The history and field estimates therefore hold on the entire prospective interval. Extend the retained field b_J by zero after H_+ , and define its *full actual* first growing amplitude by

$$C_q = \frac{\alpha}{\varepsilon} e^{\lambda t_0} \mathcal{A}(\Phi) - \frac{1}{\varepsilon} \int_0^{H_+} e^{-\lambda(t-t_0)} \left[\int T b_J(t) \cdot \nabla W \, dx \right] dt. \quad (162)$$

This includes the own seed and every retained actual older, descendant and higher-sector input. Equivalently it is $\lim_{s \rightarrow \infty} e^{-\lambda s} \mathcal{A}(g_{10}(s + t_0))$ for this zero-extended first coefficient. Retained sectors other than ± 4 have zero direct growing projection, but remain in its full spatial coefficient and in the nonlinear recursion. Its difference from the actual main-child discounted transform is $O(q^{\chi_4} L^D)$ for some fixed $\chi_4 > 0$, by the preceding power estimates. The main-child transform converges to the canonical value by (158); that convergence is only $o(1)$ here, since the finite ancient-truncation error may be logarithmic. Hence

$$C_q = e^{i\phi_c} C_* + o(1), \quad |C_q| \geq |C_*|/2.$$

This ordering avoids using a future transform to exclude an earlier stop.

Universal coefficient reset. Use Lemma 17.3 with main field equal to the actual immediate-child fourth harmonic. Its post-reference C^R norm is $C(1+s)^4 e^{C\delta s}$, so $h_* = \sigma/2$ is admissible for small q . Every other retained field has weighted remainder norm $\eta_q \leq q^{\chi_3} L^D$ by the preceding strict power estimates. The pattern rotation is the vector pushforward/scalar rotation of Section 17; all angular sectors are rotated separately. The full coefficient in (162) equals (146), with the field zero-extended at H_+ . Choose a fixed $0 < d < \min(\sigma - 2\eta, \sigma/2)$. Lemma 17.3 proves, with the explicit derivative reserve, convergence to $F_j(C_q)$ for each pure coefficient. At $s = S_q + O(1) \asymp L$ its transient is a positive q -power. The actual main waveform need not be close to a canonical waveform for this statement: only its finite input ball and future envelope are used. Its canonical closeness was used for the lower bound on C_q . Formula (151), with the positive-viscosity correctors and $\|W_{\kappa_i} - W\|_X \leq C\kappa_i H_+$, gives a state error $C\delta^{M+1} + q^{\chi'} L^{D'}$ against the universal polynomial. Lemma 17.4 matches its *full* protected coordinate by the two-real-variable inverse, and identifies the matching all-finite-jet ancient state. This proves (157) once the exact first cap is found. Neither the incoming waveform error nor an incoming reset exponent enters this intrinsic transient exponent.

Protected first crossing. The global density envelope excludes a cap before $H_+ - 2L_0$ if L_0 is increased. In the ensuing fixed window the settled first growing coordinate is $\varepsilon e^{\lambda s} C_q + O(a^2) + o(a)$. The polynomial future tail and the small weighted remainders make the error uniform there. Its moduli at the two ends straddle δ . The actual adjoint-coordinate identity gives, when the coordinate is in $[3\delta/4, 5\delta/4]$,

$$\mathcal{A}' = \lambda \mathcal{A} + O(\delta^2) + o(\delta).$$

For the instantaneous radial-base forcing, use (159), not merely the weighted history: the collar field is $O(q^6 L^D)$ plus the small older field. For the exact foreign cross-advection pairing use Lemma 14.1:

$$\int |f| |b| |\nabla T| \leq C \chi_q \|f\|_1 = o(\delta), \quad \chi_q = C \left(\beta q + \frac{\delta q^2 + q^4 + \varepsilon_{\text{core}}}{\beta} \right).$$

This uses the global test weight and the actual older cores; it requires no claim that the receiver's own heat-base defect is Gaussian in its core. The heat-profile foreign correction is $O(\kappa_i H)$, using the global radial bound on b and the weighted smooth heat Taylor bound. It is $o(\delta)$; the remaining

heat errors and the quadratic self error have the stated orders too. Thus the three separate pointwise criteria of (152) hold with $\rho_q \rightarrow 0$. Corollary 17.5, applied with half-width $L_0/2$, proves the protected first crossing and its lower and upper slopes. Its center uses $|C_q|$; replacing that by $|C_*|$ changes the center by $o(1)$. Consequently this narrower window is contained in the controlled prospective interval for small q ; no forcing beyond H_+ is used. This gives the asserted time and phase asymptotics. Uniqueness of recrossing is asserted in the protected window; no later Euler or Navier–Stokes return to the level is excluded.

The exact universal polynomial’s coordinate is $a[1 + O(a)]$ in its growing phase gauge. Its modulus derivative is $1 + O(a) > 0$. Matching the actual first coordinate therefore selects its time/amplitude and phase gauges uniquely; the matching error has the same orders as (157). This step uses the exact cap, rather than identifying it with the linear preparation time.

Finally, (156) gives integrated receiver nonradial drift $O(\delta)$; the younger global velocity gives $O(\beta q H)$, and the older field is smaller. The positive receiver envelope has Gaussian tails outside a fixed ball and bounded fixed moments at the cap by the radial exit lemma and global moment growth. The same uniform preparation and crossing estimates hold under the stated homotopies. Positive-viscosity continuous dependence, followed by the strictly positive crossing slope, proves cap-time and cap-phase continuity. Its own cap clock is $[6/\sigma + C_0/\beta]L + O(\log L) < C_0L$; the additional reference advance is only $O(\log L)$, so the same strict clock margin remains. The lower clock bound persists since $t_i \geq (6/\sigma)L - O(\log L)$. This completes the actual receiving-stage implication. $\square$

Remark 18.2 (Scope and audit point). The proposition is intended for insertion only after the actual packet, postcap source-history, finite-coefficient reset and deepest selected-stage lemmas have been assembled with matching constants. Its proof does not supply the deepest source cap. The numerical exponent 6.43 belongs to that separate selected-stage estimate. In particular, a global theorem cannot be inferred merely from calling this implication an induction step: one must exhibit the base stage, preserve its exact-cap hypotheses and clocks at every depth, and complete the phase shooting.

19 Uniform constants, the descending induction, and phase selection

We record the constant selection explicitly. An assertion that every finite stage has some finite constant would not suffice for the construction below. In this section the selected seed has amplitude exactly α ; other seed amplitudes may lie in $[0, \alpha]$.

Proposition 19.1 (A common invariant class of actual stages). *Fix $p > 2$, the smooth profile and a sufficiently small fixed reference orbit, normalized once by a time shift and rotation so that $\mathcal{A}(\Omega_{\text{ref}}(0)) = \delta_{\text{ref}} > 0$. There are choices of $K \in (6, 643/100)$, fixed $\beta > 1$, $q > 0$, and constants $C_{\text{past}}, C_{\text{reset}}, R_0, c_0, C_0, C_2 > 0$, all independent of the selected index N , such that the following holds. Put $\alpha = q^K$, $\delta = (\log(1/q))^{-4}$, $A_j = \beta^{j-1}$, $R_j = q^{j-1}$, and $\nu_N = q^{3/2}A_N R_N^2$. For arbitrary seed phases and amplitudes in $[0, \alpha]$, with the N th amplitude exactly α , every actual colour $j = N, N-1, \dots, 1$ has a protected first cap. In its normalized older-core rotating frame, its cap time τ_j from the physical initial time obeys*

$$c_0 \log(1/q) \leq \tau_j \leq C_0 \log(1/q) < C_2 \log(1/q),$$

and its preparation satisfies

$$\|w_j(t) - e^{\kappa_j t \Delta} W\|_X \leq C_{\text{past}} \delta e^{\sigma(t-\tau_j)}, \quad 0 \leq t \leq \tau_j.$$

Its state in its exact-cap gauge lies within $C_{\text{reset}} \delta^{M+1} + q^\chi (\log(1/q))^D$ of the corresponding state on Ω_{ref} , for fixed finite $M, \chi > 0, D < \infty$. The positive envelope is localized uniformly throughout preparation and at its fixed-reference time. Its fixed moments, fixed-radius Gaussian tails and parent-radius Gaussian tails have common bounds. The first caps have common protection constants at each fixed physical scale and depend continuously on the seed parameters.

Proof. Choose K , for example $K = 31/5$. The generic profile constants, including a one-sided X comparison exponent C_W uniform on a fixed small ancient segment and the finite residual exponent C_1 , are fixed first. Fix a small reference on that segment; the finite expansion orders are selected below after the clock and amplitude ratios, and before the invariant class constants. The later advance from the shrinking actual cap to this reference uses one-sided comparison; it does not require the selected source's terminal expansion amplitude to exceed the reference amplitude. Choose a large interval for β so that all strict inequalities in the selected-source, inactive and receiving estimates hold, including

$$K > 6 + \frac{K}{\beta - 1}, \quad C_2 \lambda_B < K - 6, \quad C_2(\lambda_P + C_4/\beta) < 1, \quad C_2 \sigma/\beta < K - 6.$$

Here choose finite clock constants with strict slack

$$\max \left\{ K/\sigma, \frac{6\beta}{\sigma(\beta - 1)} \right\} < C_0 < C_2.$$

One can first use uniform upper bounds over a large interval for β , choose the clock slack, and then enlarge that interval if required by the generic inactive rates. The bilateral canonical transform is analytic and not identically zero, so a β in this interval can be chosen with $C_* \neq 0$. Increasing the finite weight D then makes the inactive rates as small as required. All these parameters are fixed before q .

Fix a broad localization ball B_{R_0} containing the initial profile and small ancient segment with a fixed radial margin. We verify that a common input class, rather than an inherited arbitrary constant, feeds the finite receiving recursion. Suppose provisionally that a child is in a trial class with past-envelope constant C_{past} and cap-reset constant C_{reset} . Its own nonradial velocity integrates to at most $CC_{\text{past}}\delta$ before its cap. The younger velocity contributes $C\beta q C_2 \log(1/q)$, and the older harmonic remainder is smaller. For any fixed trial constant these vanish as $q \downarrow 0$. The radial exit argument therefore gives the same localization ball and moment constants, independent of this trial constant, once q is small. This holds at every preparation time, since the integrated radial drift up to that time has the same bound. For the stronger parent-radius tail, run a separate exit argument at radius c/q in source units. The older stopped-core radial coefficient there is $C\delta/\beta$, and the source and descendants have bounded total speed. Their age is $O(\log(1/q))$, so the deterministic radial displacement is much smaller than q^{-1} . The exit probability is $C \exp[-cq^{-2}/(\kappa_j C_2 \log(1/q))]$. Conditional global moment growth adds only fixed powers of q^{-1} ; the same Gaussian, with reduced exponent constant, controls the parent-radius weighted tails.

Let s_q be the reference orbit's exact δ -cap time. Advancing to the fixed reference state takes $-s_q = O(\log \log(1/q))$. On this segment canonical nonradial velocity integrates to $O(\delta_{\text{ref}})$, and

the actual comparison error has integral tending to zero if M is large enough. Choose δ_{ref} small within the fixed radial margin. A second positive-envelope localization at the reference state hence gives the same fixed radius and universal moment constants, without a factor $\log \log(1/q)$.

In the reference source clock, the pre-small-cap contribution to the weighted forcing norm is at most

$$CC_{\text{past}}\delta^{1-1/\beta}. \quad (163)$$

Indeed the past envelope is $C_{\text{past}}\delta e^{\sigma v}$ in small-cap coordinates $v \leq 0$, while the reference discount contributes $e^{-\sigma s_q/\beta} = O(\delta^{-1/\beta})$. The cap-to-reference segment is bounded by the fixed ancient envelope and the vanishing forward-comparison error. The positive reference history is bounded by the universal moment estimate under the parent stop. Consequently every needed spatially differentiated forcing norm is bounded by

$$M_{\text{input}} \leq M_{\text{ref}} + CC_{\text{past}}\delta^{1-1/\beta} + o(1).$$

The constant M_{ref} depends only on the fixed reference/profile and fixed radius, not on either trial constant or depth. The small error depends on the trial reset constant, but its coefficient tends to zero. The main signed transform is uniformly within $o(1)$ of $e^{i\phi_c}C_*$; fixed-window comparison followed by the two discounted tails proves this uniformly on the trial class. Thus $|C_q| \geq |C_*|/2$ once a sufficiently long prospective receiving interval has been reached. The upper input bound, which is valid on any shorter stop, is used first to reach that interval.

Select all finite expansion orders before computing the class constants. Choose $M+1 > C_W/\sigma + 2$ and $(M+1)\sigma > C_1$; choose enough actual harmonic terms and viscosity orders that their generic residual losses are smaller than the entire growing envelope on $0 \leq t \leq C_2 \log(1/q)$. In particular the omitted harmonic power P satisfies $P > 6 + C_2\sigma/\beta + C_2(C_1 - \sigma)_+ + 1$ for the receiving stage, and the analogous selected-source inequality with $a(0) = q^K$. There are only finitely many required derivatives and sectors.

Compute finite-expansion bounds using the fixed input ball $M_{\text{input}} \leq M_{\text{ref}} + 1$. Choose C_{past} larger than twice the resulting cap-relative envelope constant, divided by the fixed lower growing coefficient, and larger than twice the selected-source past constant. Now choose q so that the term (163) and all other input errors are at most one. The output past envelope improves this same C_{past} ball. The fixed comparison and coefficient bounds give a finite output cap-reset constant independent of the incoming reset error: the source waveform changes C_q , which exact-cap time and phase normalization remove; the universal coefficient reset has only intrinsic positive- q -power transients. Choose C_{reset} larger than twice this output constant and the selected-source constant. Decrease q again so that its inherited reset error advances to the fixed reference by $o(1)$. This constructs a genuinely invariant class of constants.

Choose a common positive exponent χ smaller than the intrinsic base and receiving exponents and take a common finite logarithmic power. No input reset exponent is successively lost.

Theorem 16.2 gives the actual base $j = N$ in this invariant class. Its normalized cap clock is $(K/\sigma) \log(1/q) + O(\log \log(1/q))$. Assume the class holds for $j+1$. Proposition 18.1, with the common input ball just established, supplies it for j . The receiving normalized clock from the physical initial time obeys

$$\tau_j \leq \frac{\tau_{j+1}}{\beta} + \frac{6}{\sigma} \log(1/q) + O(\log \log(1/q)).$$

The geometric sum and strict clock slack keep it below $C_0 \log(1/q)$ after a final decrease of q . The receiving growth interval gives a common positive lower clock $c_0 \log(1/q)$. All core,

tail, moment and inactive estimates therefore stay inside their C_2 windows, including the short reference advance. Descending induction proves the assertions for every finite N with the same constants. Only finite older prefixes are stopped; all infinite younger packets were controlled by their convergent mass, L^p and velocity sums.

The estimates hold throughout a fixed interval beyond each prospective cap and give both an early margin and a strictly positive radial slope. The quantitative first-cap lemma yields common protection constants. Positive-viscosity continuous dependence and the protected crossing imply continuous cap times and phases under seed homotopies. All estimates were uniform in those phases and in all nondistinguished amplitude reductions. $\square$

Proposition 19.2 (Surjective outer phase for each selected viscosity). *With the preceding fixed parameters, fix $N \geq 2$, all phases except the N th, and put all seed amplitudes equal to α . The map taking $\vartheta_N \in \mathbb{R}/(2\pi\mathbb{Z})$ to the unit phase of the outer colour at its first cap for viscosity ν_N is continuous and surjective.*

Proof. Multiply every seed except the distinguished N th seed by $r \in [0, 1]$. The invariant actual-stage proposition applies throughout this homotopy. At $r = 0$ the initial datum is the radial stack plus just that seed. Rotational covariance of the unforced equation and its colour equations shows that changing its phase rotates the entire solution, leaves the cap time unchanged and rotates the outer cap coordinate once. Thus the outer phase map has degree one (with an immaterial sign if the opposite rotation convention is used). Common protected caps give a continuous circle-valued homotopy as r goes to one. Its degree stays nonzero, and a continuous circle map of nonzero degree is surjective. For completeness, a map omitting a point factors through the circle minus that point, an interval, and is homotopic to a constant; it has degree zero. This proves the last assertion without a differentiability assumption. $\square$

Theorem 19.3 (A single datum with two viscosity accumulation points). *For each $2 < p < \infty$ there is a real compactly supported vorticity $\omega_0 \in L^1 \cap L^p(\mathbb{R}^2)$ with zero circulation, and a finite $T > 0$, such that the unforced Navier–Stokes vorticities with this same initial condition have two different accumulation points as $\nu \downarrow 0$ in $C([0, T]; L^1 \cap L^p)$. Both accumulation points are renormalized solutions of unforced Euler with initial vorticity ω_0 .*

Proof. Use the fixed parameters in Proposition 19.1. All actual seeds will have the same fixed amplitude α . Set the outer seed phase to zero and provisionally every other phase to zero. The compact admissible stack lemma gives all required norm, support and circulation properties for every phase choice. Choose $T > C_2 \log(1/q)$ plus the common outer protection window. Let

$$a^\nu(t) = \int T(x) w_1^\nu(t, x) dx$$

be the bounded smooth adjoint observable of the outer passive colour. There is no older-core rotation at this outer scale. The invariant stage theorem gives cap times in one compact subinterval of $(0, T)$, a common precap margin and common positive radial slope near the first crossing of radius δ . The observable is uniformly Lipschitz, since its scalar equation gives

$$|(a^\nu)'| \leq \|w_1(0)\|_1 [\|u^\nu\|_\infty \|\nabla T\|_\infty + \nu \|\Delta T\|_\infty].$$

The velocity bound is uniform by the fixed-data L^1, L^p contractions and the kernel split, and $\nu_N \leq 1$ after choosing q small.

We choose sparse indices $N_k \rightarrow \infty$ and their phases successively. At the k th step keep all earlier assigned phases and vary only ϑ_{N_k} . Proposition 19.2 permits its outer cap phase to be exactly $+1$ for even k , and -1 for odd k . Changing a phase at depth N changes the initial $L^1 \cap L^p$ norm by at most $C\alpha A_N(R_N^2 + R_N^{2/p}) \rightarrow 0$. Choose each next index so far out that all subsequent tail phase changes alter every earlier selected-viscosity outer observable by less than the desired summable tolerance. This is possible by continuous dependence at each fixed positive viscosity and absolute convergence of the remaining initial tail. In detail, after step k there are finitely many positive viscosities and continuity radii; choose N_{k+1} so that the norm of the entire freely variable tail is below all those radii. The fixed smooth outer colour is the same in every datum. Its scalar continuity follows from the fixed-viscosity mild equation. For two velocities, put h for their outer-colour difference and v for their velocity difference. The heat-kernel gradient estimate gives, for $r = 1, 2$,

$$\|h(t)\|_r \leq C\nu^{-1/2} \int_0^t (t-s)^{-1/2} [\|u(s)\|_\infty \|h(s)\|_r + \|v(s)\|_\infty \|w_1(s)\|_r] ds.$$

Volterra comparison gives uniform colour continuity at fixed $\nu > 0$ from uniform velocity continuity; no actual colour gradient is assumed. At step k choose a continuity radius $r_k > 0$ at that step's assigned datum, for its selected viscosity and target accuracy 2^{-k} . Require the worst-case norm of every variable phase tail starting at N_{k+1} to be below $\min_{i \leq k} r_i$. Protected first-cap continuity then gives the stated phase tolerances. Tolerances may be chosen so the k th final cap phase differs from its target by at most 2^{-k} .

The resulting infinite phase assignment defines one fixed ω_0 . No viscosity-dependent initial datum is used. The selected viscosities $\nu_{N_k} = q^{3/2}(\beta q^2)^{N_k-1}$ tend to zero. The fixed-data compactness theorem yields subsequences of the even and odd selected solutions converging strongly in $C([0, T]; L^1 \cap L^p)$ to Euler solutions ω^+ and ω^- . Their velocities converge strongly, and the passive-colour theorem identifies the limits of w_1' as the unique bounded transport colour with the fixed initial outer density, determined by the total Euler path. Thus their observables converge uniformly on $[0, T]$.

Extract also convergent cap times. Common precap protection and radial slope ensure each limiting time is the first crossing of radius δ for its limiting observable: before the common cap window there is a strict margin, and inside it the integrated radial lower slope excludes an earlier crossing. The limiting cap coordinates are respectively $+\delta$ and $-\delta$. If $\omega^+ = \omega^-$, uniqueness of the bounded passive colour in that total velocity gives the same observable. It would then have the same first cap time and coordinate, contradicting these two values. Therefore the total Euler paths differ. The compactness theorem already gives their strong initial trace and renormalization, completing the proof. $\square$

References

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- [2] G. Ciampa, G. Crippa, and S. Spirito, *Strong convergence of the vorticity for the 2D Euler equations in the inviscid limit*, <https://arxiv.org/abs/2008.12133>. Theorem 3.2 concerns strong convergence to an identified renormalized limit; it does not assert uniqueness of that limit.