

Comparison with the supplied referee report

9 October 2026

Assessment and scope

The supplied report concerns an earlier version of the fixed-datum planar Navier–Stokes cascade. I compared every numbered objection with the current manuscript, traced its prerequisites, and made the corrections listed below. The supplied report itself has not been altered.

Conclusion of this comparison. No remaining major mathematical defect corresponding to M1–M5 was identified in the revised arguments checked. M2, M4 and the substantive content of M3 and M5 were already addressed. M1 still contained a compressed $o(1)$ at the crucial induction step; it is now replaced by an explicit vanishing bound. A standalone continuation corollary, an explicit moment-tightness display and constant and cap-protection formulas make the relevant implications directly checkable. The closest-work attribution requested by M6/B2 was missing and is now supplied. The spectral wording in C4 is also corrected.

The report therefore does not, after these changes, provide a demonstrated reason to replace the construction or weaken its main theorem. This is a conclusion about the objections examined, not an independent certification of every statement in the full paper. No counterexample to the main theorem was established. Absence of a counterexample is not a proof of it.

The manuscript’s Theorem 1.1 continues to assert two distinct strong $C([0, T_f]; L^1 \cap L^p)$ viscosity accumulation paths for one fixed compactly supported, zero-circulation datum, for every finite $p > 2$. The present edits do not strengthen that statement. Locations below use the new manuscript’s section and result numbering; old report line numbers no longer locate the corresponding arguments reliably.

Point-by-point disposition

Point	Status	Current location and action
M1	Substantively supported; quantitative consolidation added	Proposition 10.1; equations (206)–(209); Table 3. Defined the common input norm, displayed the trial-dependent advance error and all five strict receiving powers, and specified one finite parameter order.
M2	Already addressed	Lemma 9.1, especially equations (193)–(194). The weighted absolute history norm, inherited-error amplification, three time pieces and finite controlled zero extension are explicit.
M3	Substantive stop argument already present; extraction added	Proposition 9.2 and new Corollary 8.3. The upper input ball gives a strict density-stop improvement before the full signed transform is used.

Point	Status	Current location and action
M4	Already addressed; tightness consequence displayed	Lemmas 7.8 and 7.11, with new equation (168). Both child and source/deep positive envelopes are treated; fixed and parent radii are separate. The reference advance has integrated displacement $O(\delta_{\text{ref}})$.
M5	Already addressed; constants consolidated	Corollary 8.6, Theorem 7.7 and Proposition 10.1. The final proof gives the fixed-viscosity continuity radius and the entire-variable-tail condition. Explicit physical-scale protection constants were added.
M6/B2	Previously missing; corrected now	Introduction, profile section and bibliography. Added the exact two-interface attribution and distinguished the three closest nonuniqueness results from fixed-datum unforced viscosity selection.
B1	Already corrected	Lemma 3.1, the sparse-phase lemma and final proof. Circulation passes by strong L^1 ; compact support follows from the common support ball of the admissible family.
C1	Already corrected	The scaled density estimate keeps $A(R^2 + R^{2/p})\delta$ for the physical $L^1 \cap L^p$ norm; $AR^2\delta$ is an L^1 estimate.
C2	Already corrected	The inactive-prefix discussion identifies the estimate for the actual packet force without claiming a numerically stronger bound.
C3	Already corrected	The literal mathematical text <code>epsilon(t)</code> was replaced by $\epsilon(t)$.
C4	Previously ambiguous; corrected now	The profile's Fredholm argument now specifies that the angular multiplication operator has purely imaginary spectrum. This assertion does not concern the full linearized generator.

Verification of the crucial implications

M1: a finite, depth-independent invariant class. Write

$$L := \log(1/q), \quad \delta := L^{-4}, \quad r := C_W/\sigma, \quad \ell_q := -s_q = O(\log L), \quad \epsilon_0 := C_2\sigma/\beta,$$

$$e_B := C_2\lambda_B < K - 6, \quad e_P := C_2(\lambda_P + C_4/\beta) < 1.$$

The receiving-stage estimates, including the finite negative-endpoint discount, give positive powers

$$2 - \epsilon_0, \quad 4 - \epsilon_0, \quad K - 4 - e_B - \epsilon_0, \quad 2 - e_P - \epsilon_0, \quad K - 6 - \epsilon_0.$$

They are all strictly positive because $\epsilon_0 < \min(K - 6, 1/2)$. Half their minimum defines $\eta_{\text{in}} > 0$. Spatial differentiation acts on the finitely many fixed stream bases, not on time derivatives of the actual moments. The input norm is the maximum of the scaled initial seed norm and the weighted C^R retained-field integral. The explicit bound is

$$M_{\text{input}}(H) \leq M_{\text{ref}} + E_{\text{in}}, \quad H \leq \min(H_+, T_s),$$

$$E_{\text{in}} := CC_{\text{past}}\delta^{1-1/\beta} + E_{\text{adv}} + Cq^{\eta_{\text{in}}}L^{d_{\text{in}}},$$

$$E_{\text{adv}} := C\delta^{-r-1/\beta}(1 + \ell_q)[C_{\text{reset}}\delta^{M+1} + q^\chi L^D + q(1 + \ell_q)].$$

For each fixed trial class this tends to zero: $M + 1 > r + 2$ implies $M + 1 - r - 1/\beta > 1$, and every positive q power dominates the fixed logarithmic factors. The universal input radius is computed

before the trial constants; the later q threshold may depend on these fixed constants but not on the stage depth or seed phases.

The finite-memory estimate normalizes the pure coefficients to $F_j(C_q)$, with an intrinsic exponentially decaying transient and a positive-power field remainder. The incoming error changes C_q ; exact matching of the full cap coordinate removes that change from the outgoing gauge. Thus its exponent is not repeatedly degraded by the number of receiving stages. This is the essential distinction from propagating the incoming comparison error directly into every output reset. The added table records the finite order of choices and the factor-two slack in the invariant class.

M2: absolute history control rather than cancellation. The current source-history lemma controls

$$\int_{\mathbb{R}} e^{-\sigma u/\beta} |B_q(u) - B_4(\Omega_{\text{ref}}(u))| du \longrightarrow 0,$$

where B_q is extended by zero outside the finite controlled actual history. Its negative preparation part is $O(C_{\text{past}}\delta^{1-1/\beta})$, the advance is bounded by the displayed E_{adv} , and the controlled positive part has

$$|B_q(u)| \leq C(1+u)^4 e^{C(1+C_s)\delta u/\beta} + \text{Gaussian remainder}.$$

For fixed C_s , the exponential is at most two on $u \leq C_U\beta L$ for later small q , since $\delta L = L^{-3}$. The discounted tail is consequently bounded by $C(1+J)^4 e^{-\sigma J/(2\beta)}$. Fixed-window comparison is used before sending J to infinity. Shorter density-stopped intervals give only upper bounds; full convergence is invoked after the prospective interval is completed. No claim about the actual uncontrolled infinite future is needed.

M3: receiver closure without a signed lower bound. The finite comparison hypotheses include the actual colour equation, initial seed, derivative input bounds, collar mismatch and global radial-growth bound. Those hypotheses are verified on every shorter stopped interval before evaluating the complete transform. They imply

$$\|w_i(t) - W_{\kappa_i}(t)\|_X \leq C_a a(t), \quad a(t) := q^6 e^{\sigma(t-t_0)},$$

with C_a computed in the fixed input ball independently of the trial stop constant. Since

$$a(H_+) = \frac{e^{\sigma L_0}}{|C_*|} \delta, \quad C_s > \frac{2C_a e^{\sigma L_0}}{|C_*|},$$

the bound improves to $C_s\delta/2$ through $\min(H_+, T_s)$. Continuity excludes a stop there. Only afterwards does Proposition 9.2 use the complete controlled history to establish $C_q = e^{i\phi_c} C_* + o(1)$ and $|C_q| \geq |C_*|/2$. The fixed canonical C_* sets the prospective horizon; this does not assume a nonzero actual C_q . Corollary 8.3 isolates this implication.

M4: moments and two different localization radii. For both normalized positive envelopes, geometric summability gives the initial mass bound

$$M_{\text{sup}} \leq \frac{\|W\|_1 + \alpha \sup_{\theta} \|\Phi_{\theta}\|_1}{1 - \beta q^2}.$$

The proved sixth-moment bound gives, for an independent cutoff radius,

$$\sup_{j, \text{seeds}, P} \int_{|y| > R_{\text{cut}}} |y|^4 P(t_{\text{ref}}, y) dy \leq C_6 M_{\text{sup}} R_{\text{cut}}^{-2} \longrightarrow 0.$$

Together with strong L^1 on a fixed ball, this justifies whole-moment convergence. Strong L^1 alone would not do so. The separate parent-radius estimate has Gaussian exponent of order $q^{-2}/(\kappa C_2 L)$; it is not inferred from the fixed-radius estimate. The canonical velocity integrates to $O(\delta_{\text{ref}})$ during the advance, rather than a constant speed times $O(\log L)$.

M5: protection and the complete future phase tail. After reducing the source and receiver windows to a common normalized $d_{\text{cap}} > 0$, the manuscript takes

$$e := d_{\text{cap}}/2, \quad c := \sigma\delta/2, \quad h := \min\{\delta/4, ce\}.$$

At physical scale j , the normalized observable has constants

$$e_j = e/A_j, \quad h_j = h, \quad c_j = A_j c, \quad L_{\text{obs},j} = A_j L_{\text{obs}}.$$

The outer scale has $A_1 = 1$, so its constants are uniform in the selected depth. The sparse proof defines a positive fixed-viscosity continuity radius r_k by half the supremum of admissible radii. This is legitimate because their property is downward closed and contains a nonempty interval. Choosing the entire freely variable tail smaller than $\min_{i \leq k} r_i$ preserves every earlier selected output. It does not require a viscosity-uniform continuity radius.

M6/B2: attribution and distinctions. The report's attribution is correct: equation (38) of Castro, Faraco, Mengual and Solera has an inner constant c and outer constant -1 . At $r_1 = \rho$, $r_2 = 1$, $\rho^2 = 3/5$ and $c = 2/3$, $W_0 = -\rho^2 \bar{w}$ exactly. Their two-interface spectral prototype is now explicitly credited; the current proof rederives its normalization and uses a different vorticity mollification. The added references also distinguish admissible-velocity nonuniqueness, some- $p > 1$ vorticity flexibility, and forced viscosity selection with perturbed data. None is cited as proving the present main theorem. The relevant profiles and stated scopes were checked against the primary sources, rather than inferred from titles alone.

Additional verification and remaining limits of the assessment

Independent scoped checks covered the interface matrix and finite-sector spectrum, the canonical ancient orbit and nonzero transfer, core and Gaussian-energy estimates, the selected-cutoff cold budget, inactive rows, rough-leak comparison, compactness and fixed passive-colour identification. Their checked calculations and hypotheses did not yield a new blocker. In particular the use of a positive infinite packet does not assume a summed L^∞ bound: its mass, L^p and moments are controlled, while signed velocities use individual colour contractions.

These checks do not establish an unrestricted infinite-angular-mode propagator, arbitrary-horizon cold-tail erasure, arbitrary restart estimates, or control of the actual infinite future after a parent stop. Those broader assertions are not included as proved results. The present review also does not claim a fresh exhaustive verification of every classical external theorem cited elsewhere in the manuscript.

Final disposition. All supplied numbered comments have been accounted for. No theorem weakening or major reconstruction was justified by the checked objections. The missing attribution and remaining quantitative/organizational clarifications were implemented. The non-unique-selection argument is coherent at the scrutinized dependencies; calling the entire manuscript independently certified would exceed this comparison's actual scope.

Sources checked for the added comparison

1. Á. Castro, D. Faraco, F. Mengual and M. Solera, arXiv:2404.15995. <https://arxiv.org/abs/2404.15995>.
2. F. Mengual, arXiv:2304.09578. <https://arxiv.org/abs/2304.09578>.
3. E. Bruè, M. Colombo and A. Kumar, arXiv:2408.07934. <https://arxiv.org/abs/2408.07934>.
4. D. Albritton, M. Colombo and G. Mescolini, arXiv:2507.19257. <https://arxiv.org/abs/2507.19257>.