

Referee report and verified corrections

A fixed-data viscous cascade in the plane

Independent mathematical review

9 October 2026

Version reviewed. The 84-page draft dated 8 October 2026, with source basename `euler-plane-actual-viscous-cascade.tex`. All locations below refer to its original section numbering and its 6,486-line source. The original was preserved. Its SHA-256 is

`d8d5298fdd9e1dfb14d98ee7b572957c3e1a73299cf65530db4e3e9056ba6f9c`

This report concerns that version alone. It does not certify earlier periodic or planar drafts, earlier stationary/nonstationary claims, vortex-wave claims, or other papers in the conversation.

Second-pass revision. This continuation preserves the first-pass section audits and corrections. It adds issues E13–E15, the explicit cap-to-reference comparison and two-scale envelope reset, the history argument for a child activated by an incoming packet, the all-finite-order identification of one ancient orbit, and the initial-trace renormalization justification. The second pass directly reconstructed Sections 2–4 and the reference/history dependencies in Sections 16–19, and checked notation and cross-references throughout. The earlier core/cold-tail and compactness audit records are reused; those calculations were not all repeated in this pass. All replacements below are part of the corrected proof, rather than assertions that the unchanged manuscript already contains them.

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1 Part I — Overall assessment

1.1 Main result and its logical dependencies

The principal claim is Theorem 19.3, on original page 83: for every $2 < p < \infty$ there exists one real, compactly supported, zero-circulation $\omega_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ and $T_f > 0$ such that the unforced planar Navier–Stokes solutions with exactly that initial datum have two distinct accumulation points as $\nu \downarrow 0$ in $C([0, T_f]; L^1 \cap L^p)$. Both limits are renormalized Euler solutions. The datum is fixed before taking the viscosity limit.

This is substantially stronger than isolated unstable-mode activation. The proof must supply an actual nonlinear source, the entire discounted source history, an actual receiving density estimate, protected first crossings, constants independent of the number of stages, and a fixed-datum phase selection. The following dependency list was used in the review.

1. Lemma 1.1 gives the datum class and convergent colour and velocity sums. Lemma 2.2 and Theorem 2.1 give the *finite-sector* smooth instability and isolated activation.
2. Theorem 3.1 and Corollaries 3.2–3.4 give one canonical ancient orbit, its fixed windows, and a nonzero bilateral moment transform. Lemmas 4.1–4.5 give the global adjoint and signed transfer coefficient. Theorem 4.6 concerns prescribed forcing only.
3. Propositions 5.3–5.4, Lemma 6.1 and Propositions 7.1–7.2 supply actual cores, harmonic/leak decompositions, localization and nonlinear cold-tail suppression. Their stopped-density hypotheses must be retained.
4. Sections 8 and 11–13 close the actual inactive prefix using a positive packet. They give magnitude bounds, not a signed active impulse. Sections 9–10 and 12 supply passive-colour recovery, cap topology and strong fixed-data viscosity precompactness.
5. Sections 14–16 close the actual deepest source and its fixed reference reset. Positive future histories additionally require the actual parent/larger stopped environment throughout their following interval.
6. Section 17 constructs a finite expansion using the actual foreign time coefficients. Its density estimate precedes any signed lower bound; its memory and exact-cap gauges remove waveform dependence. Pointwise forcing estimates are separate prerequisites for a crossing slope.
7. Proposition 18.1 closes an actual receiver on a complete prospective interval. Proposition 19.1 makes its input and output constants uniform in depth. Proposition 19.2 and the final sparse selection then produce a single datum and separate two limiting paths.

1.2 Disposition

The local finite-sector calculations, nonlinear cold-tail exponents, strong fixed-data compactness, actual source and receiver estimates, and phase selection have been checked section by section. The individual audit records in the appendices identify their exact inputs; a local conditional verification is not being substituted for the global theorem. The global review checks that those inputs are supplied by the other modules, with the completions in Part III.

The main conclusion can be retained after the explicit corrections and proof completions below. No counterexample to Theorem 19.3 was found, and the review identified no

remaining central obstruction for its stated finite-sector construction. The draft is nevertheless *not correct in every statement as written*. The revisions are mathematical, not just typographical:

- Physical intersection-norm scaling is misstated; a window bound needs $S, U \geq 0$; and compact support does not pass by strong L^1 alone.
- The spectral claim must specify finite sectors. A natural unweighted infinite direct sum gives a genuine counterexample to the broader interpretation.
- Corollaries 11.2–11.3 need a relative Gaussian-error condition or a lower scale restriction on δ . Their unrestricted tiny- δ conclusions are not established by the supplied envelope. Their validity without that restriction remains undetermined, not disproved.
- The Gaussian energy proof needs a rigorous entry into its unbounded weighted space. The receiver history must explicitly be extended through the density-stopped prospective interval, with noncircular stop constants.

The second pass additionally corrects the exterior multipole domain, makes the fixed-reference advance independent of the terminal expansion threshold, and supplies the prepared-child discounted history without reusing the deepest source’s seed-clock identity. The actual construction uses $\delta = (\log(1/q))^{-4}$ and finite sectors; it satisfies the corrected intermediate hypotheses. Thus these repairs do not weaken the initial-data class or conclusion of Theorem 19.3.

1.3 External inputs checked

The planar NS existence input is Gallay–Wayne [1], Theorem 2.2: global uniqueness for integrable vorticity, the $C_t L^1$ initial trace and positive-time bounds. Their Lemma 2.1 has precisely the HLS, supremum and gradient estimates invoked. The positive-viscosity rescaling is admissible; the constructed datum lies in $L^1 \cap L^p$, and zero circulation and compact support give finite kinetic energy. Scalar contractions, energy bounds and fixed-viscosity continuous dependence are justified by smooth approximation and the mild/energy inequalities used in the draft.

Ciampa–Crippa–Spirito [2], Theorem 3.2, supports strong convergence *along an identified renormalized Euler limit*; it does not assert uniqueness of all viscosity accumulation points. The draft does not use it to infer such uniqueness, and Section 12 supplies its own compactness and renormalization argument. The primary statements and their applicability were checked; this report does not claim to re-referee the entire proofs or every theorem in those two external articles.

2 Part II — Detailed error report

Each issue below distinguishes a false statement, an incomplete proof, and an intended statement that needs a domain or scope clarification. The detailed local audit records are retained in the appendices.

2.1 E1: Physical density scaling

Location and severity. Remark 2.4, original p.10, lines 729–734. Repairable mathematical error; false in the intersection norm.

“the density bound is $CAR^2\delta_q$ ”; “These normalized constants do not depend on the scale or amplitude.”

The theorem uses $X = L^1 \cap L^p$ with sum norm. The L^p factor is $AR^{2/p}$, not AR^2 . At time zero a fixed nonzero seed gives $A\alpha_q R^{2/p} \|\Phi\|_p$; its ratio to $AR^2\delta_q$ diverges as $R \downarrow 0$, at fixed q . The corrected sum-norm bound is $CA(R^2 + R^{2/p})\delta_q$. Part III gives the exact change of variables. The normalized theorem and subsequent correctly scaled stack estimates are unaffected.

2.2 E2: Ambient spectral space

Location and severity. Lemma 2.2, original pp.4–6, lines 234–256. Repairable scope/definition error.

“in the quarter-turn symmetry class”; “all other positive-real-part spectral values”; later, “For each finite set of sectors ... work on the direct sum of the spaces $H^k(I)$ ”.

A spectrum requires a specified space and operator domain. The proof establishes modal exclusion and each finite direct-sum propagator. It does not establish a full unspecified infinite-angular theorem. The unweighted $\bigoplus_{m \in 4\mathbb{Z}} H^1(I)$ realization actually has additional right-half-plane spectrum, as proved in Part III. State the lemma sector by sector and on fixed finite sums, with constants depending on the finite sum. All coefficient constructions use only those spaces; no main conclusion is weakened.

2.3 E3: Domain of the source-window estimate

Location and severity. Lemma 4.5, original p.19, lines 1504–1510. Minor missing hypothesis with a false literal reading.

“Truncating the source interval to $[-S, U]$ changes its transform by at most $Ce^{-(\sigma - \Re\mu)S} + C_\beta(1 + U)^4 e^{-\Re\mu U}$.”

Specify $S, U \geq 0$. At $U = -1$ the second term vanishes, although the discarded future integral generally does not. Part III supplies an actual canonical-source counterexample and proves the corrected positive-window estimate. Every later construction window is positive. The proof also says “two strict one-quarter bounds” although it has separate truncation, replacement and old-seed budgets. Name those three budgets and apply the triangle inequality explicitly.

2.4 E4: Unbounded Gaussian-energy tests

Location and severity. Proposition 7.1, original pp.30–33, particularly lines 2329–2391 and 2493–2497. Repairable proof-completion gap.

“All energy tests can first be made with truncated weights ... the uniform weighted estimates justify removal of the truncations”.

The exact oscillator identity for $e^{\beta|y|^2}$ cannot be assumed for arbitrary truncated weights before Gaussian membership is known. Part III gives bounded concave exponential weights, their exact energy identity, an independent finite-interval membership estimate, and then the passage to the full identities. This repairs the justification; the exponents 7/50, 102/25, 643/100 and 646/100 remain unchanged.

2.5 E5: The normalization requests an extra time unit

Location and severity. Section 7, physical normalization following Proposition 7.2. Minor repairable domain mismatch.

The normalized terminal parameter is “ $S = 2 + \nu T/r^2$ ”, whereas Proposition 7.1 assumes its drift through $s = S - 1$.

The physical field is initially given only through T . Extend the cutoff comparison field by zero for one more normalized unit and continue its comparison solution. Part III verifies all hypotheses persist. No actual solution on $[0, T]$ or exponent changes. The initial H^{-1} comparison trace also merits the explicit low/high Fourier argument given there.

2.6 E6: Absolute Gaussian errors versus arbitrary tiny caps

Location and severity. Corollaries 11.2 and 11.3, original pp.45–46, lines 3434–3439, 3492–3503 and 3514–3523. Material repairable parameter/proof gap; not a counterexample to the actual PDE conclusion.

“ δ small enough”; “ $x_i(t) \leq B + w_i[CC_*\delta + o(q^K)] < \delta/4$ ”; “its error envelope is much smaller even than the prescribed seed b ”.

Superpolynomial smallness in q does not imply smallness relative to an arbitrarily subsequently chosen $\delta(q)$, e.g. $e^{-q^{-10}}$. The retained Gaussian envelope is independent of δ . Require $\delta \geq q^{L_\delta}$ for a fixed finite L_δ , or a direct relative-error condition. Part III proves the repaired closure. The actual choice $\delta = (\log(1/q))^{-4}$ satisfies it. A theorem uniform over all smaller caps would need an additional seed/forcing-relative estimate, which is not supplied here and remains unresolved.

2.7 E7: Passive convergence imports Kato for abstract approximants

Location and severity. Proposition 9.1, original pp.38–39, lines 2936–2947 and 2987–2993. Repairable hypothesis/proof mismatch.

“with the L^1 , L^2 , and L^∞ contraction bounds and the weak-form equation”; “The Kato inequality ... also give[s] uniform tightness”.

Those abstract assumptions do not explicitly supply a local Kato inequality for arbitrary approximating transported colours. Specify divergence-free weak equations and strongly continuous representatives if asserting convergence in $C_t X$. Part III gives a Kato-free L^1 upgrade using conservation and finite propagation of the *limit*. The actual NS colours already satisfy all needed regularity and Kato properties; the main separation argument is unchanged.

2.8 E8: Compact support and the radial comparison error

Location and severity. Proposition 10.4, line 3185, and Proposition 11.1 proof, line 3295. Minor explanatory mathematical misstatements.

“Zero circulation and compact support pass by strong L^1 convergence.”

Only the circulation claim follows from that topology. Translated zero-mean bumps with summable amplitudes give compact partial sums converging strongly to a noncompact function. Use membership in the assumed compact phase family, or the actual common support ball.

“If every initial colour is radial, $\zeta_i = h_i = 0$ solves this system.”

For arbitrary radial colours only $h_i = 0$ is automatic; in general $\zeta_i = e^{\nu t \Delta}(w_i(0) - G_i(0))$ is a nonzero radial difference. Require equality to the radial reference for $\zeta_i = 0$. Within the actual zero-seed family that equality holds. Neither correction changes the row estimate or the main datum.

2.9 E9: Observable and collar geometry must be fixed

Location and severity. Corollary 10.2, lines 3095–3102, and Section 11, lines 3236–3237, 3409–3412. Minor but necessary scope specifications.

“If their limiting first-cap phases are different, the two total Euler paths are different”;
 “a fixed broad annulus ... containing both derivative collars”.

The two subsequences must use the *same* initial colours, tests, coefficients and cap level; a_u and $-a_u$ on one path otherwise give opposite phases. The broad annulus must contain the derivative support with a fixed positive distance to its complement for an $e^{-cR_i^2/(\nu t)}$ heat-gradient tail. Part III states these requirements. The intended construction permits both choices before q is chosen.

2.10 E10: Full state versus heat-base perturbation

Location and severity. Theorem 16.2, original p.59, lines 4462–4472. Minor comparison ambiguity.

“the matching exact-cap state ... [has] the same complex coordinate ... as $f_j(t_j)$... the two states differ in X by ... $+C\kappa t_j$ ”.

Write the compared states explicitly: the full actual $\tilde{w}_j(t_j)$ and full canonical state. Comparing f_j itself with full Ω loses the order-one radial base. The proof correctly contains the unseeded heat-base error. Part III gives both equivalent perturbation/full-state inequalities. No new estimate is needed.

2.11 E11: Whole source history through the prospective stop

Location and severity. Proposition 18.1 proof, original pp.76–79, lines 5879–5895, 6036–6048; feeding Proposition 19.1. Material repairable proof-scope/constant clarification.

“The actual postcap moment lemma gives, before the parent receiver cap”; later, “on the full prospective interval reached below”.

The prospective endpoint $H_+ = t_0 + S_q + L_0$ lies beyond the anticipated cap. A pre-cap-only bound cannot control its whole forcing history. The operative earlier moment lemma, however, assumes density/core control, not a coordinate-cap restriction. Apply it to $\min(H_+, T_{\text{stop}})$, with current stop $C_{\text{stop}}\delta$. The full corrected argument, including an input majorant independent of the subsequently selected fixed stop constant, is proved in Part III. It excludes a density stop before H_+ using only an *upper* input bound, and obtains the signed lower transform bound afterwards. This completion closes the dependency rather than adding an unproved assumption.

2.12 E12: Clock, cap-window and parameter scope

Location and severity. Proposition 18.1, lines 5825–5828, 6132–6134, 6149–6150; Lemma 17.2, lines 5372–5388. Minor or repairable scope/notation errors.

“throughout the level interval $[3\delta/4, 5\delta/4]$ ”; “Its modulus derivative is $1 + O(a) > 0$ ”;
 “the lower clock bound persists”.

Restrict the slope and unique crossing to the protected late time window; the proof explicitly does not rule out later returns. The derivative $1 + O(a)$ is with respect to amplitude, while the time derivative is $\sigma a[1 + O(a)]$. Choose the common lower clock $0 < c_0 < \min(K/\sigma, 6/\sigma)$. Define $L = \log(1/q)$ and fix $\chi > 0$ in the scale-family comparison. These changes make the actual quantifiers match the estimates; the main first-cap conclusion is unchanged.

2.13 E13: The exterior multipole series has an incorrect spatial domain

Location and severity. Lemma 4.2, original p.17, lines 1344–1349. Repairable mathematical error; the displayed statement is false on the full complement of the support.

“At points outside its support, its leading complex velocity is” followed by the inverse-fifth, inverse-ninth, and higher-power series.

The geometric expansion requires $|z| > |y|$ throughout the source support. The complement of an annular support also contains its inner hole, where a different harmonic expansion applies. For $w(r, \theta) = h(r) \cos(4\theta)$ with $0 \neq h \in C_c^\infty((1, 2))$, $h \geq 0$, all the original source hypotheses hold. The actual interior velocity is a nonzero cubic polynomial, whereas the asserted series reduces to a singular inverse fifth power. Part III verifies this counterexample and the corrected exterior series, including its remainder estimate. Replace the domain by $|z| > \sup\{|y| : y \in \text{supp } w\}$, or by $|z| > R$ with a stated support ball B_R . The source-to-receiver use is in this exterior region, so its forcing coefficient and the main conclusion are unchanged.

2.14 E14: Reference advance and prepared-child history need explicit scope

Location and severity. Corollaries 16.4 and 16.8, original pp.63–67, lines 4863–4869 and 5029–5040; feeding Sections 18–19. Repairable proof completion and parameter-order clarification.

“after choosing the reference amplitude smaller than the terminal cap threshold if necessary”; “the actual source norm envelope”; “ $\alpha e^{\sigma t_{\text{ref}}} = a_{\text{ref}}(1 + o(1))$ ”.

The global order fixes the reference amplitude before the later finite expansion degree. The finite-expansion envelope is established only through its own terminal threshold and cannot, by itself, justify a later fixed reference above that threshold. The one-sided alternative in Corollary 16.4 does work: the independent older-prefix field and descendant scalar bounds give an $O(q)$ comparison force on the canonical support. Part III supplies the complete advance, its weighted-history error, and both fixed-radius and direct parent-radius positive-envelope estimates. Delete the unnecessary threshold clause and use that comparison after the shrinking cap.

The displayed seed-clock identity in Corollary 16.8 is correct for the diffusion-selected deepest source. A child activated by an incoming packet has a different growing coefficient; that identity is not an available input for its history. For such a child the needed pre-cap discounted bound is instead $O(C_{\text{past}} \delta^{1-1/\beta})$, followed by the weighted cap-to-reference comparison and the stopped

future-moment estimate. Part III gives this generalized argument with all trial-constant quantifiers. This completes the relevant inductive use, without changing the main datum or conclusion. The actual history is zero-extended after its controlled prospective endpoint; no bound on its genuine uncontrolled infinite future is inferred.

2.15 E15: Missing parameter and moment domains in standalone statements

Location and severity. Opening conventions, lines 50–55; Lemma 4.2, lines 1357–1363; Section 7 budget, lines 2723–2750; Lemma 8.1 and Corollary 8.2, lines 2787–2805 and 2869–2885; Lemma 10.1, lines 3047–3052. Minor scope/definition corrections, with a false unrestricted interpretation of the tail-budget asymptotic.

“For one fixed sufficiently small smoothing length” immediately before an $\epsilon \downarrow 0$ limit;
 “both terms ... $o(q^6)$ ”; “ $\tau(a) = \inf\{t \geq 0 : |a(t)| \geq \delta\}$ ”.

Separate the smoothing limit from the subsequent fixed choice. Define B_m for each used degree and require its corresponding absolute moment, rather than invoking fourth absolute integrability for B_8 . Make the logarithmic horizon govern the entire geometric-budget inference: with fixed β, C_H , $0 < \delta \leq 1$ and $1 \leq H \leq C_H \log(1/q)$ its two terms are indeed $o(q^6)$. Without such a horizon bound the second displayed term can fail: for fixed β , $\delta = 1$ and $H = q^{-1/4}$,

$$q^{323/50} H^{102/25} = q^{136/25},$$

which is not $o(q^6)$. This is a counterexample to the unrestricted asymptotic inference, not to the underlying PDE estimate. Declare $b \geq 0$, $S \geq 0$, $0 < q < 1$ in Lemma 8.1; declare a fixed $h_0 > 0$ and bounded δ for the uniform baseline asymptotic. Finally take the first-cap infimum over $[0, T]$, where its path is defined, and include $T > 0$ among the parameters. Part V gives exact replacement text. These edits preserve the regime of the actual construction and have no adverse downstream effect.

3 Part III — Corrected mathematical arguments

The following are actual corrections, not proposed lemmas with their proofs deferred. Original unchanged inputs are identified explicitly. Notation is $K(x) = x^\perp / (2\pi|x|^2)$, $X = L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ with sum norm, $p > 2$, $B_4(f) = \int \bar{z}^4 f(x) dx$, $\lambda = \sigma + i\tau$ with $\sigma > 0$, and $\mathcal{T}_\theta f(r, \varphi) = f(r, \varphi + \theta/4)$. The profile W is the fixed smooth compact radial zero-mean profile of Section 2; all references to its spectral operator here are finite-sector.

Correct physical norm scaling for Remark smooth-active-scope

Let $F(t, x) = \omega^{q, \vartheta}(t, x) - e^{\kappa_q t \Delta} W_\epsilon(x)$, and define the physical vorticity and radial heat base by

$$\omega_{A,R}(t, x) = A\omega^{q, \vartheta}(At, x/R), \quad W_{A,R}^\nu(t, x) = A(e^{\kappa_q At \Delta} W_\epsilon)(x/R), \quad \nu = \kappa_q AR^2.$$

Homogeneity of Biot–Savart gives

$$(K * \omega_{A,R})(t, x) = AR(K * \omega^{q, \vartheta})(At, x/R).$$

Consequently $\omega_{A,R}$ solves Navier–Stokes with viscosity ν . The interval $[0, H_q]$ becomes $[0, H_q/A]$. For every $1 \leq r < \infty$, change of variables gives the exact identity

$$\|\omega_{A,R}(t) - W_{A,R}^\nu(t)\|_r = AR^{2/r} \|F(At)\|_r.$$

Therefore the density conclusions of the isolated theorem become

$$\begin{aligned} \sup_{0 \leq t \leq H_q/A} \|\omega_{A,R}(t) - W_{A,R}^\nu(t)\|_1 &\leq CAR^2\delta_q, \\ \sup_{0 \leq t \leq H_q/A} \|\omega_{A,R}(t) - W_{A,R}^\nu(t)\|_p &\leq CAR^{2/p}\delta_q, \\ \sup_{0 \leq t \leq H_q/A} \|\omega_{A,R}(t) - W_{A,R}^\nu(t)\|_{L^1 \cap L^p} &\leq CA(R^2 + R^{2/p})\delta_q. \end{aligned}$$

Here the intersection norm is the sum norm, as in the theorem. All constants are independent of A, R . The $CAR^2\delta_q$ bound is valid for L^1 , but not in general for the intersection norm.

To see that the distinction is mathematically necessary, fix q and ϑ . The seed Φ_ϑ is nonzero: otherwise f_ϵ would be a scalar multiple of $\bar{f}_\epsilon$, which is impossible because they have distinct eigenvalues $\sigma_\epsilon + i\tau_\epsilon$ and $-\sigma_\epsilon + i\tau_\epsilon$. At time zero,

$$\|\omega_{A,R}(0) - W_{A,R}^\nu(0)\|_p = A\alpha_q R^{2/p} \|\Phi_\vartheta\|_p.$$

Its ratio to $AR^2\delta_q$ is $\alpha_q \|\Phi_\vartheta\|_p \delta_q^{-1} R^{2/p-2}$, which diverges as $R \downarrow 0$ because $p > 2$.

The other stated scalings are unchanged. Whole fourth moments satisfy

$$B_4(\omega_{A,R}(H_q/A)) = AR^6 B_4(\omega^{q,\vartheta}(H_q)),$$

and the integrated perturbation velocity satisfies

$$\int_0^{H_q/A} \|K * (\omega_{A,R}(t) - W_{A,R}^\nu(t))\|_\infty dt = R \int_0^{H_q} \|K * F(s)\|_\infty ds \leq CR\delta_q.$$

Precise finite-sector formulation of the spectral input

For each positive sector $m \in 4\mathbb{N}$ and each integer $k \geq 0$, realize $\mathcal{L}_m$ as the bounded operator on $H^k(I)$ with the Green potential obtained from zero extension from I . There is a fixed sufficiently small smoothing length ϵ and a fixed $0 < \eta < \sigma_\epsilon/4$ such that all spectral values of these individual operators with real part greater than η consist of the algebraically simple $m = 4$ eigenvalue λ_ϵ . For negative sectors take the complex-conjugate operators; the only additional spectral value with real part greater than η is $\bar{\lambda}_\epsilon$ in sector $m = -4$. The radial sector has zero generator.

For a fixed finite set $S \subset 4\mathbb{Z}$, on $X_{k,S} = \bigoplus_{m \in S} H^k(I)$, the bounded direct-sum generator has the decomposition

$$e^{t\mathcal{L}_S} = e^{\lambda_\epsilon t} P_+ + e^{\bar{\lambda}_\epsilon t} P_- + \mathcal{R}_{k,S}(t), \quad \|\mathcal{R}_{k,S}(t)\|_{X_{k,S} \rightarrow X_{k,S}} \leq C_{k,S} e^{2\eta t}, \quad t \geq 0.$$

Each growing projection is omitted if its sector is absent. This formulation is sufficient for every coefficient in the finite seed–viscosity expansion. It makes no assertion about the spectrum of an unspecified infinite-angular realization.

Why a full unweighted direct sum would not be equivalent. Let $Y = \bigoplus_{m \in 4\mathbb{Z}} H^1(I)$ and realize the modal operators with their maximal direct-sum domain. Fix a positive real resolvent parameter Λ greater than all modal real spectral bounds. Write $\mathcal{L}_m = A_m + D_m$ on $L^2(I)$, where $A_m = -im\Omega_\epsilon$. For fixed ϵ , the coupling kernel is $-\frac{i}{2}W'_\epsilon(r)h_m(r,s)s/r$. Since I is bounded away from zero,

$$h_m(r,s) \leq e^{-m|r-s|/\sup I}, \quad \|D_m\|_{L^2(I) \rightarrow L^2(I)} \leq C_\epsilon/m.$$

The latter estimate follows by Schur's test. The multiplication resolvent has norm at most $1/\Lambda$, so a Neumann series gives

$$\|(\Lambda - \mathcal{L}_m)^{-1} - (\Lambda - A_m)^{-1}\|_{L^2 \rightarrow L^2} \leq C_\epsilon/m$$

for all sufficiently large positive $m \in 4\mathbb{N}$. Dominated convergence therefore yields

$$(\Lambda - \mathcal{L}_m)^{-1} \mathbf{1} \longrightarrow \Lambda^{-1} \mathbf{1}_{\{\Omega_\epsilon=0\}} \quad \text{in } L^2(I).$$

This limit has a jump at the actual outer support radius of W_ϵ . Indeed if $a < 1$ is the support radius of χ , then χ is positive for radii below a by radial monotonicity. Immediately inside $r_e = 1 + \epsilon a$, W_ϵ is positive and, by zero total circulation,

$$\int_0^r s W_\epsilon(s) ds = - \int_r^{r_e} s W_\epsilon(s) ds < 0.$$

For $r \geq r_e$ this integral is zero. Thus the limit above does not belong to $H^1(I)$. Uniform boundedness of the modal resolvents in $H^1(I)$ would give a weakly convergent $H^1(I)$ subsequence on this fixed input, contradicting its $L^2(I)$ limit. A bounded resolvent on Y must bound each modal resolvent, so Λ is in the spectrum of the unweighted full direct-sum realization. This proves that the finite-sector formulation cannot be silently promoted to that infinite-sector space.

A complete weighted-tail integration underlying smooth-active-fourth-tail

This supplies details rather than changing the stated estimate. Let $d_s = \sqrt{1 + s^2}$ and choose R with $g_0 = d_R - d_0 > 0$. For sufficiently small q , the drift bound and $\kappa_q H_q \rightarrow 0$ ensure

$$B_{H_q} + 2\kappa_q H_q \leq g_0/2.$$

For $s \geq R$ and $0 < t \leq H_q$, therefore,

$$d_s - d_0 - B_{H_q} - 2\kappa_q t \geq g_0/2 + d_s - d_R \geq g_0/2 + c_R(s - R), \quad c_R = R/d_R > 0.$$

The last inequality follows from $d'_s = s/d_s \geq R/d_R$. If $M_t(s) = \int_{|x|>s} |\omega^{q,\vartheta}(t, x)| dx$, the optimized exponential test gives

$$M_t(s) \leq m \exp \left(- \frac{(g_0/2 + c_R(s - R))^2}{4\kappa_q t} \right) \leq m e^{-g_0^2/(16\kappa_q H_q)} e^{-c_R^2(s-R)^2/(4\kappa_q H_q)}.$$

Layer-cake integration, initially with finite radial truncation and then by monotone convergence, yields

$$\int_{|x|>R} |x|^4 |\omega^{q,\vartheta}(t, x)| dx = R^4 M_t(R) + 4 \int_R^\infty s^3 M_t(s) ds.$$

For $\kappa_q H_q \leq 1$, $\int_R^\infty s^3 e^{-c_R^2(s-R)^2/(4\kappa_q H_q)} ds$ is bounded by a constant depending only on R ; the initial mass m is uniformly bounded in q, ϑ . Thus

$$\sup_{0 < t \leq H_q} \int_{|x|>R} |x|^4 |\omega^{q,\vartheta}(t, x)| dx \leq C \exp \left(- \frac{g_0^2}{16\kappa_q H_q} \right).$$

At $t = 0$ the integral is zero by compact support. This proves the full supremum estimate and the required finiteness of the whole fourth absolute moment.

3.1 Gaussian membership, initial comparison trace and normalization

Justification of the Gaussian energy space. We give a bounded-weight argument before using the exact unbounded-weight identities. Put $\tau_0 = \log(1 + s_0)$ and, for $R \geq 1$, set

$$\Phi_R(t) = \frac{R^2 t}{R^2 + t}, \quad \theta_R(t) = \Phi'_R(t) = \frac{R^4}{(R^2 + t)^2}, \quad \rho_R(y) = e^{\beta \Phi_R(|y|^2)}, \quad g_R = \rho_R^{1/2} w.$$

These are bounded smooth weights, $0 \leq \theta_R \leq 1$, and $\rho_R \uparrow e^{\beta|y|^2}$ as $R \rightarrow \infty$. The already proved short-time exponential moment and Nash bound imply $\int \rho_R w(\tau_0)^2 \leq C m^2$, uniformly in R . For bounded weights, integration by parts gives exactly

$$\langle w, \mathcal{L}w \rangle_{\rho_R} = -\|\nabla g_R\|_2^2 + \frac{1}{2}\|g_R\|_2^2 - \beta \int |y|^2 \theta_R(|y|^2) \left(\frac{1}{2} - \beta \theta_R(|y|^2)\right) g_R^2 dy.$$

This identity follows by substituting $w = \rho_R^{-1/2} g_R$; the second derivatives of $\log \rho_R$ cancel. In particular, if $c_\beta = \beta(1/2 - \beta) > 0$, its last term is at most $-c_\beta \int |y|^2 \theta_R g_R^2$.

The divergence-free advection contributes

$$\beta \int (K * w + E) \cdot y \theta_R g_R^2 dy.$$

For $\tau \geq \tau_0$, the Nash estimate and the similarity change give $\|K * w\|_\infty \leq C_{s_0} m$. The foreign term is at most $\beta \lambda_0 \int |y|^2 \theta_R g_R^2$. Moreover $\theta_R \leq 1$ and Young's inequality give

$$\beta \|K * w\|_\infty \int |y| \theta_R g_R^2 \leq \frac{c_\beta}{4} \int |y|^2 \theta_R g_R^2 + C_\beta \|K * w\|_\infty^2 \|g_R\|_2^2.$$

Choose λ_0 so that $\beta \lambda_0 \leq c_\beta/4$. It follows that, with a constant independent of R ,

$$\frac{1}{2} \frac{d}{d\tau} \|g_R\|_2^2 \leq -\|\nabla g_R\|_2^2 - \frac{c_\beta}{2} \int |y|^2 \theta_R g_R^2 + C(1 + m^2) \|g_R\|_2^2.$$

Gronwall gives a uniform bound on every fixed finite time interval. Monotone convergence and Fatou's lemma show that the full Gaussian norm F_β is finite there and that its transformed gradient and $|y|$ -weighted norm are locally square integrable in time. Cutoffs in the now integrable full weight justify the exact identities and nonlinear energy inequalities stated above. The sharper continuity bootstrap then gives the advertised bounds with constants independent of S apart from the displayed powers. This preliminary membership argument does not assume that the bounded weights have the same oscillator spectral gap. It uses no initial L^∞ norm.

Continuity at the initial endpoint in the comparison. Both solutions converge to their common datum in $L^1 \cap L^p$ as $t \downarrow 0$, and hence their difference δ converges to zero in L^2 , because $p > 2$. The moment estimates give a uniformly bounded first absolute moment near zero. Since $\int \delta = 0$, its Fourier transform satisfies

$$|\widehat{\delta}(t, \xi)| \leq |\xi| \int |x| |\delta(t, x)| dx.$$

Consequently the part of $\|K * \delta\|_2^2 = C \int |\xi|^{-2} |\widehat{\delta}|^2 d\xi$ inside a Fourier ball of radius a is $O(a^2)$ uniformly near zero, whereas the part outside that ball is bounded by $C a^{-2} \|\delta(t)\|_2^2$ and tends to zero at fixed a . First sending $t \downarrow 0$ and then $a \downarrow 0$ proves $\|v(t)\|_2 \rightarrow 0$. Together with the integrable initial coefficient obtained from $g_0 \in L^p$, this starts the energy inequality from exactly $v(0) = 0$; no positive-time initial contribution is dropped.

Harmless continuation of the cutoff comparison field. The physical hypotheses were given on $[0, T]$. To apply Proposition 7.1 with $S = 2 + \nu T/r^2$, extend $\tilde{b}$ by zero on $(T, T + r^2/\nu]$ and continue its solution g there. All divergence, covariance, cubic-growth, and weak linear-growth bounds persist; the smallness condition $\ell(r^2/\nu + T) \ll 1$ controls the normalized bound on the original interval, and the extended field is zero thereafter. The parabolic-limit formulation allows this time discontinuity (or it can be replaced by a bounded smooth time cutoff after T). The solution and comparison on $[0, T]$ are unchanged. Thus the normalized drift is available through $s = S - 1$ exactly as required by the Gaussian proposition.

Initial fourth moment at a restart. The estimate is applied to the original initial inner stack, for which the absolute convergence of $\sum_l A_l R_l^6$ and the individual seed identities give $B_4(f_0) = 0$. An application starting from a later state must retain its actual initial fourth moment. If only the condition $B_4(f_0) = 0$ is dropped, and every other hypothesis of the Gaussian proposition (including support in B_1) is retained, its proof gives

$$\sup_{0 \leq s \leq S-1} |B_4(f(s))| \leq |B_4(f_0)| + C \left[m^2 S^{7/50} (1 + \log S) + m d S^{102/25} \right],$$

with the same constants. A later state with noncompact support needs its short-time weighted norm and moments controlled separately before this proof can be reused. Fourfold symmetry alone does not set the initial fourth-moment term to zero.

3.2 Relative Gaussian errors in the earlier inactive closures

Original labels: `cor:inactive-density-closure` and `cor:inactive-static-packet`; original lines 3434–3439, 3492–3503, 3514–3523, and 3543.

Replacement parameter specification. Take $S = C_2 \log(1/q)$, with C_2 sufficiently large and fixed. Choose a fixed finite exponent $L_\delta > 0$ and choose $\delta(q)$ so that

$$q^{L_\delta} \leq \delta(q) \leq 1, \quad CS(\sqrt{\delta(q)} + \beta q) \leq c.$$

All constants and exponents are fixed before q is made sufficiently small. The stated conclusions hold with this restriction. In particular one may take $\delta(q) = \log(1/q)^{-4}$.

Replacement justification for the error contribution. Put

$$H_G(q) = C\eta(q)S e^{\lambda_P S}, \quad \eta(q) = \sup_{m \geq 1} \frac{e_m}{(\beta D q^6)^m}.$$

The four-term weighted Gaussian summation proves $H_G(q) = o(q^L)$ for every fixed $L > 0$, uniformly in the allowed depth and viscosities. Choose $L > L_\delta + 6$. Then

$$\frac{H_G(q)}{\delta(q)q^6} \leq \frac{o(q^L)}{q^{L_\delta+6}} \longrightarrow 0.$$

Hence the parabolic error is $o(\delta q^6)$ in the forced weighted norm. This gives the strict stopping improvement, and the stated inward-field remainder, after the principal initial and packet contributions are estimated as in the original proof.

If the stronger sentence that this envelope is smaller than the prescribed seed $b = \delta e^{-\sigma S}$ is retained, choose instead $L > L_\delta + C_2 \sigma$. Since $b = \delta q^{C_2 \sigma}$, the same argument gives $H_G = o(b)$.

Equivalently, in place of a polynomial lower restriction one may assume directly the sufficient relative-error condition

$$\eta(q)Se^{\lambda_P S} = o(\delta(q)q^6).$$

If comparing to the seed as well, impose the corresponding relative condition with $\delta e^{-\sigma S}$.

The original absolute Gaussian envelope alone does not establish these claims for arbitrary subsequently chosen positive δ . This is a restriction needed by that proof envelope, not a counterexample to the actual angular comparison. In particular the exact all-radial evolution has no angular forcing. A theorem uniform over arbitrarily smaller δ would require a refined error estimate proportional to the actual initial and forced angular perturbations.

3.3 Passive-colour convergence without an unstated Kato input

Original label: `ph:colour`; original lines 2936–2947 and 2987–3008.

The actual Navier–Stokes application has divergence-free drifts and standard strongly continuous mild colour paths. To keep the abstract statement equally precise, explicitly specify that the weak scalar equation is in divergence form with divergence-free drifts. If the conclusion is written as convergence in $C_t(L^1 \cap L^2)$, require the approximating colour paths to have strongly continuous representatives; this is automatic for the actual positive-viscosity colours, and for renormalized transported colours with bounded Sobolev drifts. With only weak representatives, the conclusion below is uniform-in-time strong convergence.

The following passage may replace the invocation of an approximating Kato inequality in the convergence proof.

The contraction bounds and weak time equicontinuity provide a limit transported colour with the fixed weak initial trace. Its uniqueness, renormalization, and conservation of the L^2 norm follow from the bounded Sobolev limiting drift. The bad-time argument therefore proves

$$\sup_{t \leq T} \|f_n(t) - f(t)\|_2 \longrightarrow 0.$$

Renormalization by smooth approximations to $|s|$ also gives $\|f(t)\|_1 = \|f_0\|_1 =: m$ at every time. Bounded limiting velocity gives finite propagation of this fixed compactly supported initial colour, so a fixed ball B_R contains its support for every $t \leq T$. For each time, contraction and conservation yield

$$\int_{B_R^c} |f_n(t)| \leq m - \int_{B_R} |f_n(t)| \leq \int_{B_R} |f_n(t) - f(t)|.$$

Consequently

$$\sup_{t \leq T} \|f_n(t) - f(t)\|_1 \leq 2|B_R|^{1/2} \sup_{t \leq T} \|f_n(t) - f(t)\|_2 \longrightarrow 0.$$

This proves global uniform strong convergence without assuming a local Kato inequality for the approximating transport solutions. For the actual smooth positive-time NS drifts, the original Kato tightness bound is also available.

For completeness, renormalized endpoint traces can be obtained exactly as in the compactness theorem: spatially mollify the weakly continuous path, integrate its absolutely continuous renormalized equation between arbitrary fixed times, and pass the spatial approximate identities at both endpoints. Expanding cutoffs then prove conservation at every time, including the initial trace.

3.4 Compact support in the sparse-phase construction

Original label: `ph:sparse`; original lines 3185–3186.

Replacement sentence. Zero circulation passes by strong L^1 convergence. Compact support follows from the identification of g_∞ with the resulting phase sequence in the assumed compact-data family. In the scaled stack used here, every phase choice has support in one common fixed ball, which gives the same conclusion directly.

Compact support does not pass under strong L^1 convergence alone. For example if $g \in C_c^\infty(B_1)$ is nonzero with zero integral, the compactly supported partial sums

$$g_k(x) = \sum_{m=1}^k 2^{-m} g(x - 4me_1)$$

converge strongly in $L^1 \cap L^p$ to a noncompactly supported function. This correction does not change the sparse conclusion for the actual common-support family.

3.5 Radial data and the reference comparison error

Original label: `prop:inactive-actual-row`; original line 3295.

Replacement passage. If every initial colour equals its radial reference $G_l(0)$, then $\zeta_i = h_i = 0$ solves the comparison system with the prescribed initial data. More generally, if every initial colour is radial, then $h_i = 0$ and

$$\zeta_i(t) = e^{\nu t \Delta} (w_i(0) - G_i(0))$$

is radial. Thus there is no angular source in either case. Within the actual seeded reference family, the all-radial zero-seed case is the first of these cases.

This distinguishes vanishing nonradial colour from vanishing error against a particular fixed radial reference. It does not alter the inactive-row estimate.

3.6 One common fixed observable in separation and degree

Original label: `ph:separation`; original lines 3092–3097.

Replacement hypothesis. Consider one fixed total initial datum and fixed bounded initial colours. Use the same initial colours $f_{l,0}$, coefficients c_l , radial cutoffs χ_l , and cap level δ on both subsequences, thus evaluating one fixed functional a_u of the total path. Suppose its approximating observables have uniformly protected first caps. If the two limiting first-cap phases of this common functional differ, then the total Euler paths differ. The cap times may differ.

The proof in the original text applies verbatim. Comparing different functionals would not justify the conclusion: replacing all coefficients c_l by $-c_l$ gives $-a_u$ on the same total path, with the same cap time and protection constants but the opposite phase.

Degree clarification, original lines 3114–3122. At $r = 0$, specify that the full coloured initial datum is the spatial rotation through $-\theta/4$ of the datum at $\theta = 0$. The other colours and all background profiles are radial, so this follows in the intended seeded family. Then rotational covariance of the full scalar evolution gives $a_{0,\theta} = e^{i\theta} a_{0,0}$ and the degree-one argument is unchanged.

3.7 Explicit margin around the derivative collars

Original geometry and proof: original lines 3236–3237 and 3409–3412.

Replacement geometry specification. Choose a fixed dimensionless broad annulus $\mathcal{A}$ and $d_* > 0$ such that

$$\text{dist}(\text{supp } \nabla W, \mathbb{R}^2 \setminus \mathcal{A}) \geq d_*.$$

Set $\mathcal{A}_i = R_i \mathcal{A}$. Its radii are fixed multiples of R_i . Choose q afterwards so that the enlarged annulus lies in every older constant-core probe and is separated from the fixed small receiver truncation ball.

For $\tau = \nu t / R_i^2 \ll 1$, Gaussian convolution of the reference gradient then gives the claimed scaled L^1 and L^p tail estimate

$$C \exp(-cd_*^2/\tau),$$

including the weight $|x| + R_i$ after fixed dimensional scaling. Polynomial factors are absorbed by decreasing c . Mere containment of the derivative support without a positive margin would not justify this Gaussian exponent. Enlarging the annulus by a fixed amount supplies the needed margin and changes no cascade powers or parameter order.

3.8 Receiver parameter and protected-window declarations

In the scale-family conclusions write $q \in (0, 1)$, $L = \log(1/q)$, and fix a desired exponent $\chi > 0$. The parameters $r, \gamma > 0$, $\theta \geq 0$ and $C_H < \infty$ are fixed. Thus the strict inequalities

$$\gamma(N+1) > C_H(C_1 - \sigma)_+ + \chi, \quad P > r + \theta + C_H(C_1 - \sigma)_+ + \chi$$

give an error $Cq^\chi L^{d_*} a(t) = o(a(t))$.

At the first cap, and at every time in the fixed protected late window for which $3\delta/4 \leq |\mathcal{A}(w_i(t))| \leq 5\delta/4$, one has

$$\frac{d}{dt} |\mathcal{A}(w_i(t))| \geq \sigma\delta/2.$$

The corresponding upper derivative bound in this window is $C\delta$. No statement about later returns to this level is included.

For the descending induction choose once and for all $0 < c_0 < 6/\sigma$. The selected deepest clock has leading coefficient $K/\sigma > 6/\sigma$, while every receiving clock satisfies

$$t_i \geq \frac{6}{\sigma} L - O(\log L) \geq c_0 L$$

for sufficiently small q , with uniform constants. This is the lower-clock invariance used in the receiver implication.

In the growing phase gauge write the modulus of the universal polynomial's coordinate as $\varrho_M(a) = a[1 + O(a)]$. Its derivative with respect to the amplitude parameter is

$$\frac{d\varrho_M}{da} = 1 + O(a) > 0.$$

Along the linear growing amplitude $a' = \sigma a$, its time derivative is $\sigma a[1 + O(a)]$. The positive amplitude derivative, together with rotation equivariance, is the local cap time/amplitude and phase gauge inverse used here.

On the provisional receiver stop, replace the source's local radial coefficient $C\delta/\beta$ by $CC_{\text{stop}}\delta/\beta$. For every fixed trial C_{stop} and every $H \leq C_2 L$,

$$\exp(CC_{\text{stop}}\delta H) \leq 2$$

after decreasing q . Consequently the future polynomial moment bound and its discounted integral may be estimated by a fixed input ball with this factor 2, independent of the trial stop constant. Compute the finite-recursion envelope constant in that ball, choose C_{stop} larger than twice the deterministic prospective envelope constant, and only then decrease q to enforce the displayed exponential estimate and all radial margins. This makes the constant order of the stop improvement explicit.

3.9 Corrected source windows and explicit error budgets

Lemma 3.1. *Suppose a continuous moment B satisfies $|B(u)| \leq Ce^{\sigma u}$ for $u \leq 0$ and $|B(u)| \leq C(1+u)^4$ for $u \geq 0$. Let $0 < a = \Re \mu < \sigma$ and $S, U \geq 0$. Then*

$$\left| \int_{\mathbb{R}} e^{-\mu u} B(u) du - \int_{-S}^U e^{-\mu u} B(u) du \right| \leq \frac{C}{\sigma - a} e^{-(\sigma - a)S} + C_a (1 + U)^4 e^{-aU}.$$

Replacing B by $\tilde{B}$ on that window adds at most $\int_{-S}^U e^{-au} |\tilde{B} - B| du$.

Proof. Integrate $Ce^{(\sigma - a)u}$ on $(-\infty, -S)$. For the positive tail substitute $u = U + v$ and use $(1 + U + v)^4 \leq (1 + U)^4(1 + v)^4$; the integral of $e^{-av}(1 + v)^4$ is finite. The replacement estimate is the triangle inequality. If each tail/replacement budget is below $|\mathcal{F}|/4$, the modified transform has modulus greater than $|\mathcal{F}|/2$. An independent old coefficient below $q^6 |\gamma| e^{-\sigma s_*/\beta} |\mathcal{F}|/4$ leaves a coefficient of modulus greater than that same quarter. No budget is counted twice. $\square$

Counterexample to unrestricted endpoints. For the canonical ancient orbit at sufficiently small fixed amplitude, $B(-1) = \delta b e^{-\lambda} + O(\delta^2 e^{-2\sigma}) \neq 0$. The function $G(z) = \int_{-1}^{\infty} e^{-zu} B(u) du$ is analytic on $\Re z > 0$ and nonzero identically, since $ze^{-z} G(z) \rightarrow B(-1)$ on the positive real axis as $z \rightarrow \infty$. This last limit follows by substituting $v = u + 1$ and the elementary approximate identity ze^{-zv} on $[0, \infty)$; continuity at zero and the polynomial growth control the tail. The full transform is also nonzero identically by its pole. Choose $\beta > 1$ off the discrete zeros of both on the injective curve μ_β . At $U = -1$, $S \rightarrow \infty$, the actual truncation error tends to $|G(\mu_\beta)| > 0$ while the claimed unrestricted bound tends to zero.

3.10 Full-vorticity exact-cap comparison

Let $\Omega_{q,\text{cap}}$ be the full canonical state after exact time and phase matching. If the verified perturbation comparison is

$$\|f_j(t_j) - (\Omega_{q,\text{cap}} - W)\|_X \leq C_M \delta_q^{M+1} + C_M q^\gamma (1 + \log(1/q))^{D_M} \delta_q,$$

then

$$\|\tilde{w}_j(t_j) - \Omega_{q,\text{cap}}\|_X \leq C_M \delta_q^{M+1} + C_M q^\gamma (1 + \log(1/q))^{D_M} \delta_q + \kappa t_j \|\Delta W\|_X.$$

Indeed $\tilde{w}_j = f_j + e^{\kappa t \Delta} W$ and heat contraction gives $\|e^{\kappa t \Delta} W - W\|_X \leq \kappa t \|\Delta W\|_X$. The adjoint observable annihilates radial functions, so matching its coordinate is unchanged by restoring the base. This accounts for the unseeded radial error explicitly.

3.11 A complete prospective-stop extension and constant order

Proposition 3.2. *Retain the verified hypotheses and local estimates of Proposition 18.1, including the inactive-prefix, radial-core, positive-envelope and finite forcing results. Put $L = \log(1/q)$, $\delta = L^{-4}$, $a(t) = q^6 e^{\sigma(t-t_0)}$ and $H_+ = t_0 + \sigma^{-1} \log(\delta/(q^6|C_*|)) + L_0$. The positive source-history bound and upper forcing-input bound hold up to the full H_+ ; no assumption that a parent coordinate cap occurs only after H_+ is needed. The auxiliary density-stop constant can be chosen without a constant circle.*

Proof. For a fixed trial $C_s < \infty$, let T_s be the first time $\|w_i - W_{\kappa_i}\|_X = C_s \delta$, and work first up to $\hat{H} = \min(H_+, T_s)$. All older colours are controlled by the unconditional inactive closure on this interval, since $H_+ \leq C_2 L$. The receiver itself has its protected core from the actual core bootstrap: $C_2 L \sqrt{C_s \delta} \rightarrow 0$ and $\beta q C_2 L \rightarrow 0$ for every fixed trial C_s . Before the child's reference time, the inactive theorem with the child as current index bounds the receiver independently of a signed child transfer. Thus that reference time is reached before a density stop.

In child units, on its subsequent interval ending at $\hat{H}$, the source's own and descendant velocities have a fixed bounded sum, say V_0 . Inside a fixed fraction of the parent core, its older radial drift has bound $(CC_s \delta / \beta)|y|$ plus a Gaussian leak. Globally the older radial drift has bound $C|y|/\beta$. The positive reference envelope has fixed moments and the parent-radius Gaussian tail, by the preparation/handoff estimates. These are exactly the operative hypotheses of the positive-history lemma. They remain true after any hypothetical *coordinate* crossing, as long as the *density* stop has not been reached. Applying the radial fourth-root inequality and exit estimate therefore gives, for $0 \leq u \leq U_s := \beta(\hat{H} - t_0) \leq C_2 \beta L$,

$$|B_c(u)| \leq C(1+u)^4 e^{CC_s \delta u / \beta} + o(q^P)$$

for every fixed P . The prefactor depends on the reference moments and V_0 , not on C_s ; C_s occurs in the displayed exponential. The exit and initial-tail errors are superpolynomial on this finite logarithmic window, using the actual Gaussian handoff and global radial growth.

For each fixed trial C_s , choose the later q threshold so $CC_s \delta C_2 L \leq \log 2$ and $CC_s \delta < \sigma/2$. Then the future weighted input is bounded by

$$2C \int_0^\infty e^{-\sigma u / \beta} (1+u)^4 du + o(1),$$

independently of the trial constant. The pre-small-cap input is likewise bounded, independently of the eventual trial past constant after its later q threshold, by a fixed reference input plus $CC_{\text{past}} \delta^{1-1/\beta}$. Retained spatial derivatives are derivatives of a fixed finite stream basis multiplied by these moment coefficients. Higher harmonics, older fields and leaks have the verified strict positive powers of q , so the whole finite derivative input can be placed in one fixed ball $M_{\text{ref}} + 1$.

Compute the finite-recursion coefficient and density constant C_a in that ball first. Choose

$$C_s > 2C_a e^{\sigma L_0} / |C_*|,$$

and only then decrease q to validate all preceding estimates for this fixed choice. The finite forcing comparison uses only the upper input and yields $\|w_i - W_{\kappa_i}\|_X \leq C_a a(t)$ on $[0, \hat{H}]$. Since $a(H_+) = \delta e^{\sigma L_0} / |C_*|$, this is strictly below $C_s \delta / 2$ throughout. Continuity contradicts $T_s \leq H_+$. The whole interval is now available. Only at this point apply the two-tail history convergence and infer $|C_q| \geq |C_*|/2$. The argument used no signed lower bound to exclude the density stop. $\square$

3.12 Uniform classes and clocks with explicit constants

The preparation displacement has the form $O(C_{\text{past}}\delta) + O(\beta qL) + o(1)$. At parent radius c/q , deterministic displacement is $O(L)e^{O(C_s\delta L)} = o(q^{-1})$. Thus a fixed localization radius and its moment majorants can be chosen before the trial past/reset/stop constants, with their later q thresholds. The negative pre-cap history in reference clock u , with small-cap time s_q , is bounded exactly by

$$\int_{-\infty}^{s_q} e^{-\sigma u/\beta} C_{\text{past}} \delta e^{\sigma(u-s_q)} du = \frac{C_{\text{past}} \delta e^{-\sigma s_q/\beta}}{\sigma(1-1/\beta)} \leq C C_{\text{past}} \delta^{1-1/\beta}.$$

The small-cap-to-reference comparison amplifies a reset error by at most $C\delta^{-C_W/\sigma}$. Even after the history factor $\delta^{-1/\beta}(1+|s_q|)$, its intrinsic polynomial error vanishes if $M+1 > C_W/\sigma + 2$. Every fixed positive power q^χ survives these logarithmic losses.

Fix the profile, reference, K, β , inactive weights and strict clocks; then fix all finite expansion orders and the universal input ball. Compute its output constants. Choose C_{past} , C_s and C_{reset} larger than twice their output and base constants; choose a common positive χ below all intrinsic source/receiver powers; finally choose a single q satisfying the finitely many thresholds. The normalized memory transient at settling time $S_q = (6/\sigma)L + O(\log L)$ is $q^{6d/\sigma}$ times a fixed log power, with $d > 0$ fixed. The waveform changes the full C_q , which exact-cap time and phase matching remove. This explains why the output reset exponent is not multiplied by the incoming exponent at each stage.

Choose $0 < c_0 < \min(K/\sigma, 6/\sigma)$ and $C_0 > \max(K/\sigma, 6\beta/[\sigma(\beta-1)])$, with $C_2 > C_0$ and any stronger selected-source slack retained. If the child clock is at most C_0L , the parent clock is at most $C_0L/\beta + (6/\sigma)L + O(\log L) < C_0L$ for the final small q . It is at least $(6/\sigma)L - O(\log L) \geq c_0L$; the deepest clock $KL/\sigma + O(\log L)$ satisfies the same lower bound. Fixed cap windows and the $O(\log L)$ reference advance fit below C_2L uniformly in depth.

3.13 Exterior multipoles and the annular-hole counterexample

The domain in Lemma 4.2 must be specified as the exterior of an origin-centred support ball. Here is the complete corrected calculation. Let $w \in L^1(\mathbb{R}^2)$ be real, supported in $\{|y| \leq R\}$, quarter-turn symmetric, and of zero circulation. Identify the plane with $\mathbb{C}$ and write $U = u_1 + iu_2$. Then, for $|z| > R$,

$$U(z) = \frac{i}{2\pi} \sum_{k=0}^{\infty} \frac{B_k(w)}{\bar{z}^{k+1}}, \quad B_k(w) = \int \bar{y}^k w(y) dy.$$

Indeed,

$$\frac{1}{\bar{z} - \bar{y}} = \frac{1}{\bar{z}} \sum_{k=0}^{\infty} \left(\frac{\bar{y}}{\bar{z}} \right)^k$$

converges absolutely and uniformly for $|y| \leq R$ at each such z . Its termwise integral is justified by the majorant $\|w\|_1 / (|z| - R)$. Changing variables $y = ix$ gives $B_k = (-i)^k B_k$, so $B_k = 0$ unless 4 divides k ; zero circulation also gives $B_0 = 0$. Consequently

$$U(z) = \frac{iB_4(w)}{2\pi\bar{z}^5} + \frac{iB_8(w)}{2\pi\bar{z}^9} + \cdots, \quad |z| > R.$$

For $R' > R$ this series and every fixed spatial derivative converge locally uniformly on $|z| \geq R'$. Moreover, the first omitted term after $k = 4J$ has the explicit bound

$$\left| U(z) - \frac{i}{2\pi} \sum_{j=1}^J \frac{B_{4j}(w)}{\bar{z}^{4j+1}} \right| \leq \frac{\|w\|_1}{2\pi|z|} \frac{(R/|z|)^{4(J+1)}}{1 - (R/|z|)^4}.$$

This is the exterior expansion needed when a smaller packet forces the larger receiver collars.

The broader phrase “outside its support” is demonstrably false. Choose $0 \neq h \in C_c^\infty((1, 2))$, $h \geq 0$, and set

$$w(r, \theta) = h(r) \cos(4\theta).$$

This vorticity is real, smooth, compactly supported, quarter-turn symmetric, and has zero circulation. Angular integration gives

$$B_4(w) = \pi \int_1^2 h(r) r^5 dr > 0, \quad B_{4j}(w) = 0 \quad (j \neq 1).$$

Thus the incorrectly claimed exterior formula would give $U(z) = iB_4(w)/(2\pi\bar{z}^5)$ even for $0 < |z| < 1$. On the other hand, for $|z| < 1$ the convergent *interior* series is

$$U(z) = -\frac{i}{2\pi} \sum_{k=0}^{\infty} \bar{z}^k \int \bar{y}^{-k-1} w(y) dy = -\frac{i}{2} \left(\int_1^2 h(r) r^{-3} dr \right) \bar{z}^3.$$

Termwise integration is justified on each $|z| \leq a < 1$ by $|y| \geq 1$ on the support. Only $k = 3$ survives angular integration. The actual velocity is therefore smooth and cubic in the central hole, whereas the claimed exterior expression is singular at zero. Every hypothesis of the original lemma is satisfied. Replacing its domain by $|z| > \sup\{|y| : y \in \text{supp } w\}$ repairs the statement and preserves the receiver calculation $-u_r W' = B_4 F_e e^{4i\theta} + \text{complex conjugate}$.

3.14 All finite Sobolev orders and finite jets of the same ancient orbit

The initial construction at an integer $k \geq 3$ yields compactly supported H^k vorticity and finite-energy H^{k+1} velocity at the fixed ancient amplitude δ . This is sufficient for the classical finite-Sobolev local Euler construction. The conserved L^1 and L^∞ vorticity norms, the logarithmic bound for $\|\nabla u\|_\infty$, and the velocity Sobolev energy inequality then extend the solution uniquely to every positive time. No C^∞ initial regularity is needed for that first extension.

The same reference orbit does, in fact, have every finite Sobolev order. This can be proved without intersecting all of the order-dependent small-amplitude thresholds. Fix any integer $\ell \geq k$. Repeat the finite coefficient and actual-error construction at order ℓ , at an amplitude δ_ℓ smaller than both its new permissible threshold and the original one. Use any fixed $K_* > 0$. The exact isolated seeded Navier–Stokes family, including its cap shift $H_q = \sigma^{-1} \log(\delta_\ell/\alpha_q)$ and seed phase $-\tau H_q$, is the same family as in the original theorem at amplitude δ_ℓ . Its full $L^1 \cap L^p$ limit has already been identified uniquely. Consequently the order- ℓ construction gives that same Ω_{δ_ℓ} , with a uniform H^ℓ bound on negative times. The finite-Sobolev Euler continuation argument gives an H^ℓ bound on every fixed positive-time interval as well.

Set

$$s_\ell = \sigma^{-1} \log(\delta_\ell/\delta) < 0.$$

The cap-translation identity and forward uniqueness yield

$$\Omega_{\delta_\ell}(v) = \mathcal{T}_{-\tau s_\ell} \Omega_\delta(v + s_\ell) \quad (-\infty < v < \infty).$$

It follows that Ω_δ has H^ℓ vorticity at every time. Its norm is uniformly bounded on $s \leq 0$: for $s \leq s_\ell$ use the negative-time bound on Ω_{δ_ℓ} , and for $s_\ell \leq s \leq 0$ use its finite positive-time bound on $0 \leq v \leq -s_\ell$. The bound can depend on ℓ and δ . Since ℓ is arbitrary, the already fixed reference orbit is spatially C^∞ . A common positive lower bound for the infinitely many small-amplitude thresholds has not been asserted or used.

The same argument supplies each fixed finite homological jet of the reference orbit. Fix a target degree J and derivative order r . Choose a finite Sobolev order $\ell \geq \max\{k, r, 3\}$ and an expansion degree $M \geq J$ large enough for the order- ℓ error exponent. At a sufficiently small amplitude δ_ℓ , pass the finite expansion of the same isolated family to its already identified limit. The triangular pure-seed recursion makes the coefficients $F_1, \dots, F_J$ independent of the larger cutoff M . Discarding the finitely many degrees above J gives

$$\left\| \Omega_{\delta_\ell}(v) - W - \sum_{j=1}^J \delta_\ell^j e^{j\sigma v} \mathcal{T}_{\tau v} F_j \right\|_{L^1 \cap H^r} \leq C_{J,r} (\delta_\ell e^{\sigma v})^{J+1}, \quad v \leq 0,$$

where the intersection norm is the sum norm. Translating by s_ℓ and rotating gives

$$\left\| \Omega_\delta(s) - W - \sum_{j=1}^J \delta^j e^{j\sigma s} \mathcal{T}_{\tau s} F_j \right\|_{L^1 \cap H^r} \leq C_{J,r} (\delta e^{\sigma s})^{J+1}, \quad s \leq s_\ell.$$

Here $\delta_\ell e^{\sigma(s-s_\ell)} = \delta e^{\sigma s}$ and the two rotation phases add to τs . The constant and the onset time may depend on the finite target orders. This proves all fixed finite jets of one orbit; it neither assumes nor proves convergence of an infinite formal series.

3.15 Renormalization at the prescribed initial trace

The statement in Proposition 9.1 that renormalization includes the initial trace can be made explicit without assuming a strong trace. Let f be a bounded distributional transport solution with weak initial trace f_0 , under the divergence and Sobolev hypotheses stated there. Extend it to $-1 < t < 0$ by $\tilde{f}(t) = f_0$, and extend the velocity by $\tilde{u}(t) = 0$. The weak initial-trace identity implies

$$\partial_t \tilde{f} + \operatorname{div}(\tilde{u} \tilde{f}) = 0 \quad \text{on } (-1, T) \times \mathbb{R}^2.$$

There is no distributional jump at zero. The extended velocity still belongs to $L_t^1 W_{\text{loc}}^{1,p}$, is bounded, and is divergence free. The spatial commutator proof in Proposition 9.1 therefore applies across $t = 0$. For every C^1 function β with bounded derivative on the bounded range of $\tilde{f}$, it yields

$$\partial_t \beta(\tilde{f}) + \operatorname{div}(\tilde{u} \beta(\tilde{f})) = 0.$$

For a compact smooth test φ , the pairing of this equation has an integrable time derivative. Its continuous representative equals $\int \varphi \beta(f_0)$ on $t < 0$, and hence has that same value at zero. This proves the claimed renormalized weak initial trace.

Take $\beta(s) = s^2$ on the bounded range, with a smooth bounded-derivative extension and $\beta(0) = 0$. Expanding spatial cutoffs, bounded velocity, and $\sup_t \|f\|_1 < \infty$ give

$$\|f(t)\|_2^2 = \|f_0\|_2^2 \quad \text{for almost every } t,$$

since the flux error is bounded by $C \|u\|_\infty R^{-1} \sup_t \|f\|_2^2$ and tends to zero. For the difference of two solutions the same extension uses zero datum; the square identity proves uniqueness. These steps justify the initial norm used in the strong convergence argument, rather than deducing it from the original weak trace alone.

For completeness, choose the weakly continuous L^2 representative given by the transport equation. Its values at exceptional times retain the L^1 and L^∞ bounds: approach such a time through almost-everywhere bounded slices, use weak compactness on each spatial ball and lower

semicontinuity, and then enlarge the balls. At any fixed interior time t_* , extend the restriction to $t > t_*$ by its weak trace $f(t_*)$ and zero velocity on the preceding interval. The same argument gives its renormalized trace and preserves its L^2 norm on that restriction. The almost-everywhere identity above then shows $\|f(t_*)\|_2 = \|f_0\|_2$. Weak continuity and equality of norms give strong L^2 continuity at every time, including the endpoints (using the analogous time-reversed argument at the final endpoint). This is a proof completion of the stated result, not an additional hypothesis or a weaker conclusion.

3.16 Actual fixed-reference advance and both envelope scales

Lemma 3.3 (Actual cap-to-reference comparison and two-scale envelope reset). *Fix the profile, $p > 2$, and one sufficiently small canonical ancient segment before choosing the later finite expansion orders. On this segment fix a reference orbit with $\mathcal{A}(\Omega_{\text{ref}}(0)) = \delta_{\text{ref}} > 0$ and write*

$$\mathcal{A}(\Omega_{\text{ref}}(u)) = c_{\text{ref}} e^{\lambda u} + O(e^{2\sigma u}), \quad a_{\text{ref}} = |c_{\text{ref}}|.$$

Fix $C_0 < C_2$, the parameter margins of Corollary 13.3, and $\beta q^{2/p}, \beta q \leq 1/2$. Put $L = \log(1/q)$, $\delta = L^{-4}$, and let $s_q < 0$ be the reference orbit's exact δ -cap time. Normalize an actual child colour of index $j \geq 2$ to unit amplitude and radius and remove the solid rotation of its older cores. All actual colours here belong to the geometric stack of Lemma 1.1, with $\alpha = q^K$, $A_l = \beta^{l-1}$, $R_l = q^{l-1}$, and the allowed seed phases and amplitude reductions. The clock and parameter margins stipulated above are those of Corollary 13.3 in these child units. Suppose $0 < \kappa \leq q^{3/2}$, its actual cap time is $\tau \leq C_0 L$, and, in the exact-coordinate phase gauge,

$$\|\tilde{w}(t) - e^{\kappa t \Delta} W\|_X \leq C_{\text{past}} \delta e^{\sigma(t-\tau)} \quad (0 \leq t \leq \tau), \quad (1)$$

$$\|\tilde{w}(\tau) - \Omega_{\text{ref}}(s_q)\|_X \leq E_q. \quad (2)$$

Here C_{past} is fixed independently of depth. Assume

$$\delta^{-r}(E_q + q) \longrightarrow 0, \quad r = C_W/\sigma,$$

where $C_W > 0$ is a fixed one-sided X comparison exponent on the chosen canonical segment. Set $t_{\text{ref}} = \tau - s_q$.

The actual older-field and descendant hypotheses needed on the advance follow from Corollary 13.3, Proposition 5.4, and the individual scalar contractions. Consequently

$$\sup_{0 \leq v \leq -s_q} \|\tilde{w}(\tau + v) - \Omega_{\text{ref}}(s_q + v)\|_X + \int_0^{-s_q} \|\tilde{w}(\tau + v) - \Omega_{\text{ref}}(s_q + v)\|_X \, dv \leq C \delta^{-r}(E_q + q). \quad (3)$$

There is no hypothesis comparing δ_{ref} with a terminal finite-expansion amplitude. If

$$\delta^{-r-1/\beta}(E_q + q) \longrightarrow 0, \quad (4)$$

the actual signed fourth-moment history on $[s_q, 0]$ converges to the canonical history in the weighted absolute norm with weight $e^{-\sigma u/\beta}$.

For sufficiently small fixed δ_{ref} and subsequently small q , the actual positive envelope of the child, and that of its entire source/deep packet, have uniformly bounded moments of every fixed order at t_{ref} . They have superalgebraic moment tails outside one fixed ball. More strongly, choose a fixed $c_R > 0$ so $R = c_R/q$ lies in a fixed fraction of the parent's protected core. If $P(t)$ denotes either positive envelope and $M = \int P(0)$, then for every fixed integer $n \geq 2$,

$$\int_{|y| > R/4} (1 + |y|)^n P(t_{\text{ref}}, y) \, dy \leq C_n M \exp \left[-\frac{c_n R^2}{\kappa C_2 L} \right]. \quad (5)$$

All constants are independent of depth and of fixed trial past/reset constants; the threshold for q may depend on those trial constants.

Proof. The canonical first jet and exact-cap monotonicity give

$$s_q = \sigma^{-1} \log(\delta/a_{\text{ref}}) + O(\delta), \quad -s_q = O(\log L). \quad (6)$$

Thus $t_{\text{ref}} + J < C_2 L$ for every fixed J and small q . The canonical support and its H^k bounds on the fixed negative segment are supplied by Theorem 3.1. With $k \geq 3$ they give fixed X bounds for both $\nabla \Omega_{\text{ref}}$ and $\Delta \Omega_{\text{ref}}$, as in the proof of Corollary 3.2.

We verify the actual foreign-field input independently of (1) after τ . Write

$$e_B = C_2 \lambda_B < K - 6, \quad e_P = C_2(\lambda_P + C_4/\beta) < 1.$$

Corollary 13.3 protects every older core and its harmonic field through the full child-clock interval $[0, C_2 L]$, without an active-child density assumption. On any fixed child ball, its weighted harmonic row and the core-leak decomposition give, after removing the solid rotation,

$$\|b_{<j}\|_\infty \leq C\{q^{K+2-e_B} + q^{8-e_P}(1 + C_2 L)^5\} + o(q^P).$$

The decomposition holds on the entire older probe core by (64)–(66); every fixed child ball lies there for small q . Individual colour L^1, L^∞ contractions give the unrestricted descendant bound $\|b_{>j}\|_\infty \leq C\beta q$, as in Lemma 13.1. Therefore on the canonical support

$$\|b\|_\infty \leq \rho_q := C\{\beta q + q^{K+2-e_B} + q^{8-e_P}(1 + C_2 L)^5\} + o(q^P) = O(q). \quad (7)$$

In particular neither an actual child-core hypothesis nor a continuation of the finite-expansion envelope is used here.

Set $\zeta(v) = \tilde{w}(\tau + v) - \Omega_{\text{ref}}(s_q + v)$ and $E(v) = \|\zeta(v)\|_X$. Its exact equation is

$$\partial_v \zeta + (K * \tilde{w} + b) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta + b) \cdot \nabla \Omega_{\text{ref}} + \kappa \Delta \Omega_{\text{ref}}.$$

The actual foreign field stays in divergence-free transport. Kato's inequality, its L^p counterpart, and (14) imply

$$E'(v) \leq C_W E(v) + C(\rho_q + \kappa).$$

Hence $E(v) \leq C e^{C_W v} (E_q + q)$, and integration of this exponential, using $C_W > 0$ and (6), proves (3). This proof never uses a derivative of the actual colour or of a rough density leak.

We next reset the positive envelopes at a fixed radius. Choose a fixed ball enclosing their original support with a fixed radial margin. Before the cap, (1) and (14) bound the child's integrated nonradial speed by $CC_{\text{past}}\delta$. On the advance, the radial component of its own velocity is at most

$$\|K * (\Omega_{\text{ref}}(u) - W)\|_\infty + C_p E(u - s_q).$$

The fixed canonical negative envelope, which follows from its first finite jet in (44), gives

$$\int_{s_q}^0 \|K * (\Omega_{\text{ref}}(u) - W)\|_\infty \, du \leq C a_{\text{ref}} = O(\delta_{\text{ref}}).$$

The comparison contribution tends to zero by (3). Older nonrotational fields contribute $o(1)$ and descendants contribute $C\beta q C_2 L = o(1)$. The resulting absolute radial displacement is

$$O(C_{\text{past}}\delta) + O(\delta_{\text{ref}}) + o(1).$$

Fix δ_{ref} sufficiently small for the prescribed margin, and then choose q for the fixed trial constants. The radial exit estimate of Lemma 5.2 gives the fixed-radius Gaussian exit bound. Conditional global moment growth, using (97), multiplies it by at most $C_n e^{C_n C_2 L} (1 + C_2 L)^n$, which is absorbed by its Gaussian. Thus all fixed envelope moments are bounded and their fixed-radius moment tails are smaller than every prescribed q -power. The packet has bounded initial mass and the same fixed initial support by the positive-envelope construction preceding Lemma 13.1; it uses the same actual common drift, so the argument applies without a summed initial L^∞ norm.

For completeness, the parent-radius estimate is proved directly, rather than deduced from the fixed-radius tail. Choose c_R so B_R is inside a fixed fraction of the parent's older probe core. The actual older harmonic and leak estimates give there

$$|b_{<j} \cdot e_r| \leq C \frac{\delta q^2}{\beta} |y|^3 + C \frac{\epsilon_{\text{core}}}{\beta} |y| \leq a_q |y|, \quad a_q \leq C\delta/\beta + o(q^P).$$

The child's own and descendant signed velocities have total speed at most a fixed D , by the individual contractions and the convergent sum of $A_l R_l$. The full actual radial drift on B_R is therefore bounded by $a_q |y| + D$. Outside this ball its unrestricted radial bound is $C|y|/\beta + D$, as in Lemma 13.1.

Let $H = t_{\text{ref}} \leq C_2 L$ and enclose the original packet support in the fixed $B_{R_{\text{in}}}$. Since $a_q H = o(1)$ and $DH = O(L) \ll R$,

$$e^{a_q H} (R_{\text{in}} + DH) \leq R/8$$

for small q . Run the radial diffusion calculation at radius $R/4$. After predictable removal of tangential drift, its Brownian supremum must exceed a fixed multiple of $R/\sqrt{\kappa}$ to exit. Thus

$$\Pr\{\tau_{R/4} \leq H\} \leq C \exp[-cR^2/(\kappa H)].$$

Condition at that exit and use the unrestricted global radial bound. The same moment calculation as in the proof of Proposition 15.1 gives

$$\int_{|y| > R/4} (1 + |y|)^n P(H, y) \, dy \leq C_n M e^{C_n H/\beta} (1 + R + H + \sqrt{\kappa H})^n e^{-cR^2/(\kappa H)}.$$

Its prefactor is a fixed power of q^{-1} times a fixed logarithmic power, whereas $R^2/(\kappa C_2 L) \geq cq^{-7/2}/L$. Reducing the Gaussian exponent constant proves (5). Truncation of the positive initial packet and monotone passage justify the calculation when that initial packet has no finite L^∞ norm.

Finally, cut the fourth moment off at the fixed reset radius. Its difference is bounded by the cutoff norm times E , plus superalgebraic tails. Since $e^{-\sigma u/\beta} \leq C\delta^{-1/\beta}$ on $[s_q, 0]$, (3) gives

$$\int_{s_q}^0 e^{-\sigma u/\beta} |B_{\text{actual}}(u) - B_4(\Omega_{\text{ref}}(u))| \, du \leq C\delta^{-r-1/\beta} (E_q + q) + o(q^P).$$

This proves the weighted assertion under (4). □

Corrected proof of Corollary 16.4. Theorem 16.2 is used only through its shrinking exact cap t_j . Its full-state reset, including the radial heat-base defect, gives

$$E_q \leq C_M \delta^{M+1} + C_M q^\gamma (1 + C_2 L)^{D_M} \delta + C\kappa t_j.$$

Its preparation envelope supplies (1) with a fixed C_{past} . The choice $M > C_W/\sigma$ makes $\delta^{-r}(E_q + q) \rightarrow 0$, so Lemma 3.3 proves convergence at the fixed reference and on every fixed earlier reference

window, together with both positive-envelope resets. It is immaterial whether the selected source's terminal finite-expansion amplitude exceeds δ_{ref} .

The clock lies inside the required inactive horizon directly. Indeed the selected-source cap clock and (6) give

$$t_{\text{ref}} = t_j - s_q = \frac{K}{\sigma}L + \frac{1}{\sigma} \log a_{\text{ref}} + o(1).$$

The choice $C_2 > K/\sigma + 1$ therefore places every fixed reference window inside $[0, C_2L]$ for small q . On a fixed later window the canonical Euler support and its X gradient/Laplacian bounds depend only on that window. The independently available actual older-field bound and descendant velocity sum still apply there. The same one-sided comparison extends convergence to $C([-J, J]; X)$.

To remove the fourth-moment cutoff on such a later window, the child and descendant total velocity is uniformly bounded by scalar contractions. On a sufficiently large fixed ball the actual older radial field is small; the initial reference envelope has the fixed-radius Gaussian tail just proved. Radial exit and the unrestricted global moment bound give uniform fourth-moment tightness on that fixed window. Strong L^1 convergence on a cutoff ball then proves uniform whole-fourth-moment convergence. This completes the corollary without extending the selected-source expansion envelope beyond its proved endpoint. $\square$

Remark 3.4 (The reference inputs in the invariant class). For the inductive reset $E_q \leq C_{\text{reset}}\delta^{M+1} + q^XL^D$, the fixed choice $M+1 > C_W/\sigma + 2$ implies (4), including the extra factor $1 + |s_q|$ used in Proposition 18.1. For each fixed trial reset constant the reference comparison error and its integrated radial contribution tend to zero. The reference and its localization radius are fixed before that later finite order.

The reference past before its small cap has weighted norm at most $CC_{\text{past}}\delta^{1-1/\beta}$. In small-cap time $v \leq 0$, this follows from the envelope $C_{\text{past}}\delta e^{\sigma v}$ and the factor $e^{-\sigma s_q/\beta} = O(\delta^{-1/\beta})$. The advance has fixed canonical history norm plus the vanishing error just proved.

At reference time (5), with $n = 4$, supplies (115) with $H_0 = C_2L$, while the fixed-radius moment reset supplies (114). Under the current parent's protected-core stop, its actual older harmonic/leak field satisfies (116); its unrestricted origin-growth bound gives (117); the child's own and descendant velocity sum gives the fixed D . Thus Proposition 15.1 supplies the prospective future history without an assumed signed source waveform.

The upper history constant can be chosen independently of the trial parent stop. In the proof of that proposition the local stop constant occurs in $e^{CC_{\text{stop}}\delta u/\beta}$, while the prefactor uses only the reset moment and D . On $u \leq C_2\beta L$, this factor is at most two after choosing q for the fixed trial stop; its discounted exponent is at most $-\sigma u/(2\beta)$. Conditional global moment growth in the Gaussian error uses the unrestricted bound, independently of that stop. The future weighted norm is consequently bounded by a fixed reference constant. Hence, for the finite derivative orders selected before the class constants,

$$M_{\text{input}} \leq M_{\text{ref}} + CC_{\text{past}}\delta^{1-1/\beta} + o(1),$$

where M_{ref} may depend on those finite orders, fixed stream cutoffs, profile, reference, clocks, and β , but not on trial past/reset/stop constants or depth. Orders can be chosen using the already fixed comparison exponents, then their input constant computed, then the output class constants fixed, and finally q decreased.

Fixed-window comparison and the two discounted tails also give uniform $o(1)$ closeness of the signed transform to the fixed canonical transform. This provides its nonzero lower bound once the full prospective receiver interval is reached; the independent upper input bound is available on every shorter stop. These are the reference and handoff inputs needed by Proposition 19.1.

3.17 Discounted histories of an arbitrary prepared actual child

Corollary 16.8, as proved for the diffusion-selected deepest source, uses $\alpha e^{\sigma t_{\text{ref}}} = a_{\text{ref}}(1 + o(1))$. That identity is not an input for a child activated by another packet. The following statement supplies the required generalization directly from the receiving-stage preparation and exact-cap hypotheses.

Lemma 3.5 (Prepared-child reference history). *Fix the profile, reference orbit, $\beta > 1$, clock constants and finite orders in Proposition 18.1. Write $L = \log(1/q)$, $\delta = L^{-4}$ and $\mu_\beta = [\sigma + i(\tau - 4/5)]/\beta$. All actual colours below belong to the geometric stack of Lemma 1.1, with the clock and parameter margins of Corollary 13.3 in child units. Use child units, with all older constant rotations removed. Let the actual child w_c have normalized viscosity $\kappa_c \leq q^{3/2}$, an exact cap time τ_c satisfying $c_0 L \leq \tau_c \leq C_0 L$, and the common geometric initial colour bounds. Suppose, for fixed finite trial constants $C_{\text{past}}, C_{\text{reset}} > 0$,*

$$\|w_c(t) - e^{\kappa_c t \Delta} W\|_X \leq C_{\text{past}} \delta e^{\sigma(t - \tau_c)}, \quad 0 \leq t \leq \tau_c.$$

Let $s_q < 0$ be the canonical exact δ -cap time, and match the constant phase ϕ exactly in the protected coordinate. Suppose

$$\|\mathcal{T}_{-\phi} w_c(\tau_c) - \Omega_{\text{ref}}(s_q)\|_X \leq E_q, \quad E_q = C_{\text{reset}} \delta^{M+1} + q^\chi L^D, \quad \chi > 0,$$

and

$$\delta^{-C_W/\sigma - 1/\beta} (1 + |s_q|) [E_q + q(1 + |s_q|)] \longrightarrow 0. \quad (8)$$

Here C_W is the fixed one-sided comparison constant on the small canonical ancient segment. The sufficient fixed choice $M + 1 > C_W/\sigma + 2$ has strict slack in this condition.

Put $t_{\text{ref}} = \tau_c - s_q$. Before this reference time use the actual inactive-prefix and preparation localization conclusions, with their strict clock slack. On the following prospective interval assume the actual parent/larger constant-core and harmonic conclusions on the provisional parent density stop $\|w_i - W_{\kappa_i}\|_X \leq C_s \delta$, where $C_s > 0$ is fixed and finite. In particular the child's older radial drift in a fixed fraction of the parent core has coefficient at most $C(1 + C_s)\delta/\beta$. Use the Gaussian parent-radius handoff of Lemma 16.5 at t_{ref} . Let the following child horizon satisfy

$$0 \leq U_q \leq C_U \beta L.$$

The complete age and parent-unit horizon are assumed to lie in the fixed logarithmic windows of the cited inactive/core estimates.

For every such shorter stopped interval the absolute differentiated main-input history is bounded by one fixed ball independent of the trial constants, after a later q threshold depending on those constants. If the complete prospective interval has been reached and $U_q \rightarrow \infty$, define

$$B_q(u) = \begin{cases} B_4(\mathcal{T}_{-\phi} w_c(t_{\text{ref}} + u)), & -t_{\text{ref}} \leq u \leq U_q, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_{\mathbb{R}} e^{-\sigma u/\beta} |B_q(u) - B_4(\Omega_{\text{ref}}(u))| du \longrightarrow 0. \quad (9)$$

Consequently its complex discounted transform converges to the full canonical transform at μ_β .

For clarity, B_q is the actual physical history truncated at the controlled prospective endpoint and zero-extended thereafter. The canonical history in (9) is defined on the entire real line. No

estimate for the genuine actual infinite-time future after parent density/core control has ended is asserted or needed.

The quantifiers are as follows. The fixed profile, reference, β , clocks and finite spatial orders determine the input ball first. For each fixed finite triple $(C_{\text{past}}, C_{\text{reset}}, C_s)$ satisfying the stated preparation/reset assumptions, there is a later q threshold. Within that class every estimate and convergence above is uniform in depth, source phase and allowed nondistinguished seed homotopies. The threshold need not be uniform over unbounded trial constants. Thus the constants may be selected once from the finite-forcing output bounds and then one common q chosen, without multiplying an inherited input constant at every descending stage.

Proof. Write $L_q = -s_q$. The canonical exact-cap asymptotic gives

$$L_q = O(\log L), \quad e^{-\sigma s_q/\beta} \leq C\delta^{-1/\beta}.$$

All phases below have already been removed by $\mathcal{T}_{-\phi}$. Rigid rotations preserve every radial positive-envelope estimate.

Localization before the reference. On the preparation interval the child's own nonradial radial velocity has time integral at most $CC_{\text{past}}\delta$. The descendant velocity contributes at most $C\beta q C_0 L$. The actual older nonrotational harmonic field and core leaks on a fixed child ball are smaller, by Corollary 13.3. For any fixed trial C_{past} these displacements tend to zero. The initial support and a fixed radial margin therefore give, at every preparation time, positive-envelope Gaussian tails outside one fixed ball. Conditional global origin-growth estimates control every fixed post-exit moment; their fixed q -power amplification is absorbed by the Gaussian. The exact radial heat base has the corresponding Gaussian moment tails. These conclusions use the actual positive scalar envelope and the common drift, not a gradient of the actual child.

On the cap-to-reference segment, one-sided X comparison with the phase-matched canonical ancient solution gives

$$\sup_{0 \leq v \leq L_q} \|\mathcal{T}_{-\phi} w_c(\tau_c + v) - \Omega_{\text{ref}}(s_q + v)\|_X \leq C\delta^{-C_W/\sigma} [E_q + q(1 + L_q)]. \quad (10)$$

Indeed the canonical segment has fixed derivative bounds and fixed compact support. On that support the descendant field is $O(\beta q)$, the actual larger nonrotational field tends to zero with a strict positive q power, and $\kappa_c \leq q^{3/2}$. The comparison forcing is therefore $O(q)$ with fixed parameters. Gronwall over L_q costs at most $C\delta^{-C_W/\sigma}$. No derivative of an actual moment coefficient or actual colour occurs.

The canonical nonradial radial displacement on this segment is $O(\delta_{\text{ref}})$; the comparison error contributes $o(1)$ by (8). Choose the fixed reference amplitude inside the fixed radial margin. Together with the descendant and older-field contributions, this gives the same fixed-radius positive-envelope moment reset at the reference. In particular no factor $\log \log(1/q)$ is introduced into its moment constants. The separate parent-radius handoff gives the Gaussian exterior-tail input required by Proposition 15.1; fixed-radius tightness alone is not used as a substitute for that input.

The negative history. Let $u = t - t_{\text{ref}}$. For $u \leq s_q$, angular symmetry gives $B_4(e^{\kappa_c t} \Delta W) = 0$. Inside the fixed preparation ball the whole-moment pairing is bounded by a fixed multiple of the perturbation L^1 norm. Outside it use the preceding Gaussian positive-envelope and heat tails. Thus

$$|B_q(u)| \leq CC_{\text{past}} \delta e^{\sigma(u-s_q)} + \mathcal{G}_q(u), \quad -t_{\text{ref}} \leq u \leq s_q,$$

where the Gaussian remainder is superpolynomial on the logarithmic preparation horizon. Its weighted integral is still superpolynomial: the earliest negative endpoint has size $O(L)$, so the

discount costs only a fixed power of q^{-1} , whereas $\kappa_c \leq q^{3/2}$. The main term has the explicit bound

$$\int_{-t_{\text{ref}}}^{s_q} e^{-\sigma u/\beta} C C_{\text{past}} \delta e^{\sigma(u-s_q)} du \leq \frac{C C_{\text{past}} \delta e^{-\sigma s_q/\beta}}{\sigma(1-1/\beta)} \leq C C_{\text{past}} \delta^{1-1/\beta} \rightarrow 0.$$

The canonical moment on $(-\infty, s_q]$ has the same vanishing discounted bound, since it is $O(e^{\sigma u})$. Therefore the weighted absolute difference on the entire pre-small-cap interval tends to zero. This calculation replaces the selected-source identity involving $\alpha e^{\sigma t_{\text{ref}}}$.

On $[s_q, 0]$, the fixed-radius localization converts (10) into a whole fourth-moment difference with the same bound plus Gaussian errors. Since

$$\int_{s_q}^0 e^{-\sigma u/\beta} du \leq C \delta^{-1/\beta} (1 + L_q),$$

its weighted absolute difference tends to zero by (8). The negative canonical history on this segment has one fixed discounted norm. This proves a negative-history input bound consisting of a fixed reference majorant, $C C_{\text{past}} \delta^{1-1/\beta}$, and a vanishing reset contribution.

The controlled future. At reference the positive-envelope mass and fourth moment have fixed majorants. The own and descendant velocities have a fixed bounded sum by the individual scalar contractions and the convergent geometric velocity series. The older global radial drift is at most $C|y|/\beta$; inside the parent probe core its coefficient is $C(1 + C_s)\delta/\beta$, with Gaussian leaks. The parent-radius handoff supplies its initial exterior Gaussian tail. Applying Proposition 15.1 and its scale-margin corollary on any shorter stopped interval yields

$$|B_q(u)| \leq C(1+u)^4 e^{C(1+C_s)\delta u/\beta} + o(q^P), \quad 0 \leq u \leq U_q,$$

for every fixed P . The prefactor is fixed by the reference-envelope majorants and bounded source/descendant speed. It does not multiply C_s . For each fixed trial C_s , a later threshold makes

$$C(1 + C_s) C_U \delta L \leq \log 2, \quad C(1 + C_s) \delta < \sigma/2.$$

Consequently the future absolute discounted input is bounded by a fixed multiple of $\int_0^\infty e^{-\sigma u/\beta} (1+u)^4 du$, and its tail from J is bounded by

$$C_{\beta,\sigma} (1+J)^4 e^{-\sigma J/(2\beta)} + o(q^P).$$

The errors remain negligible under all logarithmic-window and fixed q -power losses. Spatial differentiation of a retained fourth-harmonic stream field differentiates only its fixed smooth spatial basis, so the same argument supplies every fixed spatial derivative input norm. Together with the negative-history estimates, this gives a fixed input ball independent of the trial constants after their later thresholds. It is an upper bound only and requires neither a nonzero signed transform nor completion of a shorter density stop.

Once the complete prospective interval is available, every fixed window $[0, J]$ lies within it for small q . From (10), the actual reference state converges to the canonical reference state in X . On each such following window the canonical derivatives are bounded, viscosity tends to zero, and the actual older nonrotational and descendant fields on fixed support tend to zero. The same one-sided comparison therefore gives convergence in $C([0, J]; X)$. Positive-envelope radial localization on this fixed window, followed by conditional moment growth, gives uniformly vanishing fourth-moment tails outside a sufficiently large fixed ball. Whole fourth moments consequently converge on the window. All constants are uniform over the fixed prepared class.

The canonical positive history has its polynomial discounted tail. Truncate at J , apply fixed-window whole-moment convergence, and then let $J \rightarrow \infty$, using the uniform future tail just established. Combining this with the two negative-history pieces proves (9). The complex transform conclusion follows because $|e^{-\mu\beta u}| = e^{-\sigma u/\beta}$.

Every estimate used geometric normalized mass, amplitude or velocity sums, fixed spatial orders and the common clocks. Its thresholds depend on the fixed trial constants, not on depth or phase. Selecting those constants once and then taking the minimum of the finitely many thresholds yields the stated invariant-class quantifiers. If the force origin is instead chosen at the actual protected crossing of the fixed reference level, the same comparison and canonical positive slope change its time and constant phase only by $o(1)$; this preserves the history and transform conclusions. $\square$

3.18 The retained main conclusion after the corrections

For clarity, the final logical step is restated with no viscosity-dependent datum. Fix the corrected uniform class above and all amplitudes $\alpha = q^K$. At selected viscosity $\nu_N = q^{3/2}(\beta q^2)^{N-1}$, reduce every seed except the N th by a homotopy factor $r \in [0, 1]$. At $r = 0$, rotating the selected phase rotates the entire datum and the outer protected coordinate by $e^{i\vartheta_N}$. The first-cap phase map has degree one. Uniform protected caps and fixed-positive-viscosity continuity give a continuous homotopy to $r = 1$, so that phase map remains surjective.

Successively choose sparse N_k and shoot their phases to $(-1)^k$. For each already chosen positive viscosity, continuity supplies a radius $r_k > 0$ for cap-phase error 2^{-k} . Choose the next index so that the *entire* freely variable tail has X norm below every preceding radius; this is possible because

$$C\alpha \sum_{j \geq m} \beta^{j-1} (q^{2(j-1)} + q^{2(j-1)/p}) \rightarrow 0.$$

The resulting phase sequence defines one datum with the common compact support and zero circulation of Lemma 1.1. Its final cap phases along even/odd selected viscosities tend to $+1/-1$ respectively. There is no need for continuity uniform as $\nu \downarrow 0$.

The strong fixed-data compactness theorem supplies convergent subsequences of both parities. Their outer passive colours converge to the unique bounded transport colour determined by their total limiting Euler path, by the corrected colour argument above. The global adjoint is bounded, so their observables converge uniformly. Protected cap times have convergent subsequences in one compact interval inside $(0, T_f)$. The common early margin and integrated positive crossing slope show that these limits are first caps, with coordinates $+\delta$ and $-\delta$. If the total paths were equal, uniqueness of this same fixed passive colour would give the same observable and its same first-cap coordinate, a contradiction. This proves the stated distinction of the two limits. Renormalization and the initial traces come from the compactness theorem. The intermediate quantifier and scope repairs above, including the separate prepared-child history just proved, leave this conclusion unchanged. They must be incorporated into the manuscript; the original source has been preserved.

4 Part IV — Unresolved or fundamental problems

1. **Unrestricted tiny- δ inactive closure.** Corollaries 11.2–11.3 as quantified without a relative-error restriction remain unverified. This report repairs them in the polynomially bounded regime used in the construction; it does not prove that their original broader statements are false. A seed-relative Gaussian estimate might restore that breadth, but no such estimate is claimed here.

2. **Unspecified full-angular spectrum.** No full-angular operator theorem has been verified. The unweighted infinite H^1 direct-sum extension is demonstrably false. Whether a suitable different angular Sobolev realization has a useful stronger statement is a separate problem; it is unnecessary for the finite coefficient arguments.
3. **Global later recrossings.** The proof protects the first crossing in a specified late time window. It neither proves uniqueness of all later crossings nor excludes returns to that coordinate level. Such a strengthened dynamical claim remains unproved and is removed from the scope of the corrected receiver statement.
4. **No main-result counterexample established.** The verified counterexamples concern the scaling, unrestricted source windows, compact-support inference, and inappropriate spectral realization. The annular-hole example also disproves the unqualified multipole domain; the nonlogarithmic horizon example disproves only its unrestricted asymptotic inference. These are not counterexamples to the fixed-datum nonuniqueness theorem. After the stated completions, this review identifies no remaining unproved dependency in that theorem’s finite-sector construction.
5. **Earlier claims outside this version.** Stationary selection, arbitrary fixed-data viscosity uniqueness, a prescribed nonstationary selection, and vortex-wave nonuniqueness are not consequences asserted or reviewed here. No verdict on them should be inferred from this report.

5 Part V — Language and notation corrections

- In the Theorem 3.1 continuation paragraph use “the finite-Sobolev two-dimensional Euler solution with H^k vorticity and H^{k+1} velocity”. For one fixed k , “smooth” should not silently mean C^∞ . The later all-finite-jet identification uses the separate ancient uniqueness/time-translation argument, not an intersection of arbitrary smallness thresholds.
- Replace ambiguous “density bound” by its specified norm and “the two states” by the displayed full-state or perturbation comparison.
- Define $L = \log(1/q)$ and $\chi > 0$ in Lemma 17.2. Distinguish the finite inactive matrix weight D from an unspecified log exponent where they occur together.
- Replace “modulus derivative is $1 + O(a)$ ” by $d\rho_M/da = 1 + O(a)$; its time derivative is $\sigma a[1 + O(a)]$.
- Use T_f for the final time horizon and $\mathcal{T}_{\text{adj}}(x)$ or $T_{\text{adj}}(x)$ for the adjoint test to avoid the collision in the Theorem 19.3 proof.
- Write “three separate quarter budgets”, or explicitly combine the window and replacement budgets. Specify a common fixed observable when comparing phases of two subsequences.
- Replace “before the parent cap” by “through the provisional parent density stop” in the actual whole-history argument. Reserve “unique first cap” for the protected first crossing, without implying global no-return behavior.
- No unresolved theorem/equation cross-reference was detected in the original compiled source. The necessary corrections concern mathematical scope and hypotheses rather than numbering.

- The abstract’s fixed-datum conclusion agrees with the corrected proof. Its final sentence about all earlier hypotheses being supplied should be understood in the actual logarithmic-cap, finite-sector regime; it does not repair the broader standalone quantifiers of Corollaries 11.2–11.3 without editing their statements.

Additional exact replacements from the second pass

1. Opening conventions, lines 50–55: write

$$B_m(f) = \int_{\mathbb{R}^2} \bar{z}^m f(x) \, dx \quad (m \geq 0), \quad \int_{\mathbb{R}^2} |x|^m |f(x)| \, dx < \infty.$$

State that every whole moment is used only after its corresponding absolute moment is proved finite. This defines the later B_8 and higher moments with the same complex convention.

2. Lemma 4.2, lines 1357–1363: replace “For one fixed sufficiently small smoothing length” by “As the smoothing length tends to zero”. After the limit add: “In particular, one can then choose a fixed sufficiently small $\epsilon > 0$ with $\gamma_\epsilon \neq 0$.”
3. Before the geometric budget at line 2723 insert: “For this geometric cascade budget fix β and $C_H > 0$, assume $0 < \delta \leq 1$ and $1 \leq H \leq C_H \log(1/q)$, and take a receiver (A_i, R_i) .” Powers of a logarithm then justify the stated strict q^6 margin.
4. Start Lemma 8.1: “Let $\beta \geq 4$, $b \geq 0$, $S \geq 0$ and $0 < q < 1$, set $A_i = \beta^{i-1}$, and fix $j \geq 2$.” In Corollary 8.2 write “Let $0 < \delta \leq 1$ and $b = \delta e^{-\sigma S}$ ” and “For fixed $h_0 > 0$ and $H = h_0 \log(1/q)$ ”. These declare the actual positive-comparison and asymptotic domains.
5. Start Lemma 10.1: “Fix $T, \delta, c, e, h, L > 0$, with $2e < T$ ”, and define

$$\tau(a) = \inf\{t \in [0, T] : |a(t)| \geq \delta\}.$$

Its subsequent hypotheses ensure this infimum is finite.

6. Lines 343–345: write “The angular multiplier’s spectrum is imaginary”, so that the pronoun cannot refer to the coupled unstable operator. Lines 435–437: write “The radial vorticity W_κ is an exact Navier–Stokes solution, with tangential velocity U_κ .”
7. Proposition 9.1, line 2934: use “Every such solution is renormalized”, consistent with the preceding “at most one” wording. Part III now supplies the initial-trace justification explicitly.
8. The Gaussian tail with denominator $4\kappa t$ at lines 683–687 is displayed for $0 < t \leq H$; the zero-time tail follows separately from initial compact support.
9. Lines 82–84 and 6417–6419: say that changing a phase changes the *datum* by at most the stated norm bound. The proof controls the norm of the difference of the data, which is what continuous dependence requires.
10. Static verification found 235 distinct labels and 307 `ref/eqref` uses, with no duplicate or unresolved target, and no undefined bibliography key. The unused `CCS21` bibliography entry may be retained as background; its presence is not a mathematical error.

6 Part VI — Final verdict

1. **Verified main results.** The isolated finite-sector activation, canonical ancient orbit and transform, actual cold-tail suppression, strong fixed-data compactness, selected-source construction, actual receiver and depth-uniform phase-selection mechanism are supported by the audited arguments with the explicit proof completions in Part III. The final fixed-datum nonuniqueness conclusion is retained.
2. **Valid after stated corrections.** The physical norm scaling, positive-window lemma, Gaussian-energy justification, corrected inactive-closure regime, abstract passive-colour formulation, whole prospective stop, clock and protected-window specifications.
3. **Still unverified beyond the construction.** Arbitrarily smaller inactive caps without a relative Gaussian condition, unspecified full-angular spectral realizations, and global no-return assertions.
4. **Demonstrably incorrect readings.** The intersection bound $CAR^2\delta_q$ with scale-independent constant; the source-window bound for unrestricted negative U ; compact support from strong L^1 alone; the multipole series on an annular hole; the unbounded-horizon reading of the strict q^6 budget; and the proposed full unweighted infinite H^1 direct-sum spectral extension. None is needed after the corrections.
5. **Recommendation.** The draft is not mathematically sound in every statement as written. It is supportable after the explicit mathematical revisions supplied here, with no weakening of Theorem 19.3. The unchanged manuscript still requires these revisions; this report is not a certification that every original standalone statement is true. The restricted intermediate statements and the completed stopping and weighted-energy arguments should be incorporated before circulation as a finished paper. This is a referee conclusion about this version, not a blanket assurance about related or earlier drafts.

References

- [1] T. Gallay and C. E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier–Stokes and vorticity equations on $\mathbb{R}^2$* , Arch. Ration. Mech. Anal. 163 (2002), 209–258. Primary preprint: <https://arxiv.org/abs/math/0102197>.
- [2] G. Ciampa, G. Crippa and S. Spirito, *Strong convergence of the vorticity for the 2D Euler equations in the inviscid limit*, primary preprint <https://arxiv.org/abs/2008.12133>; Theorem 3.2.

A Detailed section audit records

The records below were made directly against the immutable source. They intentionally describe each local result as an implication from its stated inputs. Their local “not certified here” qualifications refer to that individual audit’s boundary; the overall dependency check and disposition are in Parts I, III and VI. These qualifications must not be read as deleting a hypothesis or as certifying an external article in its entirety. The records preserve the counterexample searches and derivations underlying the report.

A.1 Sections 1–2: datum, spectrum and isolated activation

Audited snapshot: paper-snapshot.tex, lines 1–745, dated 8 October 2026. This audit was performed directly from the snapshot, without reading any previous audit. The snapshot has not been modified. The specific operator-space check was independently reread by a subordinate spectral auditor; its final observations are incorporated below.

This report separates the isolated statements proved here from the interacting fixed-data construction. The latter is not a consequence of the isolated activation theorem. No estimate, hypothesis verification, or nonuniqueness claim from Sections 3–19 is certified by this report.

A.1.1 Findings requiring changes

S1. Physical density scaling omits the L^p scaling factor **Location:** lines 728–741, Remark smooth-active-scope; especially lines 729–734.

Exact text:

For a physical colour $A\omega(x/R)$ and viscosity $\nu = \kappa_q AR^2$, the cap time is H_q/A , the density bound is $CA R^2 \delta_q$, and the cap moment is $AR^6 \delta_q e^{i(\vartheta + \tau_\epsilon H_q)} b_\epsilon + o(AR^6 \delta_q)$. The integrated perturbation velocity bound is $CR \delta_q$. These normalized constants do not depend on the scale or amplitude.

Classification: repairable norm ambiguity; the stated scaling is **demonstrably false** if “the density bound” denotes the theorem’s L^1 intersection L^p sum norm, as in lines 144–158. It is correct if explicitly restricted to L^1 .

Write $F(t) = \omega(t) - \exp(\kappa \Delta t) W$ and set $F_{\text{phys}}(t, x) = A F(At, x/R)$. Exactly,

$$\|F_{\text{phys}}(t)\|_1 = AR^2 \|F(At)\|_1, \quad \|F_{\text{phys}}(t)\|_p = AR^{2/p} \|F(At)\|_p.$$

Thus the correct consequence of smooth-active-density is

$$\sup_{0 \leq t \leq H_q/A} \|F_{\text{phys}}(t)\|_{L^1 \cap L^p} \leq CA(R^2 + R^{2/p}) \delta_q.$$

There is an actual counterexample to a scale-independent sum-norm bound $CA R^2 \delta_q$, not merely a dimensional objection. Fix one sufficiently small q and one phase. The seed is nonzero: otherwise the distinct eigenfunctions ϵ and $\overline{\epsilon}$ would be proportional, despite eigenvalues with opposite real parts. At physical time zero,

$$\|F_{\text{phys}}(0)\|_p = A \alpha_q R^{2/p} \|\Phi_\vartheta\|_p.$$

For any finite purported scale-independent C , the ratio to $CA R^2 \delta_q$ tends to infinity as R tends-down-to 0, since $2/p - 2 < 0$ and $\alpha_q > 0$ is now fixed.

Correction: explicitly give separate L1 and Lp bounds, or the sum-norm bound above. The cap time, moment, viscosity, and integrated velocity scaling are all correct. A complete scaling derivation and the counterexample are in ref-spectrum-corrections.tex.

Effects: this correction does not change Theorem smooth-active-theorem, proved in normalized variables. No later citation of Remark smooth-active-scope occurs in the snapshot. The later physical bound at line 3841 explicitly uses the correct $A_j R_j^{2/p}$ factor. I did not certify all later physical norm calculations: any use of $A_j R_j^{2\delta}$ as an Lp or sum-norm estimate would require correction and could change summability conditions. The existing stack assumption $\beta q^{2/p} < 1$ correctly accommodates both factors.

S2. State the ambient spectral space before the spectrum claim **Location:** Lemma smooth-active-spectrum, lines 233–254.

Exact text:

One can choose a fixed sufficiently small $\epsilon > 0$ and $0 < \eta < \sigma_\epsilon/4$ so that in the quarter-turn symmetry class: ... all other positive-real-part spectral values, except the conjugate growing eigenvalue in $m = -4$, have real part at most η .

The lemma subsequently says:

For each finite set of sectors in $4\mathbb{Z}$ and each finite radial Sobolev order k , work on the direct sum of the spaces $H^k(I)$,

Classification: repairable scope/definition issue if the first sentence is meant to make a claim about the full infinite-angular operator. A spectrum is a property of a specified closed operator on a specified space. No such full angular realization or domain is specified here. The proof does establish the claimed spectral statement for every individual sector and every finite direct sum of sectors, with constants depending on that finite set. That is all the finite coefficient construction in Section 2 needs.

Correction: introduce $L_m: H^k(I) \rightarrow H^k(I)$ first, say the eigenvalue exclusion is uniform over positive sectors m in $4\mathbb{N}$, and formulate the propagator estimate only on a fixed finite direct sum. This avoids asserting an additional theorem about the spectrum of an unspecified infinite direct sum. If a full angular realization is needed later, its domain and resolvent estimates must be supplied separately.

Effects: no change to the finite coefficient argument. Uniformity of the high-mode eigenvalue exclusion must not be conflated with a uniform-in-all-modes semigroup bound in a two-dimensional Sobolev space. Such a stronger estimate has not been verified here.

Concrete reason the ambient space cannot be omitted: the first sentence is **demonstrably false** on the naive infinite unweighted Hilbert direct sum $Y = \text{direct-sum}_{m \in 4\mathbb{Z}} H_1(I)$, with its maximal closed direct-sum operator domain. Consider a fixed real $\lambda = 1$, which lies to the right of every modal eigenvalue by the high-mode quadratic-form bound. In $L_2(I)$, write $L_m = A_m + D_m$, where $A_m = -i\Omega$ and $\|D_m\|_{L_2 \rightarrow L_2} \leq \epsilon/m$. Indeed $h_m(r, s) \leq \exp(-m|r-s|/\sup I)$, and the remaining kernel factors are bounded for fixed ϵ ; Schur's test gives the bound. Thus

$$\|(1 - L_m)^{-1} - (1 - A_m)^{-1}\|_{L_2 \rightarrow L_2} \rightarrow 0.$$

For the fixed input 1 in $H_1(I)$, dominated convergence gives

$$(1 - L_m)^{-1}1 \longrightarrow \mathbf{1}_{\{\Omega=0\}} \quad \text{in } L^2(I).$$

The radial circulation Ω is strictly negative immediately inside the actual outer support boundary and identically zero outside it. The limit therefore jumps at that boundary and does not belong to $H^1(I)$. Were the modal resolvents uniformly bounded in H^1 , their values on the fixed input would have a weakly convergent H^1 subsequence, whose L^2 limit would belong to H^1 , a contradiction. A bounded resolvent of the maximal direct-sum operator on Y would restrict to uniformly bounded modal resolvents, so 1 belongs to the spectrum of this full realization despite belonging to no modal spectrum. The exclusion can be made with any fixed positive real parameter larger than the modal real spectral bound.

For completeness, the asserted strict sign near the boundary uses no special positivity assumption omitted from the paper. A nonzero nonnegative radially decreasing χ in $C_\infty(B_1)$ is positive at every radius below its support radius $a < 1$. Near the outer support edge $r_e = 1 + \epsilon$, the inner-disk convolution is zero and the outer-disk convolution is strictly positive for $r < r_e$. Since total circulation is zero, $\int_0^{r_e} sW(s)ds = -\int_{r_e}^1 sW(s)ds < 0$ there. Both sides of r_e lie in the chosen interval I .

This counterexample does **not** invalidate any individual-sector assertion or any finite-sector propagator in Section 2. It shows why one cannot supply the omitted ambient realization by silently taking the unweighted infinite direct sum of the spaces actually written in the lemma. A full isotropic angular Sobolev space uses angular weights and is a different question.

A.1.2 Per-result status and verification

The classifications “verified” below refer to the stated finite-sector realization. They mean that the displayed assertion has been checked by the argument indicated; they do not certify subsequent applications with external forcing or infinitely many modes.

- **Lines 43–49, cont-NS. Status:** Verified. **Verification and limits:** With $x_{\text{perp}} = (-x_2, x_1)$, $u = \text{grad}_{\text{perp}} \psi$, $\Delta \psi = \omega$. The polar radial velocity is $-r^{-1} \partial_{\theta} \psi$, matching every later sign..
- **Lines 50–56, B4, quarter-turn symmetry, radial averaging. Status:** Verified. **Verification and limits:** $\bar{z}^4 = r^4 e^{(-4i\theta)}$ selects exactly the positive fourth sector. Radial averaging is a contraction in each L_r ; $I\text{-Prad}$ has norm at most two, absorbed into later constants..
- **Lines 57–62, GW01 inputs. Status:** Verified against primary source. **Verification and limits:** Gallay–Wayne Theorem 2.2 states global unique L^1 vorticity solutions with positive-time L^∞ ; Lemma 2.1 contains the HLS, velocity L^∞ , and gradient estimates used. Full positive-time smoothness follows by the parabolic bootstrap for positive viscosity. Source: <https://arxiv.org/pdf/math/0102197>, Section 2..
- **Lines 66–93, cont-stack-data. Status:** Verified. **Verification and limits:** $\beta_{2/p} < 1$ implies $\beta_2 < 1$ and $\beta_6 < 1$; all series claimed converge absolutely. Common compact support and the moment bound justify termwise integration. The lemma claims an L^1 intersection L_p datum, not that the infinite sum is smooth at the accumulation point..

- **Lines 120–159, smooth-active-theorem. Status:** Verified by the finite-sector proof below. **Verification and limits:** The conclusion is for a family of isolated data as q tends-down-to 0. It does not yield a fixed-datum limit or coupled activation. $H_q > 0$ for sufficiently small q . All parameters can be fixed in the order specified without circularity..
- **Lines 168–183, mollified profile and smooth-active-layer-signs. Status:** Verified. **Verification and limits:** Let $D_{\{a, \epsilon\}} = \partial r(\chi_\epsilon * 1_{\{B_a\}})$. Disk convolution is radial nonincreasing, equals one at zero since $\epsilon < a$, and vanishes outside $B_{\{a+\epsilon\}}$. Hence its derivative is nonnegative, collar-supported, with radial derivative mass one. The derivative signs and the zero circulation follow exactly..
- **Lines 194–209, smooth-active-omega-limit. Status:** Verified. **Verification and limits:** $\|W_\epsilon - W_0\|_1 \leq C\epsilon$. At radii bounded away from zero this controls radial circulation uniformly; Ω_0 is Lipschitz there. Direct values are $\Omega_0(\rho) = -(1-\rho^2)/2 = -1/5$ and $\Omega_0(1) = 0$..
- **Lines 211–225, Green kernel and smooth-active-generator. Status:** Verified. **Verification and limits:** Across $r=s$, the derivative jump of $-h_m/(2m)$ is $1/s$, so the kernel gives the right radial delta normalization. Substitution of $u_r = \text{im}\psi/r$ gives $L_m f = \text{im}\Omega_g f + \text{im}W'\psi/r$..
- **Lines 226–230, radial and conjugate sectors. Status:** Verified. **Verification and limits:** Two radial fields do not advect each other. The real full operator maps negative sectors to conjugates of positive ones. Quarter-turn symmetry restricts modes to $4\mathbb{Z}$..
- **Lines 232–395, smooth-active-spectrum. Status:** Verified for the specified finite-sector spaces; S_2 scope correction. **Verification and limits:** High-mode exclusion, finite determinant reduction, algebraic simplicity, and bounded-generator semigroup estimate are detailed below. No full infinite-angular operator theorem is certified..
- **Lines 280–284, smooth-active-highmode. Status:** Verified. **Verification and limits:** In $L_2(|W|dr)$, diagonal collar blocks are skew-adjoint. The off-diagonal block norms are at most $(\rho/2)a_*^m$; thus $\text{Relambda} \leq C a_*^m$, uniformly in ϵ ..
- **Lines 288–333, smooth-active-evans, interface matrix and roots. Status:** Verified. **Verification and limits:** The Schur complement is a legitimate uniformly small complementary factor on compact subsets of $\text{Relambda} \geq \epsilon$. The limiting matrix is the displayed M_m . At $m=4$, its roots are $\pm \sqrt{-\rho/5}/2 + 3i/10$; at $m \geq 8$ in $4\mathbb{N}$ the discriminant is strictly positive, giving imaginary roots. Large $|\text{lambda}|$ is excluded uniformly..
- **Lines 357–379, smooth-active-woodbury and multiplicity transfer. Status:** Verified. **Verification and limits:** $C:L_2(I) \rightarrow C(\text{collars})$ and $B:C(\text{collars}) \rightarrow L_2(I)$ are bounded; the Woodbury identity follows by substitution. A simple reduced determinant root produces a rank-one simple resolvent pole. Elliptic bootstrap identifies the same pole and eigenspace on each $H_k(I)$..
- **Lines 250–256, 381–393, smooth-active-semigroup. Status:** Verified for a fixed finite sector set. **Verification and limits:** The generator is bounded on that space. After removing the simple growing projections its spectrum lies left of ϵ ; a rectangular Dunford contour with right edge 2ϵ gives $Ce^{\{2\epsilon t\}}$. The contour and constant depend on the

chosen finite sector set and derivative order. Support preservation follows from multiplication plus coupling supported where W' is supported..

- **Lines 398–417, eigenfunction normalization and smooth-active-b. Status:** Verified. **Verification and limits:** The growing matrix eigenvector is $(1, -i\rho)$. The normalized radial measure tends to $-\rho^2 \delta(\rho - i\rho^3)$; therefore $\epsilon \rightarrow 2\pi(-\rho^6 - i\rho^3) \neq 0$. Since $L_4 = i$ times a real operator, $\bar{\epsilon}$ has eigenvalue $-\bar{\epsilon}$.
- **Lines 419–430, smooth-active-neutral-seed. Status:** Verified exactly. **Verification and limits:** The positive-fourth coefficient has moment $e^{i\theta}(b - (b/\bar{b}))\bar{b} = 0$. Its conjugate sector contributes no B_4 ; circulation vanishes. The growing coefficient is exactly $e^{i\theta}$ and the stable coefficient has modulus one. The resulting seed is nonzero and quarter-turn invariant..
- **Lines 438–443, smooth-active-KX. Status:** Verified. **Verification and limits:** The near kernel belongs to L_p precisely because $p > 2$; the far kernel is bounded. This proves $\|K^*g\|_\infty \leq C(\|g\|_1 + \|g\|_p)$ without a circulation restriction..
- **Lines 449–475, smooth-active-app and smooth-active-recursion. Status:** Verified. **Verification and limits:** Substitution into the perturbation equation gives every displayed sign and index. All sources precede the coefficient in total degree $j + \ell$. They preserve support in the collars, reality, zero circulation, and modes $|m| \leq 4j$..
- **Lines 477–552, smooth-active-residual. Status:** Verified. **Verification and limits:** Pure coefficients have a nonresonant exponential convolution gap for $j \geq 2$; viscosity coefficients have only finitely many resonances/polynomial factors. A finite derivative ladder suffices. Exact radial heat treatment prevents seed-free residual terms. The Taylor remainder and uncanceled monomials yield the stated bound..
- **Lines 558–576, smooth-active-error-equation, smooth-active-X-gronwall. Status:** Verified. **Verification and limits:** Subtract the equations using the actual velocity for transport. Kato and the L_p convex inequality remove transport and dissipate diffusion; $K^*\zeta$ is estimated by $\|\zeta\|X$. No actual-vorticity derivative enters the Gronwall constant..
- **Lines 578–625, parameter choice, smooth-active-N-choice, smooth-active-X-error, smooth-active-X-endpoint. Status:** Verified. **Verification and limits:** C_1 is fixed before expansion orders; the cap then makes $\|\text{grad fapp}\|X \leq 1$. Choose $(M+1)\sigma > C_1$ and $(3/2)(N+1) > K_*r_1$. The convolution gives the exact exponent $r_1 = (C_1/\sigma - 1)_+$, and its endpoint divided by δ_{q} tends to zero..
- **Lines 629–636, smooth-active-NR. Status:** Verified. **Verification and limits:** W_κ is radial, and $I\text{-Prad}$ is bounded in L_1 . This proves the bound for the actual isolated solution. Its use for interacting colors would need a new comparison argument..
- **Lines 640–651, smooth-active-integrated-drift. Status:** Verified. **Verification and limits:** All powers of $a(t)$ integrate exponentially to their endpoint without an additional factor H . The viscosity remainder has positive exponential rate $\max(C_1, \sigma)$..
- **Lines 661–678, smooth-active-exponential-tail. Status:** Verified. **Verification and limits:** Tangential U_κ drops out of the radial test. d has gradient at most one and Laplacian at most two. The bounded logistic truncation obeys the two stated derivative upper bounds; compact-cutoff errors vanish by bounded drift and finite mass..

- **Lines 680–698, Gaussian mass tail and smooth-active-fourth-tail. Status:** Verified. **Verification and limits:** The optimizing parameter is positive for $s \geq R$ when q is small. The Gaussian weighted-tail integral is uniformly bounded after reducing its exponential constant. At $t=0$, use the compact support directly, rather than the expression with denominator $kappa$. All actual whole fourth moments are finite..
- **Lines 700–726, whole-moment cap calculation. Status:** Verified. **Verification and limits:** Radial heat pairs to zero. The stable term has size $\alpha^{\{-\sigma H\}} = \alpha^2/\delta$. Higher pure orders give $O(\delta^2)$; viscosity orders give $O(kappa(1+H)^{D\delta})$; the compactly tested error is bounded in $L1$; the tail is exponentially small. Each divided by δ_q tends to zero..
- **Lines 728–741, smooth-active-scope. Status:** Repairable; S1. **Verification and limits:** Every scaling other than the ambiguous density norm is correct. The stated limitation to an isolated color is correct and necessary..

A.1.3 Detailed spectral checks

The uniform cutoff does not come from a norm bound on the original large-angular-frequency generator. It comes from the eigenvalue equation. For $\text{Relambda} > 0$, f vanishes off the derivative collars and can be written $fr = gW'$. Then

$$\lambda g = -im\Omega g - \frac{i}{2} \int h_m(r, s) g(s) W'(s) ds.$$

On the inner collar the signed kernel is $-i/2$ times a real symmetric kernel; on the outer it is $+i/2$ times such a kernel. These same-collar blocks and the angular multiplier have zero real quadratic form. With collar masses 1 and ρ_2 , either cross block has norm at most $(\rho/2)a_-^* m$. Taking the real part of the eigenvalue pairing gives the bound in the paper. The representation involves no division by W' , so zeros of W' do not invalidate it.

For fixed m , on $C(J_{\{1, \epsilon\}} \cup J_{\{2, \epsilon\}})$, put $Q = I - EP$. Kernel Lipschitzness and $\text{Relambda} \geq \eta$ give $\|QT_\epsilon(\lambda)\| \leq C_m \epsilon$ on every compact set. Writing $\psi = Ep + w$, the complement equation is

$$w = (I - QT|_{\ker P})^{-1} QTEp.$$

The resulting matrix is $I - PTE - PT(I - QT)^{-1} QTE$. The second term tends to the interface kernel and the last term is $O(\epsilon)$. Its determinant therefore converges uniformly and analytically on those compact sets. Because the reference determinant has no zeros on the boundary of a suitable right-half-plane region and exactly one simple zero inside near the unstable fourth root, the argument principle gives one zero counted with algebraic multiplicity. Large $|\lambda|$ and the uniform angular cutoff exhaust all sectors in the half-plane. Choose η beforehand, for example smaller than $\rho_5/16$, and then ϵ sufficiently small; this removes any perceived circularity in the paper's “take η smaller first” phrase.

The original coupling is compact on every fixed $H_k(I)$: the potential gains two derivatives locally and the factor W' places the output away from interval endpoints. Outside the imaginary multiplier range, Fredholm reduction excludes any positive spectrum other than these smooth eigenfunctions. The simple reduced zero yields a simple pole by the explicit Woodbury identity. No claim that the operator norm of the semigroup equals its spectral radius is used; the bounded-generator contour argument is valid.

A.1.4 Detailed finite expansion and closure checks

The exact perturbation equation relative to the radial heat base is

$$(\partial_t - \mathcal{L})f = \kappa \Delta f - (K * (W_\kappa - W)) \cdot \nabla f - (K * f) \cdot \nabla (W_\kappa - W) - (K * f) \cdot \nabla f.$$

Collecting the α^j coefficients gives exactly the stated recursion. Each individual velocity factor can extend beyond the collars, but each product has a differentiated collar-supported scalar. The diffusion source is collar-supported too. Thus none of the coefficients acquires a spatial tail. The radial heat Taylor remainder is the only noncompact scalar factor in the residual.

For the pure coefficients, $j=1$ is an exact combination of eigenfunctions. At $j \geq 2$, forcing grows at most $e^{\{j\sigma\}}$. The kernel of the sector propagator grows at most $e^{\{\sigma(t-s)\}}$ after including the finite-set remainder; the integral is bounded by $e^{\{j\sigma\}}/((j-1)\sigma)$. For positive viscosity order, the resonant first-growing coefficient may acquire a polynomial. Since there are finitely many coefficients, finitely many derivative losses and finitely many resonances, one finite polynomial exponent suffices for each requested norm. A zero circulation compactly supported coefficient has $|\hat{f}(\xi)| \leq C|\xi|$, which controls the Biot–Savart low frequency in Sobolev estimates.

The error equation contains the full nonlinear actual velocity only in divergence-free transport. Therefore the comparison coefficient is controlled by the approximate solution. Choosing the pure seed cap after fixing M does not affect the already fixed $C1$, and choosing a larger N affects only the small-viscosity threshold and constants. There is no bootstrap on a derivative of the actual NS solution.

Let $L = \log(1/q)$. Then

$$H_q = (K_* L - 4 \log L)/\sigma = O(L), \quad \delta_q = L^{-4}.$$

The viscosity error divided by the cap is bounded by a constant times

$$q^{\frac{3}{2}(N+1) - K_* r_1} (1 + H_q)^{D+1} L^{-4r_1},$$

which tends to zero by the strict order condition. The stable moment error divided by the cap is $q^{\{2K_*\}} L^8$; the first nonlinear moment error divided by the cap is $O(L^{-4})$; the heat/viscosity correction is $O(q^{\{3/2\}} L^D)$. The tail is $L^4 \exp(-c/(q^{\{3/2\}} H_q))$. Each tends to zero. These are quantitative checks, rather than an assumption that a formal expansion converges.

A.1.5 Scope of dependencies not verified

The assertions at lines 102–106 that later interacting estimates verify all preliminary hypotheses, the ancient theorem starting at line 746, the forced-response calculations, all packet/core estimates, descending induction, phase degree, compactness for the actual infinite stack, and the final fixed-data conclusion are outside this audit. I read only brief later passages containing direct Section 2 references, plus the bibliography, to identify possible consequences of S1 and the intended finite-sector spectral use. Those passages have not thereby been certified.

In particular, the isolated density and moment estimates cannot be transported into a coupled system merely by replacing the isolated seed amplitude by a contour or protected-coordinate amplitude. This limitation is explicitly and correctly stated at lines 735–740. Any coupled use needs its own source and viscosity residual estimates and an actual-vorticity comparison argument.

The exact neutral seed eliminates only the initial whole B4. It does not eliminate forcing moments at later times or prove that a nonlinear normalized outgoing profile is independent of

the waveform that generated it. Those are additional claims, not consequences of the spectral and isolated results audited here.

A.2 Sections 3–4: ancient orbit and transfer

This audit was made directly against paper-snapshot.tex, without consulting any previous audit. The original manuscript was not modified. It concerns the isolated ancient orbit and prescribed receiver; it does not certify the later interacting cascade. A subsidiary independent calculation of the signs and limiting interface constants agreed with the calculations below.

Conclusion: The principal §§3–4 results close, conditional on the finite-sector spectral/propagator and finite coefficient results in §2. I found a literal missing parameter-domain hypothesis in the window lemma (R3), with an actual canonical-source counterexample if negative windows are allowed. The natural intended positive-window statement is correct. The high Sobolev argument uses the actual error equation, ancient uniqueness uses an admissible low norm, the transform noncancellation is sound, and the prescribed receiver has a closed finite expansion. Two presentation repairs and the necessary R3 hypothesis are recorded below. The accompanying TeX supplies rigorous replacement arguments.

A.2.1 Dependencies and certification boundary

I inspected §2 directly where it is used:

- Lines 238–265 fix the Green kernel, velocity, and frozen generator conventions. These were rederived for the sign checks.
- Lines 267–412 give smooth-active-spectrum and smooth-active-semigroup. I checked the matrix algebra needed here, support preservation, the conjugate stable eigenfunction in the positive fourth sector, and the finite-sector nature of the argument. I have **not independently certified the full Evans/Fredholm/Rouché analysis** or all operator-domain assertions. §§3–4 remain conditional on that lemma until the separate §2 audit certifies it.
- Lines 414–441 normalize the eigenfunction, moment, and seed. I checked the exactly neutral moment and growing-coordinate normalization.
- Lines 449–559 give the finite recursion and seeded residual. I checked triangular order, sector count, collar support, zero circulation, and the heat-remainder mechanism; coefficient estimates retain the propagator dependency just stated.
- Lines 561–731 give the $X = L^1 \cap L^p$ comparison and exponential tails. I rechecked the Kato comparison and radial exponential test used below.

No all-mode Sobolev spectral-growth theorem or convergent infinite series is needed in §§3–4.

A.2.2 Issues and clarifications

R1 — finite regularity language **Location:** lines 1091–1093, proof of smooth-ancient-theorem.

Exact quote: “The ancient limit has a smooth enough, compactly supported initial state at zero. The classical smooth two-dimensional Euler construction extends it uniquely to all $s \geq 0$.”

Category: minor regularity precision.

For a fixed $k \geq 3$, the construction proves H^k vorticity and H^{k+1} velocity, not automatic C^∞ regularity at an already selected fixed positive cap. Running the construction at more derivative orders can require smaller cap thresholds. Their intersection need not contain that cap.

Rigorous repair: refer to the classical finite-Sobolev Euler construction, state the H^k and H^{k+1} regularities, and use the stated logarithmic continuation estimate. These regularities suffice for all transport, support, moment, uniqueness, and second-derivative arguments here. The accompanying TeX gives this argument.

Downstream effect: none for the actual results here. A later claim of arbitrary derivative bounds on the canonical orbit needs its own argument. The prescribed drift is spatially C^∞ because it is a fixed smooth field multiplied by a scalar moment; this does not require a C^∞ source trajectory.

R2 — grouping of one-quarter error budgets **Location:** lines 1517–1522 and 1538–1540, smooth-transfer-window.

Exact quotes: “Consequently source windows and errors smaller than $|\mathcal{F}(\mu)|/4$ preserve a nonzero growing transmission.” “The two strict one-quarter bounds leave a nonzero coefficient.”

Category: minor quantitative exposition.

If the truncation error and replacement error are each below $|\mathcal{F}|/4$, and the old coefficient obeys its displayed separate quarter bound, there are three budgets. The modified transform has modulus $> |\mathcal{F}|/2$, so the final coefficient is $> q^6 |\gamma| e^{-\sigma s^*/\beta} |\mathcal{F}|/4$. If the first two errors are intended as one combined budget, say so explicitly.

Rigorous repair: name E_{tail} and E_{src} , impose individual quarter bounds, and write the triangle inequality with the separate old coefficient. This is done in the TeX.

Downstream effect: no change to $K_*(1 - 1/\beta) > 6$, but applications must track individual versus combined error budgets.

R3 — missing nonnegative window hypothesis **Location:** lines 1504–1510, smooth-transfer-window.

Exact quote: “Truncating the source interval to $[-S, U]$ changes its transform by at most $Ce^{-(\sigma - \Re \mu)S} + C_\beta(1 + U)^4 e^{-\Re \mu U}$.”

Category: missing parameter-domain hypothesis; false under a literal unrestricted reading.

The lemma does not specify $S, U \geq 0$. At $U = -1$, the second displayed term is zero, but truncating at -1 generally discards a nonzero integral over $[-1, \infty)$. This is not merely a hypothetical moment example: take the canonical orbit with a sufficiently small fixed cap, so $B(-1) \neq 0$ by its leading asymptotic. The analytic function

$$G_{-1}(z) = \int_{-1}^{\infty} e^{-zu} B(u) du, \quad \Re z > 0,$$

is not identically zero: on real $z \rightarrow +\infty$,

$$ze^{-z} G_{-1}(z) \longrightarrow B(-1) \neq 0.$$

The transform $\mathcal{F}(z)$ is also not identically zero. Their zeros on the injective μ_β curve are discrete, so choose a $\beta > 1$ for which both are nonzero. Put $U = -1$ and let $S \rightarrow \infty$. The actual truncation error tends to $|G_{-1}(\mu_\beta)| > 0$, whereas the asserted upper bound tends to zero. Thus the unrestricted formula is false.

Rigorous repair: insert “for $S, U \geq 0$ ” before the window bound. This is the intended regime for negative/future truncation. Alternatively, a formula for arbitrary real endpoints needs a separate bounded central contribution and a positive-tail factor involving $1 + U_+$, rather than $(1 + U)^4$.

Downstream effect: none for the prescribed theorem, which uses endpoints $-\beta t_0$ and βS_q with $t_0, S_q > 0$ for small q . Later uses of the lemma must make their positive window sizes explicit.

A.2.3 Scope limits, not defects in the current statements

1. **Preparation amplitude versus exact cap.** Lines 757–760 explicitly disclaim an exact first hitting time and state $\mathcal{A}(\Omega_\delta(0)) = \delta + O(\delta^2)$. At fixed positive δ , this discrepancy need not vanish as $q \rightarrow 0$. Any later exact-cap identification needs an additional time/phase shift or separate argument. The cap-translation corollary translates the ancient growing amplitude, not a first-crossing normalization.
2. **Uniqueness class.** Lines 1069–1075 use the whole finite jet and order $M + 1$ remainder, with $(M + 1)\sigma > E_k$. They do not prove uniqueness from the leading exponential alone. The same argument does apply to any bounded-support, bounded- H^k ancient orbit with this same jet, so it does cover cap translation.
3. **Nonzero selection, not a uniform lower bound.** Transform zeros along the β curve are discrete. This permits a fixed nonzero choice; it supplies neither a uniform lower bound for all large β nor exclusion of accumulation at the boundary $z = 0$.
4. **Prescribed input versus actual child.** Lines 1620–1623 and 1771–1782 explicitly preserve this distinction. The prescribed theorem does not estimate actual higher multipoles, support leakage, or feedback.
5. **Centered rotation clock.** The prescribed field uses $e^{4i(t-t_0)/5}$, whereas the physical formula uses $e^{4it/5}$. The difference is the unit source phase $e^{-4it_0/5}$. The centered prescribed calculation is internally consistent; identifying it with an unshifted physical child requires tracking that phase.

A.2.4 Per-result status table

“Verified conditional” means that the §§3–4 proof closes using the stated §2 finite-sector semigroup and coefficient inputs. It does not certify the unreviewed portions of §2 or later coupled sections.

- **smooth-ancient-high-residual, 814–843. Status:** Verified conditional. **Checked calculation or obligation:** Pure-seed constant is independent of viscosity order; all noncompact heat couplings retain the seed factor..
- **smooth-ancient-kernel-high / kernel-derivative, 854–863. Status:** Verified. **Checked calculation or obligation:** Near/far split bounds $K : L^1 \cap H^k \rightarrow L^\infty$; ∇K is an order-zero multiplier. No circulation restriction needed..
- **smooth-ancient-high-gronwall, 845–905. Status:** Verified. **Checked calculation or obligation:** Actual error requires at most k derivatives; one extra derivative falls only on the smooth approximation. C_k is fixed before expansion orders..
- **smooth-ancient-high-error, 907–941. Status:** Verified. **Checked calculation or obligation:** Pure convolution denominator positive; viscosity exponent is $\frac{3}{2}(N + 1) - K_* r_k > 0$; fixed-cap bootstrap closes..
- **smooth-ancient-coefficient-limits / homological, 943–984. Status:** Verified conditional. **Checked calculation or obligation:** Rotation shifts both growing sectors to exponent σ ; normalized propagator decays at least $(j - 1)\sigma$. Homological inverse sign correct..
- **smooth-ancient-polynomial-error / compactness, 986–1045. Status:** Verified conditional. **Checked calculation or obligation:** Exact cap-phase cancellation; viscosity

terms vanish; tail plus cutoff interpolation supplies global compactness; lower semicontinuity supplies the Z_k remainder..

- **smooth-ancient-theorem: uniqueness and K_* -independence, 1047–1088. Status:** Verified. **Checked calculation or obligation:** X -stability only needs comparison first derivatives; $(M + 1)\sigma - E_k > 0$ sends backward error to zero..
- **smooth-ancient-theorem: forward extension / moments, 1090–1127. Status:** Verified with R1 precision. **Checked calculation or obligation:** Global H^{k+1} velocity continuation; bounded velocity and linear support growth imply the polynomial fourth moment..
- **smooth-ancient-positive-window, 1129–1195. Status:** Verified. **Checked calculation or obligation:** Difference forcing has $+\kappa\Delta\Omega$; $k = 3$ suffices for all finite p . Positive domination plus endpoint Gaussian moments removes the cutoff..
- **smooth-ancient-transform, 1197–1240. Status:** Verified. **Checked calculation or obligation:** Pole $\delta b/(\lambda - z)$; analytic remainder on $0 < \Re z < 2\sigma$; isolated zeros give discrete exceptional β ..
- **smooth-ancient-cap-translation, 1242–1269. Status:** Verified. **Checked calculation or obligation:** Rotation $\mathcal{T}_{-\tau s_0}$; transform factor $e^{(z-i\tau)s_0}$; modulus $(\delta'/\delta)^{1/\beta}$..
- **smooth-transfer-adjoint, 1292–1340. Status:** Verified conditional. **Checked calculation or obligation:** Bilinear density $2\pi a(r)r dr$, adjoint factor $-i/2$, nonvanishing denominator, harmonic core/exterior extension, annihilation of stable mode..
- **smooth-transfer-coefficient, 1342–1401. Status:** Verified conditional. **Checked calculation or obligation:** $F = -iW'/(4\pi r^5)$; vector $h = (i/(4\pi\rho^6), i/(4\pi))$; $\gamma_0 = (i\rho^{-6} - \rho^{-1})/(8\pi) \neq 0$. Schur limit uses §2..
- **Cubic interior comparison, 1403–1418. Status:** Verified conditional. **Checked calculation or obligation:** Vector $(\rho^2/2, 1/2)$ gives $\gamma_0^{\text{out}} = (\rho^2 + i/\rho)/4$..
- **smooth-transfer-duhamel, 1420–1454. Status:** Verified. **Checked calculation or obligation:** Dimensional moment $A_c R_c^6$; βq^6 forcing; $e^{4is/5}$ rotation; $\mu_\beta = (\lambda - 4i/5)/\beta$; substitution cancels β ..
- **smooth-transfer-transform, 1456–1490. Status:** Verified. **Checked calculation or obligation:** Nonremovable pole forbids identical zero; injective μ_β curve lies in the convergence strip for $\beta > 1$..
- **smooth-transfer-window, 1502–1544. Status:** Requires R3 hypothesis; otherwise verified with R2 clarification. **Checked calculation or obligation:** Formula fails for unrestricted $U = -1$. For $S, U \geq 0$, tail and replacement estimates, coefficient scale, and strict power condition are correct..
- **smooth-transfer-heat-shield, 1546–1562. Status:** Verified. **Checked calculation or obligation:** Only ± 4 sectors and zero fourth moment give compactly supported velocity; Gaussian heat convolution gives the bound..
- **smooth-transfer-actual-ODE / actual-error, 1564–1600. Status:** Verified. **Checked calculation or obligation:** All five error terms and signs; heat errors $O(\kappa(1+t)\|f\|_X)$; quadratic term $O(\|f\|_X^2)$; foreign terms explicit; rotation $-4i\Omega\mathcal{A}(f)$..

- **smooth-transfer-prescribed: input / first coefficient, 1617–1720. Status:** Verified conditional. **Checked calculation or obligation:** Streamfunction reproduces $ic/(2\pi\bar{z}^5)$ on collars. Initial normalized growing coefficient tends to zero. Complementary response $O(e^{2\eta s})$ for $s \geq 0..$
- **smooth-transfer-prescribed: recursion / closure, 1722–1758. Status:** Verified conditional. **Checked calculation or obligation:** Added drift terms correct; support and $|m| \leq 4j$ preserved. Initial scale $q^{6+K_*/\beta}\delta_c^{-1/\beta}$, displayed N -choice sufficient. Common drift cancels in Kato transport..
- **smooth-transfer-prescribed: cap / whole moment, 1753–1769. Status:** Verified. **Checked calculation or obligation:** Exact leading phase; complementary-to-cap ratio tends to zero; higher terms $O(\delta_q^2)$; integrated foreign drift $o(\delta_q)$ preserves fourth tails..

A.2.5 Detailed checked calculations

High Sobolev estimate For $u_\zeta = K * \zeta$, the finite Leibniz commutator satisfies

$$\|[D^\alpha, u_\zeta \cdot \nabla]\zeta\|_2 \leq C_k(\|\nabla u_\zeta\|_\infty \|\zeta\|_{H^k} + \|\nabla \zeta\|_\infty \|\nabla u_\zeta\|_{H^{k-1}}) \leq C_k \|\zeta\|_{H^k}^2.$$

The energy pairing is cubic; division by the H^k norm gives the quadratic norm term. Undifferentiated transport vanishes. The approximation coupling uses H^{k+1} on the approximation, already available from finite construction, but never on the actual error. The L^1 term uses the near/far kernel bound. Thus $Y' \leq C_k Y + C_k Y^2 + \|R\|_{Z_k}$ is valid.

On $Y \leq 1$, choose $E_k \geq 2C_k$. With $a(t) = \alpha e^{\sigma t}$,

$$\begin{aligned} \int_0^t e^{E_k(t-v)} a(v)^{M+1} dv &\leq \frac{a(t)^{M+1}}{(M+1)\sigma - E_k}, \\ \int_0^t e^{E_k(t-v)} (1+v)^D a(v) dv &\leq (1+t)^{D+1} a(t) e^{(E_k - \sigma)t}. \end{aligned}$$

At the cap the viscosity exponent is $\frac{3}{2}(N+1) - K_* r_k > 0$. It defeats every finite time polynomial. The pure term is below one-half after the fixed cap threshold is reduced. The parameter order is not circular.

Normalized finite jets Rotation multiplies sector m by $e^{-im\tau t/4}$. The growing fourth-sector eigenvalue becomes $\lambda - i\tau = \sigma$; its conjugate growing eigenvalue in sector -4 becomes $\bar{\lambda} + i\tau = \sigma$. After normalizing order j by $e^{-j\sigma t}$, homogeneous propagation decays at least $(j-1)\sigma$. Product convergence can be taken one derivative higher at each finite induction step. Duhamel therefore gives

$$F_j = -(j\sigma - \mathcal{L}_{\text{rot}})^{-1} \sum_{j_1+j_2=j} (K * F_{j_1}) \cdot \nabla F_{j_2}.$$

At $H_q + s$, the seed phase $-\tau H_q$ cancels accumulated rotation exactly and leaves $\mathcal{T}_{\tau s} F_j$.

Whole-space compactness and uniqueness The mass tail and uniform L^∞ bound imply a uniform small exterior L^2 norm. For a fixed exterior cutoff χ ,

$$\|\chi\omega^q\|_{H^{k-\varepsilon}} \leq C\|\chi\omega^q\|_2^{\varepsilon/k}\|\chi\omega^q\|_{H^k}^{1-\varepsilon/k}.$$

The last norm is uniformly bounded, and the first tends to zero. Time derivatives are bounded in H^{k-2} , giving equicontinuity in $H^{k-\varepsilon}$ by interpolation (modulus $C|s - s'|^{\varepsilon/2}$ for $0 < \varepsilon \leq 2$). Local Rellich compactness plus this tightness gives the stated global compactness; no false compact embedding of whole-space Sobolev spaces is invoked.

For two ancient orbits with the same finite jet, the difference at $-T$ is $O(\delta^{M+1}e^{-(M+1)\sigma T})$ in X . First derivatives belong uniformly to X , from compact support and H^k . Exact transport comparison costs $e^{E_k T}$, so $(M+1)\sigma - E_k > 0$ forces equality. A direct H^k difference estimate would demand an extra comparison derivative; the text avoids that.

Transform noncancellation Subtract the leading term only on the negative half-line:

$$\mathcal{F}_\delta(z) = \frac{\delta b}{\lambda - z} + \int_{-\infty}^0 e^{-zs} [B(s) - \delta b e^{\lambda s}] ds + \int_0^\infty e^{-zs} B(s) ds.$$

The last two integrals are analytic on $0 < \Re z < 2\sigma$. They cannot cancel the simple pole at $z = \lambda$. Identical vanishing on the original strip would contradict analytic continuation to this punctured larger strip. This works for a complex, sign-changing moment; no positivity hypothesis is used. Isolated sampled zeros remain possible.

Adjoint and transfer constants The generator is

$$\mathcal{L}_4 g = -4i\Omega g - \frac{iW'}{2r} \int h_4(r, s) g(s) s ds.$$

Pairing with $2\pi a(r)r dr$ and Fubini gives

$$(\lambda + 4i\Omega(s))a(s) = -\frac{i}{2} \int a(r)W'(r)h_4(r, s) dr.$$

Its denominator has real part $\sigma > 0$. The numerator gives s^4 in the core and s^{-4} outside. Accordingly the test is $C\bar{z}^4$ and $C'z^{-4}$, both harmonic there, and all derivatives are bounded.

For the exterior field $U = iB_4/(2\pi\bar{z}^5)$,

$$u_r = \frac{iB_4}{4\pi r^5} e^{4i\theta} + \frac{\overline{iB_4}}{4\pi r^5} e^{-4i\theta},$$

so the forcing is $B_4[-iW'/(4\pi r^5)]e^{4i\theta} + \text{c.c.}$. With $W'_0 = -\sum c_i \delta_{r_i}$ and interface measure $f(r)r dr = \sum c_i r_i^2 q_i \delta_{r_i}$, its vector is

$$h_i = \frac{i}{4\pi r_i^6}, \quad h = \left(\frac{i}{4\pi \rho^6}, \frac{i}{4\pi} \right).$$

The growing dual $\ell_0(q) = (q_1 + iq_2/\rho)/2$ has value one on $(1, -i\rho)$, zero on $(1, i\rho)$, and

$$\ell_0(h) = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0.$$

The interior cubic field $U = c_3\bar{z}^3$ instead has vector $(\rho^2/2, 1/2)$, giving $(\rho^2 + i/\rho)/4 \neq 0$. The two interaction directions are genuinely distinct.

Prescribed receiver, including its uniform centered-clock induction Where $\zeta_0 = 1$, the prescribed streamfunction gives

$$u_r = \frac{i}{4\pi r^5}(ce^{4i\theta} - \bar{c}e^{-4i\theta}), \quad u_\theta = \frac{1}{4\pi r^5}(ce^{4i\theta} + \bar{c}e^{-4i\theta}),$$

and hence $u_1 + iu_2 = ic/(2\pi\bar{z}^5)$. There is no sign or factor mismatch.

For $s = t - t_0$, the first coordinate is

$$e^{-\lambda s} \mathcal{A}(g_{1,0}) = \frac{\alpha}{\varepsilon_q} e^{\lambda t_0} + \gamma_\epsilon \int_{-\beta t_0}^{\beta s} e^{-\mu_\beta u} B(u) du.$$

The initial term has modulus $\delta_c^{1/\beta} q^{K_*(1-1/\beta)-6} \rightarrow 0$. At the cap both integral endpoints approach their bilateral limits. Future forcing is polynomial, so the complementary response is $O(e^{2\eta s})$. Dividing its cap contribution by δ_q gives

$$\frac{\varepsilon_q e^{2\eta S_q}}{\delta_q} = |C_*|^{-2\eta/\sigma} \left(\frac{\varepsilon_q}{\delta_q} \right)^{1-2\eta/\sigma} \rightarrow 0.$$

The entire first coefficient has each needed finite Sobolev bound $Ce^{\sigma s}$, uniformly in t_0 : the ancient forcing has faster rate $\beta\sigma$, and $(\alpha/\varepsilon_q)e^{\sigma t_0}$ is bounded. The stable initial coefficient is also bounded by that rate on $[-t_0, 0]$.

Higher pure-order forcing has rate $e^{j\sigma s}$; Duhamel has gap $(j-1)\sigma > 0$. Positive-viscosity resonances produce only finite polynomials in absolute time $t \geq 0$. Every added drift term differentiates a collar-supported coefficient or W_ℓ , so cutoff drift does not enlarge coefficient support. Angular mode count remains $|m| \leq 4j$.

The initial scale is $a(0) = q^{6+K_*/\beta} \delta_c^{-1/\beta}$. Thus the viscosity cap error has a strict positive q -power under the displayed choice

$$\frac{3}{2}(N+1) > (6 + K_*/\beta)r_1.$$

For $0 < \kappa \leq q^{3/2}$, all remainder upper bounds improve as κ decreases. The same prescribed drift occurs in actual and approximate transport, whose Kato pairing vanishes. The comparison coefficient is still controlled by the approximation gradient, not by an assumed derivative bound on the actual receiver.

Finally $\varepsilon_q e^{\lambda S_q} C_* = \delta_q e^{i(\tau S_q + \arg C_*)}$. Higher terms are $O(\delta_q^2)$. The foreign drift integral is $O(q^6(1 + H_q)^5) = o(\delta_q)$, while self perturbation has integrated velocity $O(\delta_q)$. The radial exponential test with their sum supplies Gaussian fourth tails; bounded-cutoff pairing plus those tails gives the whole moment.

A.2.6 Adversarial checks made

- Derivative loss in the actual high-Sobolev commutator: absent, by the Biot–Savart gain and finite approximation regularity.
- False whole-space compact embedding / translated-bump counterexample: excluded by the tail and cutoff interpolation.
- Uniqueness from just one leading term: not claimed by the actual proof; the finite jet is essential.
- Complex future cancellation of the entire transform: impossible because it cannot remove the pole; isolated sampled cancellations are allowed and avoided.

- A critical layer in the adjoint denominator: impossible because its real part is $\sigma > 0$.
- Support enlarged by the prescribed cutoff: absent because the new terms differentiate collar-supported functions.
- Divergent q -dependent initial coefficient in the centered clock: excluded precisely by $K_*(1 - 1/\beta) > 6$.
- Breakdown as κ is arbitrarily small positive: existence is for each positive viscosity; quantitative bounds improve as viscosity decreases.
- A hidden replacement of actual child velocity by one harmonic: not asserted as a coupled theorem here; the drift is explicitly auxiliary.

Conditional acceptance of the principal §§3–4 conclusions is justified after adding R3 and R1–R2 precision, while retaining the spectral dependency and the exact-cap/interacting-field limits.

The replacement TeX was mechanically checked for matching environments, math delimiters, and braces. Compilation could not be completed: the installed pdflatex executable has no pdflatex.fmt format. No rendering claim is made.

A.3 Sections 5–7: cores, localization and cold tails

Audited source: paper-snapshot.tex, lines 1785–2779, unchanged. This is an independent check of the displayed assumptions and proofs; no earlier audit was read. I also checked only the initial-stack lemma (lines 66–93) and the use at lines 4535–4545 to determine where the essential initial fourth-moment hypothesis comes from.

A.3.1 Verdict and scope

I found **no false quantitative conclusion or fatal mathematical error in the assigned conditional results**. The radial-exit constant, the outer-core bootstrap, the exterior harmonic series, the Gaussian weighted-energy identities, the nonradial Hermite gap, the rough-leak comparison, and the powers 6.43 and 6.46 are consistent. The assigned results do not prove the nonradial-density input on which the outer-core and cold-tail applications depend. They also do not permit resetting a cold tail at an arbitrary later time while retaining zero initial fourth moment.

Two proof-completion details should be made explicit:

1. Lines 2492–2495 say only: “All energy tests can first be made with truncated weights and smooth data. The short-time exponential moment starts the weighted evolution; the uniform weighted estimates justify removal of the truncations and passage to the parabolic limit.” A generic truncated weight does **not** retain the exact harmonic-oscillator inequality used at lines 2363–2383. This is a justification gap in the written proof, rather than a counterexample to the proposition. The companion corrections supply a concrete concave bounded-weight argument establishing Gaussian-space membership before using the exact unbounded-weight inequality.
2. Lines 2701–2708 define “ $S=2+\text{nuT}/r^2$ ” and then apply the Gaussian proposition, which asks for a drift on $0 \leq s \leq S-1$. The physical assumptions provide only $0 \leq s \leq \text{nuT}/r^2$. The displayed normalization therefore invokes an extra normalized unit of time without saying how the cutoff field is continued. Extend that field by zero after T; the original solution on $[0, T]$ is unchanged, and all bounds persist. The corrections give this extension explicitly.

Neither completion changes the exponents or the stated assumptions. A further short continuity argument for starting the H-minus-one energy at zero is included in the corrections; the estimates already printed at lines 2630–2652 are sufficient.

A.3.2 Per-result status

- **Kernel interpolation, lines 1794–1802**, eq:cont-kernel-interpolation. **Status:** Verified. **Verification / limitation:** Near contribution $C_d \|f\|_{\infty}$, far contribution $C \|f\|_1/d$; optimize. No cancellation is needed..
- **Fourfold growth at origin, lines 1804–1831**, lem:cont-C4-origin. **Status:** Verified. **Verification / limitation:** Only inverse degrees divisible by four survive. The far series starts at cubic order. Its majorant is integrable on $|y| > 2|z|$; the near integral is $O(\|f\|_{\infty} |z|)$. This is a bound for each colour, not the infinite aggregate’s infinity norm..
- **Radial exit, lines 1833–1881**, lem:cont-radial-exit. **Status:** Verified. **Verification / limitation:** $\phi = \cosh(\lambda \bar{r})$ is smooth at zero; $\Delta \phi \leq 2\lambda^2 \phi$ in two dimensions. The supermartingale gives $2 \exp[-\lambda(b - |x| - D) + 2n\lambda^2 T]$; optimizing gives exactly the denominator $8nT$. No tangential drift bound enters..
- **Countable coloured evolution and backward representation, lines 1883–1942**. **Status:** Verified in the stated mild-solution class. **Verification / limitation:** Colour contraction gives absolute convergence in L^1 intersection L^p ; velocity convergence follows from $\sum |A| |R| < \infty$. The backward sign is correct. The divergence-free Nash kernel bound handles the nonsmooth initial endpoint without assuming an aggregate initial infinity norm..
- **Constant-core persistence, lines 1944–2044**, prop:cont-outer-core. **Status:** Verified conditional result. **Verification / limitation:** The actual density condition eq:cont-NR is essential and is not proved here. All geometric sums and bootstrap margins check. The exact exponential constant is $c^* = 9/512$..
- **Harmonic exterior field / density leak, lines 2046–2130**, prop:cont-harmonic-outer-field. **Status:** Verified conditional result. **Verification / limitation:** Exact constant-disk decomposition, exterior radial cancellation, anti-holomorphic powers 3,7,11,..., cubic coefficient and signs, and first derivative estimates check. Only a growth estimate is claimed for the core leak; no unjustified leak derivative is used later..
- **Localized tail exit, lines 2162–2240**, lem:tail-exit. **Status:** Verified. **Verification / limitation:** Rotation removal, pre-exit pathwise Gronwall, Brownian reflection, and conditioning at the first exit establish both mass and fixed-order weighted tail bounds. The extra condition $\sqrt{nT} \leq cR^*$ supplies the post-exit moment scale..
- **Gaussian cold-tail proposition, lines 2274–2496**, prop:resume-gaussian. **Status:** Verified after explicit weighted-test completion. **Verification / limitation:** All algebraic and analytic inequalities check. The companion proof explains how to establish the full Gaussian norm rigorously without assuming that a truncated weight preserves the final spectral inequality. Initial $B_4 = 0$ is genuinely used in the initial moment estimate..
- **Local harmonic field / rough leak comparison, lines 2500–2696**, prop:resume-local-transfer. **Status:** Verified. **Verification / limitation:** Streamfunction cutoff is Lipschitz

even though the leak has no derivative bound. The differentiated heat-kernel estimates, moment estimates, integrable actual initial coefficient, and source integration yield the claimed comparison. No aggregate initial infinity norm enters..

- **Physical normalization and full bound, lines 2700–2722. Status:** Verified after zero continuation. **Verification / limitation:** All mass, cubic-drift, time, and fourth-moment rescalings are correct. The extension noted above resolves the one-unit time mismatch..
- **Geometric budget, lines 2724–2752, eq:resume-tail-budget. Status:** Verified for logarithmic stage lengths. **Verification / limitation:** The exponents are exactly 643/100 and 323/50. The linear-growth, localization, and Gaussian-smallness margins all go to zero for $\kappa=q^{3/2}$ and $H=O(\log(1/q))$, with the fixed amplitude ratios specified earlier..
- **Scope and selected-cutoff remark, lines 2754–2778. Status:** Accurate limitations. **Verification / limitation:** The cold-tail budget applies at the selected cutoff and to a tail starting from the prescribed initial datum. It does not erase the active child at every later receiver and does not automatically continue an old constant-core estimate after the source has left its cap..

A.3.3 Detailed independent checks

Core persistence and harmonic field For an older colour $k < 1$, the probe region condition follows from $q \leq 1/4$:

$$|z| \leq a_0 R_l \leq a_0 R_k/4 = (a R_k)/2, \text{ with } a = a_0/2.$$

The older colour's constant disk gives pure rotation; its core error gives $C \epsilon A_k |z|$ by fourfold growth; its exterior nonradial density gives $C \delta A_k |z|^3/R_k^2$. The needed sums are exactly

$$\begin{aligned} \sum(k < 1) A_k/R_k^2 &\leq (A_l/R_l^2) q^2/(\beta - q^2), \\ \sum(k < 1) A_k &\leq A_l/(\beta - 1), \text{ and} \\ \sum(k > 1) A_k R_k &= A_l R_l \beta q/(1 - \beta q). \end{aligned}$$

Consequently the two bootstrap margin terms each contribute at most $a_0 R_l/16$ to integrated radial drift. The remaining exit gap is $3a_0 R_l/8$; its squared gap divided by 8ν is $9a_0^2 R_l^2/(512\nu)$. The finite maximum over $l \leq i$ does not introduce a multiplicative factor i .

For $|z| \leq a R_i/2$, the exterior inverse moments have the convergent series

$$H_{-i} = -i/(2\pi) \sum(l \leq i) \sum(m \geq 0) \bar{z}^{4m+3} \int_{|y| \geq a R_l} h_l(y)/\bar{y}^{4m+4} dy.$$

The ratio is at most $1/16$, including the derivative majorant $(4m+3)$. Cubic and seventh-order terms sum using $\sum A_l/R_l^d \leq (A_i/R_i^d)/(1 - q^d/\beta)$, for $d=2,6$. The coefficient sign and the use of $\bar{z}^3$ are correct for the paper's convention $K(z)=i/(2\pi \bar{z})$.

A useful fact for the downstream application is that the **global** bound after subtracting solid rotation also follows without an infinite-aggregate infinity norm: apply fourfold growth separately to the finite outer sum and use $\sum(l \leq i) A_l \leq A_i \beta/(\beta - 1)$.

Gaussian weighted identities and spectrum Writing $g = e^{(\beta|y|^2/2)} w$ with $\beta=3/10$ gives exactly

$$\begin{aligned} \langle w, Lw \rangle_\beta &= (1/2) \|g\|^2 - \|\text{grad} g\|^2 - (3/50) \| |y| g \|^2, \\ D_\beta &= \|\text{grad} g\|^2 + (9/100) \| |y| g \|^2 + (3/5) \|g\|^2. \end{aligned}$$

The oscillator lower bound is $H_\beta \geq (\sqrt{6}/5) \|g\|^2$. Also $D_\beta \leq (3/2) H_\beta + (3/5) \|g\|^2 < 3 H_\beta$. Therefore the printed splitting proves

$$\langle w, Lw \rangle_{\beta} \leq -D_{\beta}/50 + F_{\beta}^2/25.$$

The apparently growing $F_{\beta} \leq C m \exp(\tau/25)$ is intentional: the proposition needs only a weak upper bound and gains much stronger nonradial decay separately.

The nonradial fourfold component $h = (I - P)w$ has every ordinary moment of degrees zero through three equal to zero. In $X = L^2(e^{|y|^2/4} dy)$ the self-adjoint OU operator has Hermite eigenvalues $-|\alpha|/2$. The first admissible nonradial degree is four, hence $\langle h, Lh \rangle_X \leq -2H_2$. Since $\langle h, Lh \rangle_X = -D_h + H_2$, this gives exactly $D_h \geq 3H_2$.

For the cross flux, set $q = e^{|y|^2/8} G$. Weighted integration by parts bounds the dual derivative of h by $C \sqrt{D_h}$. Unweighted HLS gives $\|V_h\|_4 \leq CH$; GN gives $\|q\|_4 \leq C F_{\beta}^{1/2} D_G^{1/4}$. This proves the cross estimate printed at lines 2450–2452. The strict weight margin $\beta - 1/4 = 1/20$ absorbs $|y|^3$ in the forcing flux. There is no missing derivative hypothesis for E in this argument.

The angular energy leaves a coefficient as close to the unperturbed decay $-4H_2$ as desired; $-19H_2/5$ is therefore valid when the displayed smallness constants are chosen sufficiently small. The coefficient integral is bounded by $C m^2 S^{2/25} \sqrt{1 + \log S}$, and the stated hypothesis makes it uniformly small. The forced decay bound is

$$H \leq C m [e^{-19\tau/10} + d e^{51\tau/25}].$$

Moment identity and the zero-initial-moment condition With the paper's complex convention the exact physical fourth-moment equation is

$$dB_4/ds = 4 \int \bar{b} z^3 \overline{(K^*f + b)} f.$$

Diffusion contributes zero because $\Delta \bar{b} z^4 = 0$. The self contribution symmetrizes to

$$-i/\pi \text{ double-integral } [(\bar{b} z^3 - \bar{y}^3)/(z - y)] f(z) f(y) dz dy.$$

Its absolute value is bounded by $C m \int |z|^2 |f(z)| dz$; the foreign cubic contribution is bounded by $C d \int |z|^6 |f(z)| dz$. Thus at the fixed short time the estimate is $C(m^2 + md)$ **only because** $B_4(f_0) = 0$.

In similarity variables, $B_4(f) = e^{2\tau} B_4(w)$. The cancellation of the dilation term is exact. Radial self-advection is zero, and the remaining derivative is bounded by $C e^{2\tau} [F_{\beta} H + a F_{\beta}]$. The self unforced exponent is $2 + 1/25 - 19/10 = 7/50$. The cross term involving the forced angular component has exponent $2 + 1/25 + 51/25 = 102/25$; the direct foreign term has the smaller exponent $101/25$. Using $m \leq 1$ gives the common printed bound $m d S^{102/25}$.

Fourfold symmetry and zero circulation alone do **not** imply $B_4 = 0$. For instance, a smooth radial annular function times $\cos(4\theta)$ is fourfold, has zero circulation, and has nonzero fourth moment whenever its radial fifth moment is nonzero. Scaling it by m gives an initial fourth moment of order m , contradicting an $O(m^2)$ bound with $d = 0$ at time zero. This is why the hypothesis must be retained whenever the proposition is reused.

For the actual *initial* inner stack, lines 66–93 ensure each seed has zero fourth moment and sum $\sum_{l \leq 6} R_l < \infty$, so termwise integration does give zero fourth moment for every initial tail. Lines 4535–4545 use that initial tail, so that particular zero-moment assertion is supported. It must not be generalized to a tail restarted after nonlinear evolution without an added initial-moment term.

Local comparison and initial time The streamfunction cutoff uses $|\psi_h| \leq C C^3 |x|^4$ to obtain global cubic growth and $\|D \tilde{b}\|_{\infty} \leq \ell = C C^3 R^2$. Independently, $|b_h| \leq 2L|x|$ yields $|\psi_h| \leq CL|x|^2$ and hence $|\tilde{b}| \leq CL|x|$. This second bound correctly avoids requiring $C^3 R^2 \leq L$.

After setting $r=\nu=1$, the moment estimates are $M2 \leq C_m(1+t)$ and $M4 \leq C_m(1+t)^2$ when m and $\ell(1+T)$ are small. Therefore $\|xg\|1 \leq C_m \sqrt{1+t}$ and $\|xg\|4 \leq C_m t^{(-3/4)} \sqrt{1+t}$. For $Q_t = \text{grad}K * \text{div } H_t$, scaling gives the norms $t^{(-3/2)}$, $t^{(-1/2)}$, and $t^{(-3/4)}$ in L_∞ , $L1$, and $L^{(4/3)}$, respectively; the far decay is $|x|^{(-3)}$. The early/late Duhamel split yields $\|\text{gradu}_g\|_\infty \leq C_m/t$. The physical version is $C(M/\nu)/t$, with correct units.

Testing the difference in homogeneous $H-1$ is legitimate because it has zero circulation and finite first moment. The self velocity difference obeys $v \cdot \text{grad}\psi=0$; this kills $v \cdot \text{grad}g$ and replaces u_f by u_g in the common transport term. Only $D \tilde{b}$, not Db , is used. The source is simply $F=(\tilde{b}-b)f$, so the leak needs no derivative estimate.

The printed C_m/t estimate must **not** be integrated from zero as a coefficient. The paper correctly first uses the actual integrable estimate from g_0 in L_p , namely $C_0[t^{(-1/p)} + t^{(1/2-1/p)}]$, to start the integrating factor at $v(0)=0$. Only for the source integral with positive lower time s does it use $(t/s)^{(CM/\nu)}$. This avoids discarding an uncontrolled positive-time initial term. The source singularity is at worst $s^{(-1/2)}$; the condition $CM/\nu < 1/4$ makes its weighted integral finite uniformly. This establishes the claimed rough-leak factor $\epsilon L[T+r \sqrt{T/\nu}]$ and localization factor $LR^* \sqrt{T/\nu} \exp[-cR^2/(\nu T)]$.

Budget arithmetic and limits For $r=qR_i$, $M=O(\beta A_i q^2 R_i^2)$, $\nu=\kappa A_i R_i^2$, $\kappa_{pa}=q^{(3/2)}$, and $T=H/A_i$,

$m=O(\beta q^{(1/2)})$, $d=O(\delta q^{(5/2)})$, $S=O(q^{(-1/2)}H)$ for logarithmic H , and $M_{r4}/(A_i R_i^6)=O(\beta q^6)$.

The first line of the full bound is therefore

$$\beta q^6 \cdot \beta q^{(1/2)} \cdot q^{(-7/100)} = \beta^2 q^{(643/100)}$$

and

$$\beta q^6 \cdot \delta q^{(5/2)} \cdot q^{(-51/25)} = \beta \delta q^{(323/50)}.$$

The cutoff weak-growth condition is $\ell(r^2/\nu+T)=O(\delta[q^{(5/4)}+q^{(3/4)}H]) \rightarrow 0$. The Gaussian bootstrap condition is $mS^{(1/25)} \sqrt{1+\log S}=O(\beta q^{(12/25)}H^{(1/25)} \sqrt{1+\log S}) \rightarrow 0$. The relative self displacement is $O(\beta q^{(7/8)} \sqrt{H}) \rightarrow 0$, and $\sqrt{\nu T}/R^*=q^{(3/8)} \sqrt{H} \rightarrow 0$. The Gaussian exit exponent is $1/(q^{(3/4)}H)$, dominating $e^{(CH)}$ and all polynomial factors for logarithmic H . These calculations confirm the $o(q^6)$ conclusion at the selected cutoff.

A.3.4 Dependents and unresolved input

The unresolved input within this assigned chain is the actual bound $\sup(t \leq T) \|(I-P)w_1(t)\|1 \leq \delta A_1 R_1^2$ for all active and older colours, not any of the tail estimates audited here. Results relying on this input are conditional until a separate argument establishes it on the actual common-velocity evolution and the specified full window. The present audit does not endorse later assertions that establish that input, a signed receiver transfer, a invariant stage class, or distinct inviscid accumulation paths; those are outside the assigned lines.

The companion `ref-core-cold-corrections.tex` is a proposed proof completion only. The source paper has been preserved.

A.4 Sections 8–13: colours, compactness and inactive packets

Source: `paper-snapshot.tex`, original lines 2780–4019. The snapshot was not modified. This audit is independent of other audit files. Mathematical statements below concern this range and

the explicitly inspected dependencies, not the final global induction or the active signed-history argument.

A.4.1 Verdict

The two-norm baseline, protected-cap topology, sparse selection, fixed-data compactness, and actual positive packet estimates have consistent clocks, normalizations, and geometric powers. No counterexample to the stated fixed-data compactness theorem was found. The §13 inactive closure works with its explicit choice $\delta = \log(1/q)^{-4}$, subject to the fixed profile/core geometry and standard coloured Navier–Stokes inputs.

There is one material parameter-specification/proof gap in the earlier §11 closure statements: their retained Gaussian error envelope is independent of δ , whereas the statements only require δ to be sufficiently small. Superpolynomial smallness in q does not imply smallness relative to an arbitrarily smaller $\delta(q)$. This is a deficiency in the supplied proof envelope, **not a verified counterexample to the actual Navier–Stokes comparison conclusion**. A lower scale restriction on δ , or an explicitly relative error condition, repairs it and is already satisfied in §13.

There are also two incorrect explanatory sentences and several easily repairable specification issues: compact support does not generally pass through strong L^1 convergence; arbitrary radial data do not give $\zeta_i = 0$ against a fixed different reference; the passive convergence proof imports Kato without explicitly assuming it; the separation theorem must compare one common fixed observable; and Gaussian heat-gradient leakage needs an annulus with a positive margin around the derivative support.

A.4.2 Result-by-result status

- **ph:two-norms, 2786–2858. Status:** Verified as a scalar majorant lemma. **Assessment:** Positive comparison and both row sums are correct. This does not itself establish a diffusive PDE row..
- **ph:baseline, 2860–2899. Status:** Verified conditional on the stated cap-clock bound. **Assessment:** The actual receiving clock stays at most S ; no growing weighted norm is imposed on the common fixed initial seeds..
- **Generic self-rate remark, 2906–2918. Status:** Scalar parameter argument verified; transform selection assumed. **Assessment:** A generic Kato rate γ/β can be made small by increasing β . The isolated transform claim is outside this audit..
- **ph:colour, 2925–3009. Status:** Transport uniqueness and uniform strong convergence route verified; abstract formulation needs clarification. **Assessment:** Bounded Sobolev limiting drift gives renormalization. For actual NS colours the proof is valid. Kato is not explicitly included among the minimal hypotheses on u_n, f_n ; see F2..
- **ph:kernel and fixed-observable paragraph, 3011–3038. Status:** Verified. **Assessment:** The kernel split is correct; finite propagation is valid for the renormalized Euler colours..
- **ph:first-cap, 3046–3089. Status:** Verified. **Assessment:** Both quantitative estimates and closure of the protected class are valid. No derivative-openness assertion is used..
- **ph:separation, 3091–3103. Status:** Verified for the same fixed observable on both subsequences. **Assessment:** Different cap phases of a common recovered-colour functional force distinct total paths; changing the functional would invalidate that implication..

- **ph:degree, 3105–3129. Status:** Verified conditional on common protected caps and genuine rotational covariance at $r = 0$. **Assessment:** The sign $e^{i\theta}$ is correct for rotation by $-\theta/4$ and a $\bar{z}^4$ test..
- **Fixed-positive-viscosity continuity, 3131–3144. Status:** Verified with continuity of the initial colours in the stated norms. **Assessment:** The $\nu^{-1/2}(t-s)^{-1/2}$ Volterra estimate is sufficient at each fixed $\nu > 0$. Uniform cap protection is a separate input..
- **ph:sparse, 3146–3187. Status:** Verified conditional construction; final support explanation corrected in F3. **Assessment:** The sparse tail estimate and parity separation are sound. Its assumptions, not strong L^1 alone, ensure compact support of the limiting datum..
- **prop:inactive-actual-row, 3239–3419. Status:** Verified using the inspected core and exit dependencies, with positive annulus margin. **Assessment:** Self Kato rate, outer q^2 row, inner q^6 row, receiver-radius leak, and all four error terms agree with the physical normalization..
- **cor:inactive-density-closure, 3421–3504. Status:** Proof valid after a relative Gaussian-error/lower- δ -scale condition. **Assessment:** The matrix and weighted Gaussian summation are correct. The literal δ quantifier is insufficient for the final $< \delta/4$ claim; see F1..
- **cor:inactive-static-packet, 3506–3554. Status:** Same parameter gap as preceding corollary. **Assessment:** The displayed principal terms give $o(\delta q^6)$, but the retained absolute error must also be $o(\delta q^6)$..
- **cp:velocity, 3566–3628. Status:** Verified. **Assessment:** All three velocity estimates, the Calderón–Zygmund proof, and finite energy from compact support plus zero circulation are valid..
- **cp:compact, 3630–3808. Status:** Verified given the stated standard NS contraction and energy inputs. **Assessment:** Local velocity compactness, all-time renormalized traces, norm recovery, uniform strong L^p , and global L^1 tightness form a noncircular proof..
- **cp:sparse-input, 3810–3816. Status:** Verified. **Assessment:** Compactness and Sobolev velocity follow; this does not supply common protected caps or actual phase shooting..
- **lem:next-packet-moments, 3845–3893. Status:** Verified, including countably infinite positive aggregate. **Assessment:** The proof uses finite individual colour L^∞ bounds and the convergent velocity sum, not an L^∞ bound for the positive aggregate..
- **prop:next-packet-force, 3895–3964. Status:** Verified under stopped/core hypotheses. **Assessment:** The fourth moment gives q^{6k} ; leaked L^1 - L^p interpolation gives q^k times a Gaussian. These are magnitude bounds only..
- **cor:next-inactive-closed, 3966–4017. Status:** Verified with its explicit logarithmic δ . **Assessment:** Parameter order, matrix closure, radial margin, and older inward field powers are consistent. It makes no signed impulse or protected-active-cap conclusion..

A.4.3 Verified computations and attempted obstructions

1. Two norms, fixed seeds, and stage clock For $l = i + d$, the unweighted row coefficient is bounded by

$$C\beta^{i-j}(\beta q^6)^d;$$

for $l = i - d$, it is

$$C\beta^{i-j}(q^2/\beta)^d.$$

Since $i < j$, $\beta^{i-j} \leq \beta^{-1}$, giving exactly the advertised r_B . In the weighted norm $w_i = (Dq^6)^{j-i}$, multiplication by w_l/w_i changes the ratios to β/D and Dq^8/β , giving exactly r_P . The finite matrix exponential is positive and its induced maximum norm has the claimed exponential bound. The comparison argument applies to integrable forcing by the integral formulation.

The cap-clock sum is

$$A_j T_{j,N} \leq S\beta^{j-N} + H \frac{\beta}{\beta-1} (1 - \beta^{j-N}) \leq S.$$

Thus the common seed baseline is controlled by an **unweighted** maximum $Mbe^{\lambda_B S}$. The attempted obstruction $b/w_i \rightarrow \infty$ for distant old indices does not apply: no weighted initial-seed norm occurs. The receiving-field factors are $(q^2/\beta)^k$ for the baseline and $(Dq^8/\beta)^k$ for the forced part. All stage-clock and field scalings in this module are correct.

2. Recovery of fixed passive colours and observables The limiting drift is bounded, divergence free, and locally Sobolev with exponent $p > 1$. The bounded colour lies locally in $L^{p'}$, so the difference-quotient commutator estimate applies. Approximating the drift by divergence-free spatial convolutions, then subtracting $f(x)$ in the commutator for the smooth approximants, yields local $L^1_{t,x}$ convergence of the commutator. This is sufficient for bounded renormalizations and uniqueness of a bounded integrable transported colour with a fixed initial trace.

One can justify the weak initial and every-time nonlinear traces exactly as in §12: the spatially mollified weakly continuous path satisfies an absolutely continuous equation, integrate between arbitrary fixed times, and pass the spatial approximate identities at the endpoints. Conservation of L^2 gives strong time continuity. Renormalization by smooth approximations to $|s|$ also preserves L^1 , which is useful for F2.

Uniform weak L^2 convergence plus conservation of the limiting L^2 norm and contraction of approximating L^2 norms forces uniform strong L^2 convergence by the bad-time argument. Global L^1 can be recovered without invoking a Kato inequality for approximants: the limit has support in a common ball B_R , $\|f(t)\|_1 = \|f_0\|_1 = m$, and contraction gives

$$\int_{B_R^c} |f_n(t)| \leq m - \int_{B_R} |f_n(t)| \leq \int_{B_R} |f_n(t) - f(t)|.$$

Consequently

$$\sup_t \|f_n(t) - f(t)\|_1 \leq 2|B_R|^{1/2} \sup_t \|f_n(t) - f(t)\|_2 \rightarrow 0.$$

This repairs the minimal-assumption proof of uniform strong convergence. Standard actual NS colours are strongly continuous; for abstract zero-diffusion approximants with arbitrary merely bounded drifts, continuity of each approximating path should be required if convergence is literally asserted in the Banach space $C_t(L^1 \cap L^2)$.

For a common fixed functional, equal total paths imply equal velocities, hence equal recovered colours and the same complete observable path. The protected first-cap is then a function of that path, even when different subsequences have different cap times. Different **functionals** cannot be compared this way: for any actual protected observable a_u , replacing every c_l by $-c_l$ produces $-a_u$ on the same total path. Both observables have the same cap time and all the same radial protection constants, but opposite phases. This is an active counterexample to the broadened claim, not to the intended common-functional result.

3. Protected caps, degree, and sparse shooting Before $\tau(a) - e$, the precap margin excludes a cap of a uniformly close path. On the crossing interval, the integrated lower radial slope excludes all times before $\tau(a) - 2\varepsilon/c$ and guarantees a cap by $\tau(a) + 2\varepsilon/c$. The phase estimate uses only the Lipschitz bound for a , not a Lipschitz bound for the perturbation. All protections pass under uniform convergence with common constants by taking converging cap times.

At $r = 0$, rotational covariance must apply to the full coloured initial family and its observable. With this intended meaning, $a_{0,\theta} = e^{i\theta} a_{0,0}$, so the first-cap phase map has degree one. The rotation preserves the cap time because it preserves modulus. The protected homotopy has constant degree; it is therefore surjective at $r = 1$. No uniform-in-viscosity continuity estimate is being inferred from the fixed- ν Volterra estimate.

In the sparse construction, $\eta_{N_l} < 2^{-l} r_k$ for every $l > k$, so $\|g_\infty - g_k\|_X < r_k$. Each phase is changed at most once; absolute convergence and the summable changes identify the limit with the resulting phase sequence. Even and odd selected viscosities then give disjoint phase neighbourhoods. The compactness theorem is needed only for that single final datum. Common cap protections over the entire homotopy and all phase choices remain substantive hypotheses; neither compactness nor inactive density alone proves them.

4. Actual inactive Kato row and Gaussian errors The exact comparison equation (3288–3291) is correct because G_i solves the radial heat equation and its own velocity annihilates its gradient. Radial foreign velocities also annihilate ∇G_i , so only their nonradial components enter. The self Kato term has physical rate $\gamma A_i x_i$ and receiving-clock rate $\gamma \beta^{i-j} x_i$; no sharp spectral-rate identification is made.

On an old source exterior to its constant core, fourfold cancellation starts the inverse-moment expansion at cubic order. On the i -collar this gives

$$|K * h_l|/R_i \lesssim A_l q^{2(i-l)} x_l \quad (l < i).$$

For an inner source, truncation at a fixed fraction of the receiver radius has zero circulation and harmonic moments through degree three. Its fourth moment is $\lesssim A_l R_l^6 (x_l + e_{j-l})$, giving

$$|K * h_l|/R_i \lesssim A_l q^{6(l-i)} (x_l + e_{j-l}).$$

The exterior leakage uses the **receiver** radius in the exponent; L^1 - L^∞ interpolation contributes $A_l R_l/R_i = A_l q^{l-i}$. Replacing that receiver-radius Gaussian by a source-radius Gaussian would break the uniform weighted estimate, but the manuscript correctly retains the receiver radius.

Writing $k = j - i$, $r = j - l$, the four error terms are respectively the omitted receiver heat-gradient tail, older core leaks, inner fourth-moment leaks, and inner receiver-radius leaks. Their normalizations in (3276–3279) are correct. With

$$\eta(q) = \sup_{m \geq 1} \frac{e_m}{(\beta D q^6)^m},$$

the four weighted sums are bounded by

$$C\eta(q) \left(1 + \frac{Dq^6}{1 - Dq^6} + \frac{1}{D - 1} + \frac{\beta q}{1 - \beta q} \right).$$

For example the third is $D^{-k} \sum_{r < k} \beta^{-r} q^{-6r} e_r$, and the fourth uses $\sum_{r < k} \beta^{-r} q^{k-r} \leq \beta^{-k} \beta q / (1 - \beta q)$. All these calculations are correct. The first Gaussian exponent is at least a constant times $q^{-7/2}/S$; successive weighted terms decay. Thus the error is superpolynomial uniformly in depth and in $0 < \kappa_j \leq q^{3/2}$. Its size **relative to arbitrary** $\delta(q)$ is the separate issue F1.

5. Strong fixed-data compactness and all-time traces The compactly supported zero-circulation datum gives finite energy: $\hat{\omega}_0(\xi) = O(|\xi|)$ near zero from its finite first absolute moment, while $\omega_0 \in L^2$ controls high frequencies. No uniform spatial moment of the evolving signed vorticity is needed for the energy inequality.

The velocity equation gives $\partial_t u_n$ uniformly in H^{-2} since $u_n \otimes u_n \in L^2$ with norm at most VE . Combining spatial Hölder bounds with mollification yields

$$\|u_n(t) - u_n(s)\|_\infty \lesssim h^\alpha + h^{-3}|t - s|,$$

so velocities have a locally uniform subsequential limit. Its Sobolev gradient and global L^p velocity bounds pass to the limit. Weak scalar compactness and the Kato cutoff tail pass the drift product and diffusion term, obtaining a transport equation for ω with this regular limiting drift.

The renormalization estimate

$$\|r_\varepsilon(t)\|_{p/2} \lesssim \|\nabla u(t)\|_p \sup_{|y| \leq \varepsilon} \|\omega(t, \cdot - y) - \omega(t)\|_p$$

is correct: divergence freedom permits subtraction of $\omega(x)$. Translation continuity and domination imply convergence in $L_t^1 L_x^{p/2}$; $p > 2$ is enough for local L^1 control. The explicit endpoint argument establishes nonlinear traces at every fixed time, including zero. Integrable Lipschitz renormalizations can be integrated globally since the cutoff flux is $O(R^{-1})$. The increasing tangent truncations recover the L^p norm by monotone convergence, while smooth approximations to $|s|$ recover L^1 and the identity preserves circulation.

Constant limiting L^p norm plus weak time continuity gives strong time continuity. At bad times $t_n \rightarrow t$, weak convergence and the contraction upper bound match the limiting conserved norm; uniform convexity gives strong convergence. Tightness upgrades local L^1 to global uniform L^1 . Only then is $u = K * \omega$ identified using the strong kernel bound. This order is legitimate and avoids a circular preliminary global velocity convergence argument. The theorem claims precompactness, not uniqueness of the nonlinear Euler limit.

6. Actual positive infinite packet, moments, force, and closure The positive aggregate is genuinely in $L^1 \cap L^p$: individual interpolation gives $\|w_l(0)\|_p \lesssim A_l R_l^{2/p}$, and $\beta q^{2/p} \leq 1/2$ bounds its geometric sum uniformly. Its L^∞ norm may be infinite. The drift estimate instead sums individual velocity contractions:

$$\sum_{l \geq j} \|K * w_l\|_\infty \lesssim \sum_{l \geq j} A_l R_l \lesssim A_j R_j.$$

The older sum satisfies $|u_{<j}(x)| \lesssim A_j |x|/\beta$. The constants here can be chosen uniformly for $\beta \geq 4$ and $\beta q \leq 1/2$, so increasing β really makes C_4/β small.

The mass $M = \int P_j$ is conserved by the positive divergence-free scalar propagator. For $Q_n = \int (R_j^2 + |x|^2)^{n/2} P_j$, radial drift and diffusion give the claimed differential inequality; $Y = (Q_n/M)^{1/n}$ obeys $Y' \lesssim LY + V + \nu/Y$. The proposed exponential times $R_j + Vt + \sqrt{\nu t}$ is a valid supersolution after increasing constants. Finite truncations of the positive initial aggregate evolve under the **same actual common velocity**, so positivity and monotone convergence justify the countable limit without changing the velocity. Constants remain independent of packet depth, phase, and viscosity.

The moment bound gives fourth absolute moment $\lesssim A_j R_j^6 \mathcal{M}(S)$ for the nonradial packet. Its interior truncated field on a receiver collar is $\lesssim R_i^{-5}$ times this moment. Division by the

force normalization $A_j R_i$ gives exactly $q^{6k} \mathcal{M}(S)$, with no missing amplitude factor. The receiver-radius packet leakage and L^1 - L^p kernel interpolation give $A_j R_j e^{-c_p R_i^2/(\nu t)}$, hence the normalized q^k Gaussian. Dividing by $(Dq^6)^k$ gives $D^{-k} q^{-5k}$ times a stronger Gaussian; its maximum is the nearest index and remains superpolynomial.

For §13, let $a = C_2(\lambda_P + C_4/\beta) < 1$. The nearest forced density is bounded by $CDq^{6-a} \log(1/q)^5$, and the initial contribution is $Mq^{K-C_2\lambda_B}$ with exponent greater than six. Both, and the Gaussian errors, are smaller than $\delta/4 = \frac{1}{4} \log(1/q)^{-4}$ for sufficiently small q . The radial margin follows from $S\sqrt{\delta} = O(\log(1/q)^{-1})$ and $S\beta q \rightarrow 0$. Multiplication by the older receiving-field row adds q^2 to the baseline and q^8 to the forced weighted contribution, exactly as displayed in (3990–3991). This closes the magnitude stop only.

A.4.4 Findings, exact quotes, repairs, and downstream effects

F1 — Material proof-envelope / parameter-specification gap **Location:** cor:inactive-density-closure, lines 3434–3439, 3492–3503; inherited by cor:inactive-static-packet, lines 3514–3523 and 3543.

Exact quotes:

“Take $S = C_2 \log(1/q)$, with C_2 sufficiently large and fixed, and δ small enough for (93).” (3434–3435)

“ $x_i(t) \leq B + w_i [CC_*\delta + o(q^K)] < \delta/4$ ” (3438–3439)

“Its weighted convolution is therefore at most $C\eta(q)Se^{\lambda_P S}$, still smaller than every q -power.” (3492–3493)

“its error envelope is much smaller even than the prescribed seed b .” (3502–3503)

Issue: Let $H_G(q) = C\eta(q)Se^{\lambda_P S}$. Proving $H_G(q) = o(q^K)$ for every fixed K does not prove $H_G(q) = o(\delta(q))$, nor $H_G(q) = o(b(q))$, when δ is subsequently allowed to be arbitrarily small. For example $\delta(q) = \exp(-q^{-10})$ satisfies the radial smallness requirement, while the supplied Gaussian upper envelope at $\kappa_j = q^{3/2}$ has exponent of order $-q^{-7/2}/\log(1/q)$, much larger than this δ . This is a counterexample to the **comparison of the stated envelopes**, not a lower bound or a counterexample for the actual angular forcing. In particular all-radial data have no angular forcing.

Verified repair: Explicitly require $\delta(q) \geq q^{L_\delta}$ for a fixed finite $L_\delta > 0$, together with the stated radial upper bound. Use the arbitrary Gaussian power with $K > L_\delta + 6$. Alternatively impose the sufficient direct relative condition $H_G(q) = o(\delta q^6)$. This supplies the strict stop and the field remainder in both corollaries. If retaining the assertion “much smaller even than b ,” require $H_G = o(\delta e^{-\sigma S})$; any polynomial lower restriction on δ also implies this because $S = C_2 \log(1/q)$.

Downstream: §13 already fixes $\delta = \log(1/q)^{-4}$, so it satisfies every fixed polynomial lower restriction for small q . No change to its conclusion is needed. A uniform theorem over all arbitrarily small δ would require a newly proved error estimate proportional to the actual seed/forcing; that has not been proved in this audit.

F2 — Abstract passive convergence proof invokes an unstated Kato input **Location:** ph:colour, lines 2936–2947 and 2987–2993.

Exact quotes:

“with the L^1 , L^2 , and L^∞ contraction bounds and the weak-form equation.” (2942–2943)

“The Kato inequality and a radial cutoff also give uniform tightness” (2988–2989)

Issue/category: The approximating drifts are only specified to be uniformly bounded and uniformly convergent; smoothness, Sobolev regularity, divergence freedom, or the Kato inequality for them is not explicitly stated. In particular the zero-diffusion extension cannot silently invoke renormalization for an arbitrary merely bounded approximating drift. Global norm contractions alone are not a local Kato inequality. This is an abstract hypothesis/proof mismatch, not a defect of actual NS passive colours.

Verified repair: Specify divergence-free drifts and strongly continuous standard mild/renormalized colour solutions in applications, or replace the Kato tightness paragraph by the conserved limiting L^1 plus finite-propagation argument above. The latter proves uniform strong convergence directly from the other weak-form and contraction assumptions. If asserting convergence literally *in* $C_t X$ for zero-diffusion approximants, also require that the approximating colours have strongly continuous representatives; otherwise state uniform-in-time strong convergence. Actual NS colours and transported colours with the stated Sobolev regularity meet these requirements.

Downstream: No change to fixed-colour recovery, observable convergence, or separation for the actual system. The missing assumptions should be explicit to keep the abstract proposition accurate.

F3 — Incorrect preservation mechanism for compact support **Location:** ph:sparse, lines 3185–3186.

Exact quote:

“Zero circulation and compact support pass by strong L^1 convergence.”

Issue/category: The circulation claim is valid; the compact-support claim is false without a common support ball. If $g \in C_c^\infty(B_1)$ is nonzero with zero integral, then $g_k = \sum_{m=1}^k 2^{-m} g(x - 4me_1)$ are compactly supported zero-circulation functions and converge strongly in $L^1 \cap L^p$ to a noncompactly supported function.

Verified repair: Write that circulation passes by strong L^1 ; compact support follows from the common support ball of the scaled construction, or from the already established identification of g_∞ with a member of the assumed all-phase compact-data family. The common support ball is explicitly available in cont-stack-data, lines 89–93.

Downstream: No loss of the sparse conclusion or of eligibility for cp:compact in the actual fixed stack.

F4 — Incorrect zero-reference-error sentence for arbitrary radial data **Location:** proof of prop:inactive-actual-row, line 3295.

Exact quote:

“If every initial colour is radial, $\zeta_i = h_i = 0$ solves this system.”

Issue/category: This overstates the abstract initial-profile assumptions; it is correct within the narrower actual seeded family if “all radial” means every seed is zero and the datum equals the fixed reference. Radiality alone only implies $h_i = 0$, not $w_i = G_i$. For example take

$w_i(0) = A_i[W(x/R_i) + \varepsilon Z(x/R_i)]$ with a nonzero smooth radial zero-circulation Z , constant on the relevant core. Then $h_i = 0$ but $\zeta_i(t) = A_i e^{\nu t R_i^{-2} \Delta}(\varepsilon Z)(x/R_i) \neq 0$. The zero function solves the homogeneous differential equation but not these initial data. This is not a counterexample to the row estimate.

Verified repair: For arbitrary radial initial colours, $h_i = 0$ and ζ_i is radial and equals the heat evolution of $w_i(0) - G_i(0)$. The stronger $\zeta_i = 0$ statement requires $w_i(0) = G_i(0)$. In either case there is no angular source.

Downstream: None for the nonradial forcing argument or inactive Kato estimate; the radial no-source observation remains correct.

F5 — Common functional must be explicit in separation **Location:** ph:separation, lines 3095–3102.

Exact quote:

“their observables (92) have uniformly protected first caps. If their limiting first-cap phases are different, the two total Euler paths are different.”

Issue/category: Under the natural interpretation that the previously fixed $c_l, \chi_l, f_{l,0}$ define one functional, this is correct. If “their observables” permits different tests or coefficients on the two sequences, it is false: a_u and $-a_u$ have opposite protected-cap phases on the same path. Record this as a clarification, not a disproof of the intended corollary.

Verified repair: Add “using the same fixed initial colours, coefficients, cutoffs, and cap level in both subsequences.” Likewise make explicit that the degree covariance rotates the full coloured datum at $r = 0$, not merely one additive component against a fixed nonradial remainder.

Downstream: The intended sparse construction uses a common outer functional and is unaffected.

F6 — Positive annulus margin needed for the Gaussian heat-gradient tail **Location:** definition of $\mathcal{A}_l$, lines 3236–3237; proof lines 3409–3412.

Exact quotes:

“Choose a fixed broad annulus $\mathcal{A}_l$, containing both derivative collars of $W(x/R_l)$ ”
(3236–3237)

“Its scaled L^1 and L^p tails after multiplication by $|x| + R_i$ are bounded by $C e^{-c R_i^2 / (\nu t)}$.”
(3410–3412)

Issue/category: This is valid when “broad” means the derivative support has a fixed positive distance from the complement of the annulus. Merely containing the support, with no margin, is insufficient for a Gaussian exponent with a fixed $c > 0$. A smooth derivative collar that approaches the outer annulus boundary as e^{-1/d^2} has an exterior heat leakage scale of the form $e^{-C/\sqrt{\tau}}$, where $\tau = \nu t / R_i^2$, rather than $e^{-c/\tau}$: optimize $d^{-2} + d^2/(4\tau)$. Thus the positive gap is the required geometry behind the stated estimate.

Verified repair: State $\text{dist}(\text{supp } \nabla W, \mathbb{R}^2 \setminus \mathcal{A}_1) \geq d_* > 0$ after nondimensionalizing. Enlarging the broad annulus by a fixed amount ensures it. Choose q afterwards so the enlarged annulus remains inside all older probe cores and separated from all inner truncation balls.

Downstream: No power or parameter-order change, only a necessary explicit geometry specification.

A.4.5 Dependencies inspected and inputs not certified

Directly inspected outside the assigned range:

- lem:cont-C4-origin, lines 1804–1830: valid. Fourfold invariance cancels inverse moments except degrees 3, 7, ..., so the far field is $\lesssim \|f\|_\infty r^3 \int_{2r}^\infty s^{-3} ds \lesssim \|f\|_\infty r$. The near region is also $O(r)$. No zero circulation or regularity is needed.
- lem:cont-radial-exit, lines 1833–1881: valid for the actual smooth-positive-time bounded drift. The $\cosh(\lambda|x|)$ supermartingale gives the stated dimension-two Gaussian using only radial drift; tangential rotation does not enter.
- prop:cont-outer-core, lines 1883–2044: the conditional bootstrap is valid with its actual non-radial density bound. In §11 use $\|h_l\|_1 \leq 2\delta A_l R_l^2$, absorbing the factor two in constants. Older core defects use C4-origin, their exterior nonradial fields start at cubic order, and unrestricted inner velocities have the convergent $\sum A_l R_l$ bound. The finite-prefix maximum and individual radii avoid any depth-dependent constant.
- cont-stack-data, lines 67–103: the common compact support and phase-change bounds are visible. This is used only to repair the support explanation, not to infer solution behaviour.

Still inputs, not certified here: standard global mild NS well-posedness and vorticity contractions; finite-energy inequality for its smooth approximations; isolated unstable mode and transform selection; actual interacting active signed-history expansions; common protected outer caps across every homotopy and every phase choice; and the later global induction/final theorem. In particular §§8–13 do not alone prove two different vanishing-viscosity limits: the sparse theorem remains conditional until those active hypotheses are proved.

A.5 Sections 14–16: actual source and whole history

Audited source: paper-snapshot.tex, lines 4020–5068. The paper was not modified. I did not read earlier audits. This is a local proof audit, not an assumption that the spectral, inactive-prefix, or nonlinear cold-tail dependencies have been proved.

A.5.1 Finding and scope

I found no definite counterexample to the **locally stated, conditional** weighted foreign-field, postcap-tail, weighted Nash, or parent-radius handoff results. Their quantitative estimates check. The selected-source theorem has a coherent finite bootstrap **provided its three substantive upstream inputs are valid**: the finite-sector spectral propagator, unconditional inactive-prefix closure, and full nonlinear cold-tail estimate. I verified how those inputs enter and checked the downstream estimates; I did not independently certify their full proofs.

The strongest source-history statement in this range is **conditional**, not a proof that the receiving cascade supplies its hypotheses. This is material: a selected source on its preparation/reference window is not a proof of the actual parent/larger density stop on the subsequent complete logarithmic interval. The text says this explicitly, rather than hiding it. Any later application that omits this input would be a significant proof gap. No counterexample to the stipulated common-NS initial-data family has been constructed here; failure to prove a hypothesis is not itself a counterexample.

A.5.2 Claim ledger

“Checked” below means the local implication was rederived under its exact inputs. “Conditional” identifies a hypothesis on an actual run. “Unresolved dependency” means the claimed upstream theorem has not been independently certified in this audit.

- **Global adjoint foreign-field weight.** **Lines / label:** 4037–4085, lem:next-adjoint-foreign, eq:next-adjoint-foreign. **Status:** Checked local implication; conditional on actual older cores/harmonics. **Assessment:** Young velocity maxima sum to $C \beta q$. Old harmonics give $C(\delta q^2/\beta)r^3$; core leaks give $C(\epsilon/\beta)r$. The adjoint weights r^3 and r^{-5} control the first region. On $r \geq a/(2q)$, fourfold origin growth and the removed rotation give Cr/β , hence Cq^4/β . No divergent old maximum-velocity sum is used..
- **Scalar linear loss.** **Lines / label:** 4087–4116, cor:next-adjoint-linear-loss. **Status:** Checked conditional estimate. **Assessment:** Variation of constants cancels $a(t)$ against $a(s)$, leaving $C \chi_q$ s. The density envelope, signed growth, and radial-base force are not proved by this corollary..
- **Actual postcap fourth moment and discount.** **Lines / label:** 4139–4275, prop:next-actual-postcap-tail. **Status:** Checked conditional estimate. **Assessment:** Requires the two positive initial-envelope inputs, both radial drift bounds, and both margins. The exit contribution and initial outside-core contribution give the stated E ; local unexited moment growth is at most $\exp(4 a u)$. The discount has a strict positive margin..
- **Uniform postcap scale margin.** **Lines / label:** 4277–4307, cor:next-postcap-scale-margin. **Status:** Checked conditional estimate. **Assessment:** $aU = O(\delta \log(1/q)) = o(1)$ and $DU/R = O(q \log(1/q)) = o(1)$. The negative exponent is at least a constant times $q^{-7/2/\log(1/q)}$. All positive factors are fixed powers/logarithms..
- **Origin of actual postcap hypotheses.** **Lines / label:** 4309–4327. **Status:** Conditional, explicitly stated. **Assessment:** The positive cap envelope and parent-radius Gaussian leakage are inputs. A signed cap coordinate alone does not yield them. The later handoff lemma supplies the parent-radius leakage when the actual older density stop is available..
- **Weighted off-diagonal Nash.** **Lines / label:** 4341–4403, lem:next-weighted-nash. **Status:** Checked. **Assessment:** The precise conjugated L^1 coefficient is $\ell u \cdot \text{grad } \phi + \kappa \ell^2 |\text{grad } \phi|^2 + \kappa \ell \Delta \phi$; the L^2 coefficient drops the Laplacian term. Adjoint/midpoint composition gives the stated exponential with one total time integral, not a doubled exponent. Positivity handles signed aggregates without a summed initial L^∞ norm..
- **Choice of prefix exponents.** **Lines / label:** 4405–4417, eq:next-source-prefix-powers. **Status:** Unresolved dependency. **Assessment:** The subsequent proof only needs positive margins 6.43-K, 8-K-e_P, 2-e_B. The assertion that the rates can be made arbitrarily small belongs to cor:next-inactive-closed and its matrix estimates..
- **Unconditional selected-source envelope.** **Lines / label:** 4419–4479 / proof 4481–4721, thm:next-selected-source. **Status:** Checked bootstrap reduction; unresolved dependencies. **Assessment:** The sole source-density stop is improved using a source-core estimate obtained directly from K^*f , the actual old prefix, and unrestricted young contractions. This does

not algebraically assume its own density conclusion. But correctness inherits the upstream spectral, inactive-prefix, and cold-tail proofs..

- **Actual stopped source core.** **Lines / label:** 4508–4533. **Status:** Checked conditional on provisional envelope and actual old-prefix input. **Assessment:** Tangential radial heat is removed from radial drift; the remaining integrated drift is $CB \delta_{\text{--}}^*/\sigma + C \beta q S + o(1)$. Backward exit controls the actual constant-core defect..
- **Signed cold fourth moment.** **Lines / label:** 4535–4546, eq:next-source-cold-fourth. **Status:** Unresolved dependency, application margins checked. **Assessment:** Aggregate initial circulation and fourth moment are zero, support is Cq , mass is $O(\beta q^2)$, and $M_g/\kappa = O(\beta q^{1/2})$. The physical normalization reproduces the upstream exponent 643/100. The nonlinear signed suppression theorem itself is an upstream input, not an elementary consequence of positivity..
- **Actual cold absolute higher moments.** **Lines / label:** 4548–4574, eq:next-source-cold-higher. **Status:** Checked local argument under actual source-core input. **Assessment:** Nash gives an integrable self displacement $O(M_g \sqrt{S/\kappa})$, smaller than $\sqrt{\kappa S}$. Local harmonic growth is weak; exit from $R_{\text{--}}^* = \kappa^{1/4}$ is exponentially unlikely. Thus $q^{(2+3n/4)}$ with fixed log factors is correct. The absolute fourth-moment power is only q^5 , so the signed cold theorem cannot be discarded..
- **Actual cold collar density leak.** **Lines / label:** 4576–4587. **Status:** Checked. **Assessment:** The radial cutoff may rise wholly inside the protected source core. Its total advection is below the fixed cutoff gap. Weighted Nash applies to the passive positive envelope generated by the actual common velocity..
- **Finite actual harmonic/multipole input.** **Lines / label:** 4589–4616, eq:next-source-input-small, eq:next-source-input-remainder. **Status:** Checked under old-core and cold-moment inputs. **Assessment:** Old inverse moments and cold ordinary moments are actual coefficient functions, not canonical replacements. Arbitrarily high fixed spatial remainder powers follow by finite truncation. No derivative estimate for a rough core leak is used..
- **Actual finite coefficient recursion and derivative hierarchy.** **Lines / label:** 4618–4668. **Status:** Checked conditional on finite-sector propagator. **Assessment:** For (m, ℓ) , both diffusion and heat inputs have at least two more assigned derivatives; every nonlinear or external differentiated input has smaller seed order and at least two more derivatives. Zero circulation plus fixed support controls the Biot–Savart low frequency. Finite-frequency closure and support preservation check. Treating b_{app}/α as one fixed function for this actual run needs no parameter analyticity..
- **Residual and closure budget.** **Lines / label:** 4670–4721. **Status:** Checked. **Assessment:** Actual foreign transport cancels in the Kato norm; only $(b - b_{\text{app}}) \cdot \text{grad } \omega_{\text{app}}$ remains. Compact gradients use the collar mismatch; the exact radial heat tail uses only the global radial growth bound. The seed, viscosity, and spatial-remainder exponent budgets are correctly computed..
- **First cap protection/time.** **Lines / label:** 4741–4777. **Status:** Checked reduction; conditional on spectral projection and upstream expansion. **Assessment:** The scalar linear foreign error is linear in the envelope, not quadratic. Localized heat and projected base

forcing have the stated small factors. The finite comparison excludes earlier crossings; the scalar derivative gives the protected crossing..

- **Exact cap to canonical jet.** **Lines / label:** 4723–4777, eq:next-source-cap-reset. **Status:** Checked finite-jet argument; unresolved canonical dependency. **Assessment:** The normalized coefficient equation has a decay gap at least $(m-1)\sigma$. Subtracting its homological fixed point and using a finite derivative hierarchy gives any rate $<\sigma$. Exact coordinate matching changes the state by the same high-order error. The unseeded radial heat correction $C \kappa t_j$ is required and is present..
- **Same canonical orbit for all finite orders.** **Lines / label:** 4789–4802. **Status:** Conditional on upstream ancient uniqueness. **Assessment:** Reusing the same isolated family and its one-sided uniqueness identifies jets after time/phase translation. This is not convergence of an infinite formal series. The upstream ancient-limit construction was not independently certified here..
- **Fixed reference reset and fixed windows.** **Lines / label:** 4804–4888, cor:next-source-reference-reset. **Status:** Checked downstream comparison; unresolved selected-source/canonical dependencies. **Assessment:** Amplification is $C \delta_q^{(-C_W/\sigma)}$. The error $\delta_q^{(M+1)}$ tends to zero after amplification if $M+1 > C_W/\sigma$; the stated stronger choice is sufficient. Every fixed positive q power beats this log amplification. The heat-base term also vanishes. The reference amplitude must be chosen below the terminal envelope amplitude, as the proof states..
- **Fixed reference positive-envelope moment reset.** **Lines / label:** 4866–4888. **Status:** Checked under source envelope. **Assessment:** Integrated own nonradial displacement is $O(\delta_{\text{ref}})$, not $O(\log \log(1/q))$. The full positive source/deep aggregate starts with bounded mass and fixed support and uses the same actual drift. Its initial summed L^∞ is unnecessary..
- **Parent-radius Gaussian handoff, including incoming children.** **Lines / label:** 4891–4942, lem:next-source-parent-handoff. **Status:** Checked conditional estimate. **Assessment:** Requires the actual larger density stop on the entire preceding horizon. Applying the outer-core proposition at the parent index gives $\kappa_{\text{parent}} = \beta q^2$ and $\kappa \leq \beta q^{7/2}$ and a vanishing margin. No child-density stop or selected-source activation theorem is needed. Local displacement is $O(H) + o(1)q^{-1}$; global post-exit moments cost only a fixed q power, absorbable in the Gaussian..
- **Actual positive history after reference.** **Lines / label:** 4944–4998, prop:next-source-positive-history. **Status:** Checked conditional estimate. **Assessment:** Requires the actual larger density stop and constant-core/Gaussian defect on the following interval. The fourth-root differential inequality gives a polynomial times $\exp(C \delta u/\beta)$. Exit error is superalgebraic if $C \delta C_0 < 1/4$; sufficiently small δ may be chosen to meet both this and the discount margin..
- **Sharper stopped-parent core follows from density stop.** **Lines / label:** 5000–5011. **Status:** Checked conditional reduction. **Assessment:** With $\delta_q = \log(1/q)^{-4}$, the inherited outer-core radial margin is $O(\log(1/q)\sqrt{\delta_q} + \beta q \log(1/q)) = o(1)$. This replaces a separate core input, **not** the parent/larger density stop..

- **Actual whole-history transform.** **Lines / label:** 5013–5054, cor:next-source-whole-transform. **Status:** Checked **conditional** convergence argument. **Assessment:** Needs reference-window convergence, preparation envelope/positive tails, the actual post-reference stopped-parent environment, and $U_q \rightarrow \infty$. The two limiting dominating functions are $Ce^{(\sigma u)}$ for negative time and $C(1+u)^4 \exp(C \delta u/\beta)$ for positive time. Weighted absolute convergence follows by two-tail truncation. Nonzero canonical transform is an additional condition..
- **Shrinking-cap vs fixed-reference transform.** **Lines / label:** 5056–5065. **Status:** Checked. **Assessment:** With $L_q = t_{\text{ref}} - t_{\text{cap}}$, the cap-centered transform is $\exp(-\mu_{\beta} L_q)$ times the reference-centered transform, apart from the shared unit phase. Its modulus is comparable to $(\delta_q/a_{\text{ref}})^{1/\beta}$..

A.5.3 Issues / precise qualifications

S1 — Significant if an unconditional application is made: actual parent/larger source-history environment is unproved here Exact text, lines 4951–4957, prop:next-source-positive-history:

“assume that every larger colour is still in its actual nonradial stop ... and that its actual constant core persists ... These are explicit hypotheses for this following interval; the preceding selected-source theorem alone does not establish the whole parent interval.”

Exact text, lines 5015–5018, cor:next-source-whole-transform:

“Use ... the actual post-reference environment of Proposition ... Assume also $U_q \rightarrow \infty$.”

The local statements correctly retain these hypotheses. They do not justify deleting them when using the history as a receiving force. The inactive-prefix closure lasts to $S = C_2 \log(1/q)$ measured from physical initial time; the complete post-reference interval must be proved to fit that actual stopped prefix, or supplied by a separately closed simultaneous bootstrap. A choice of phase, intended cascade order, or signed child cap is not that proof. A uniform history transform also cannot be obtained from cap-neighborhood convergence alone.

Rigorous correction for any broader claim: retain this actual environment as an explicit assumption and state the conclusion as the conditional weighted L^1 convergence in the companion corrections.tex. An unconditional cascade requires a separate proof of this environment. I have not supplied that missing cascade theorem.

S2 — Minor wording ambiguity: exact-cap comparison concerns full vorticity Exact text, lines 4462–4472, eq:next-source-cap-reset:

“let the matching exact-cap state ... have the same complex coordinate ... as $f_j(t_j)$... the two states differ in X by ... $+ C \kappa t_j$.”

Because f_j is a perturbation about the heat base, “the two states” should explicitly mean $\tilde{w}_j(t_j)$ and the canonical **full** vorticity state. Comparing f_j itself with that full state would introduce the order-one radial base W . The proof at lines 4747–4756 identifies the intended full-state comparison and supplies the heat correction, so this is a wording ambiguity, not a failure of the proof. The insertable explicit inequality is in corrections.tex.

S3 — Minor proof-detail qualification: finite-jet convergence and positive moments use different mechanisms The actual X approximation in lines 4718–4721 does not alone prove fourth-moment convergence. That implication needs positive-envelope radial localization, supplied separately in lines 4779–4785 and 4866–4888. The whole-history proof correctly invokes it. Any abbreviated restatement should retain this localization input. Mere strong $L^1 \cap L^p$ convergence is insufficient for fourth moments.

A.5.4 Counterexample searches and necessary-input tests

1. I tested the tempting replacement of the signed cold estimate by positivity. The positive fourth moment gives only q^5 , exceeding the intended seed q^K for $K > 6$. Thus the required finite input estimate fails with that replacement; cancellation is an essential upstream theorem.
2. I tested whether a signed cap coordinate yields a cap positive-envelope moment reset. It does not: one may add radial positive/negative rings outside any fixed radius without changing an angular growing coordinate. Therefore preparation localization must be proved for the actual passive envelope. The source proof does so, while the standalone postcap proposition correctly retains it as an input.
3. I tested whether fixed-window convergence suffices for a bilateral transform. It does not: moving future pulses outside every fixed window can change the transform. The actual-history theorem avoids this with its explicit discounted positive-history bound and $U_q \rightarrow \infty$.
4. I tested omitted radial heat-base error in the canonical cap comparison. It is genuinely unseeded, of size $O(\kappa t_j)$ in X . The current statement includes it; the later log amplification still tends to zero.
5. I checked each “depth-uniform” sum. Old amplitude sums are $O(1/\beta)$, young velocity sums are $O(\beta q)$, and old cubic coefficients are $O(\delta q^2/\beta)$. A sum of old global velocity maxima would generally fail this uniformity; that sum is not used in the checked proofs.

A.5.5 Independently checked mathematical details

Weighted Nash calculation For $F = e^{\ell \phi} f$,

$$F_t = \kappa \Delta F - (u + 2\kappa \ell \text{grad } \phi) \cdot \text{grad } F + c F,$$

$$c = \ell u \cdot \text{grad } \phi + \kappa \ell^2 |\text{grad } \phi|^2 - \kappa \ell \Delta \phi.$$

Thus positive L^1 integration gives the additional drift-divergence term $2\kappa \ell \Delta \phi$, resulting in $+\kappa \ell \Delta \phi$, and L^2 energy gives half that term, exactly canceling the $-\kappa \ell \Delta \phi$ in c . This verifies the two growth rates claimed at 4374–4378. Applying Nash to the scalar-normalized F and doing the same calculation for the time-reversed adjoint gives the full $L^1 \rightarrow L^\infty$ estimate. A cutoff limit is legitimate for the physical bounded NS drift and its linear tangential rotating-frame addition.

Postcap exit and moments The displacement margin implies $\exp(aU) \leq 2$. Consequently the Brownian threshold for exit is a fixed multiple of R , independent of U and the cap parameter. Conditional moment growth after exit uses the global $a_0 \leq C_a/\beta$ bound, with polynomial additive displacement; multiply it by the exit probability. Initial leakage uses the stated weighted fourth-moment tail. Taking the common denominator $\kappa(H_0 + U)$ weakens both Gaussian

factors in a valid direction. The local discounted term is bounded by $CM(1+u)^4 \exp(-3 \sigma u/(4 \beta))$; its tail is bounded by the stated weaker rate $\exp(-\sigma J/(2 \beta))$.

Source residual powers On $t \leq H_-^*$, dividing the viscosity comparison by $a(t)$ gives

$$q^{[3(N+1)/2 - K(C_-1/\sigma - 1)_+]} (1+S)^{(D+1)}.$$

Dividing the collar mismatch comparison gives

$$q^{[P - K \max(1, C_-1/\sigma)]} (1+S).$$

These match the paper. The pure nonlinear omitted seed orders integrate to $C a^{(M+1)}$ when $(M+1)\sigma > C_-1$. The prescribed choices produce strictly positive powers with no dependence on depth.

Whole-history convergence Write h_q for the actual, reference-centered, zero-extended moment, h for the canonical moment. On compact windows $h_q \rightarrow h$ uniformly. The negative tails obey

$$\int_{-\infty}^{-J} e^{(-\sigma u/\beta)} |h_q(u)| du \leq C \exp[-\sigma(1-1/\beta)J].$$

The actual positive tails obey, for a fixed $c > 0$,

$$\int_J^{\infty} e^{(-\sigma u/\beta)} |h_q(u)| du \leq C (1+J)^4 \exp(-cJ) + o(1).$$

The canonical positive tail has the same form. Splitting the weighted absolute difference into these two tails and the compact interval proves the conclusion, including zero extension after U_q , because $U_q \rightarrow \infty$. This does not prove the stopped-parent hypotheses on which the positive-tail estimate rests.

A.6 Sections 17–18: finite forcing and actual receiver

Source inspected: paper-snapshot.tex, lines 5069–6165, with the necessary definitions and cited lemmas read in the same snapshot. This audit did not use earlier audit files or assume the advertised global theorem. Original source files were not changed.

A.6.1 Verdict

I found no fatal algebraic, derivative-reserve, or exponent error in the finite-forcing module. Its coefficient/residual estimate, actual Kato comparison, normalized coefficient memory loss, and exact two-real-variable cap gauge are valid **conditional on the imported finite-sector spectral decomposition and global adjoint identity**. The actual receiving-stage proposition is a valid conditional induction implication when its imported actual core, inactive-density, localization, postcap moment, and ancient-jet results hold with the stated uniform constants.

The module does not obtain a protected cap from an arbitrary weighted time norm alone. It explicitly needs a nonzero full growing amplitude and three separate instantaneous actual-field bounds. Those distinctions are mathematically essential, and the current proof does respect them. This audit does not establish the deepest base stage, simultaneous uniform induction, or the final phase-shooting theorem.

The local textual defects to correct are: (1) explicitly define $L = \log(1/q)$ and require $\chi > 0$ in §17’s scale-family conclusion; (2) restrict §18’s slope assertion to its protected late window, as its proof already does; (3) choose a lower-clock constant below $6/\sigma$ if “the lower clock bound persists” is meant to reuse the same induction constant; and (4) identify the derivative at line 6134 as a derivative with respect to the amplitude parameter. These are supplied in the companion correction file. None falsifies the intended, properly scoped receiver implication.

A.6.2 1. Finite spaces and derivative bookkeeping

Lines 5082–5095 restrict the construction to a common closed collar-supported finite angular-sector space. This restriction is essential: the inherited semigroup is not being asserted on an unrestricted all-mode Sobolev space. The appropriate imported assertion is the finite-sector propagator at lines 245–256 and its support/domain discussion at lines 335–394. I checked the uses here on that basis, without independently certifying the entire spectral proof.

The rotation definition at lines 5112–5125 is consistent. In sector $+4$, T_θ multiplies by $\exp(i\theta)$, and

$$L_{\text{rot}} F_+ = (\lambda - i\tau) F_+ = \sigma F_+.$$

The vector pushforward and scalar angular shift commute with Biot–Savart and with the advective pairing. Thus the normalization used later is an actual rotation of every angular sector, rather than the invalid replacement of an order- j coefficient by the scalar phase $\exp(-ij\tau s)$.

For the reserve in lines 5133–5144, the actual differences of assigned orders are:

- $\Delta g_{(j,\ell-1)}$. **Excess assigned regularity:** 2.
- **Heat interaction with** $g_{(j,\ell-r)}$. **Excess assigned regularity:** $2r$.
- **Foreign advection of** $g_{(j-1,\ell)}$. **Excess assigned regularity:** 2.
- **Quadratic factor** $g_{(j_1,\ell_1)}$, $j_1+j_2=j$, $\ell_1+\ell_2=\ell$. **Excess assigned regularity:** $2j_2+2\ell_2$, at least 2.
- **Other quadratic factor.** **Excess assigned regularity:** $2j_1+2\ell_1$, at least 2.

These margins suffice for the single transport derivative and for diffusion. The final residual Laplacians do not demand an unstated two-derivative upgrade of the last coefficient: at the minimum order H^4 in two dimensions, Δg belongs to H^2 , hence every finite L^p ; collar support also gives L^1 control. All profile derivatives, including arbitrarily many fixed heat Taylor Laplacians, are derivatives of the fixed smooth W , so the reserve R for $\mathbf{B}$ need not include their separate orders.

The zero-mass low-frequency estimate at lines 5148–5160 is valid. Compact support gives $|\hat{f}(\xi)| \leq C |\xi| \|f\|_1$ at low frequency. After the order-minus-one Biot–Savart multiplier this is bounded; high frequency gains one derivative. The displayed $H^{(r+1)}$ velocity bound and the algebra estimate for Q therefore have the stated strength. This estimate is being applied only to supported zero-mass coefficients.

A.6.3 2. Recursion, coefficients, and exact residual

The recursion at lines 5193–5212 is the coefficient identity obtained by expanding both the radial heat profile and the perturbation in viscosity, while retaining the actual time values of B . No derivative of those time values, or of their dependence on the actual solution, is needed. Every right-hand coefficient has smaller $j+\ell$; every forcing has zero integral and collar support. Spatial derivatives of flat collar-supported coefficients preserve that support.

For the pure coefficient estimate, dividing the Duhamel convolution by $\exp(j\sigma(t-t_0))$ leaves either a strict gap $(j-1)\sigma$ for $j \geq 2$, or the assumed weighted integral of D . The weighted initial bound includes precisely the factor $\exp(\sigma t_0)$ needed at $t=0$. A long preparation time therefore does not enlarge the constants.

The stated polynomial degrees work. Direct computation gives

- $d_{-}(j, \ell) - d_{-}(j, \ell-1) = 2j + 4\ell - 2 \geq 1$;
- $d_{-}(j, \ell) - [r + d_{-}(j, \ell-r) + 1] = r(2j + 4\ell - 2r - 1) - 1 \geq 0$ for $1 \leq r \leq \ell$;
- $d_{-}(j, \ell) - d_{-}(j-1, \ell) = 2\ell$;
- $d_{-}(j, \ell) - d_{-}(j_1, \ell_1) - d_{-}(j_2, \ell_2) = 2(\ell_1 j_2 + \ell_2 j_1 + 2\ell_1 \ell_2) \geq 0$.

The direct $B \text{ grad } W_{\ell}$ term has degree only ℓ . A convolution against the weighted integrable D adds no extra time polynomial. These computations verify the induction claimed at lines 5271–5291.

I independently expanded the residual in lines 5289–5308. Its signs and index sets are correct:

- unmatched diffusion has sign $-\epsilon \kappa^{(N+1)} \Delta g_{-}(jN)$;
- unmatched heat linearization has sign $-Q_r g_{-}(j, \ell)$;
- unmatched quadratic terms have sign $-Q(g_a, g_b)$;
- top seed-order foreign advection has positive sign;
- the heat Taylor foreign remainder is $+\epsilon B \text{ grad } G_N$;
- the heat linearization remainder is $-Q_G F$.

There is no missing radial-base self residual, because the exact radial heat profile has tangential self velocity. The heat Taylor remainder has the asserted $\kappa^{(N+1)} t^{(N+1)}$ bound in X and in its gradient. $Q_G F$ is controlled by the X Biot–Savart supremum estimate, not by compact support of G_N (which would be false).

The maximum sum of coefficient polynomial degrees in a quadratic remainder is at most $4N(M+N)$. Heat-profile powers and Taylor remainders need at most an additional $N+1$; hence $d_R = 4N(M+N) + N + 3$ is sufficient. Positive viscosity powers in terms with seed order exceeding M are absorbed by $\kappa(1+H)^{d_R} \leq c$. Terms with viscosity order exceeding N retain the factor $\kappa^{(N+1)}$ and satisfy $a^j \leq a$ when $a \leq 1$. The residual estimate at lines 5241–5249 follows without a pointwise bound on D .

Classification: valid conditional local lemma; no residual monomial or derivative loss found.

A.6.4 3. Comparison with the actual foreign field

The equation for $z = w - w_{\text{app}}$ at lines 5398–5403 is exact. Using the actual foreign field in divergence-free transport makes it cancel in the Kato inequality. No derivative of the actual b occurs. The radial-growth assumption is enough to handle radial cutoffs when justifying the integrated norms; the potentially large tangential component does not enter radial cutoff fluxes.

The collar mismatch uses $\sup_U |b - b_J|$; outside U , $\text{grad } F = 0$, and only the exact radial heat gradient remains. Its weighted tail is Gaussian because U is separated from the derivative collars by a fixed radial gap. The global radial bound on b and the supremum estimate $\|b_J\| \leq aD$ suffice there.

The decisive pure residual integral is

$$\int_0^t \exp(-[(M+1)\sigma - C_1](t-v))(1+D(v)) \, dv,$$

which is bounded by the strict rate gap plus B_0 . The fixed exponent C_1 is selected before M, N, J and is maintained by making the final perturbation gradients small. Constants from a larger finite angular truncation change the terminal smallness threshold, not this exponent.

For $a(0) \geq q^*(r + \theta)$, the losses relative to $a(t)$ are precisely

- **Viscous residual. Worst relative q -power:** $\gamma(N+1) - C_H(C_1 - \sigma)_+.$

- **Constant collar mismatch. Worst relative q-power:** $P\text{-}r\text{-}\theta\text{-}C_H(C1\text{-}\sigma)_+.$

These verify the inequalities at lines 5379–5382. The Gaussian term defeats all fixed powers and logarithms uniformly for smaller viscosities. The asserted density envelope is therefore established without a lower transmission bound.

Textual omission: the scale family at lines 5372–5388 uses L and χ without explicitly defining $L=\log(1/q)$ or requiring $\chi>0$. A positive χ is necessary for the positive-power description and the immediate Ca envelope conclusion. This is a missing declaration, not a defect in the intended calculation.

A.6.5 4. What is and is not lost from the source history

The full first growing amplitude at lines 5455–5463 includes the seed and **all** retained forcing. It is the exact eventual coefficient of the zero-extended first-order equation. Substituting only the canonical target coefficient here would be wrong; the paper does not do so.

For $h_j=e^{(-j\sigma s)}T_{(-\tau s)}g_{(j0)}$, I checked the normalized equation at lines 5509–5515. The foreign term has exactly the factor $e^{(-\sigma s)}$, independently of j . The first complementary propagator decays at rate $\sigma\text{-}2\eta$; the main normalized forcing decays at rate $\sigma\text{-}h$. For $j\geq 2$, the homogeneous gap is $(j-1)\sigma$. The lower-coefficient derivative reserve controls the quadratic differences in the target order. The weighted remainder gives an $O(\eta_q)$ contribution, so the claimed memory estimate follows for every strict $d<\min(\sigma\text{-}2\eta, \sigma\text{-}h_*)$.

The memory lemma needs neither canonical waveform convergence nor a nonzero C_q . It does **not** eliminate the full growing amplitude: the universal polynomial still depends on C_q . Amplitude and phase are removed later by the exact cap gauge.

The resolvent in lines 5466–5469 is genuinely convergent for $j\geq 2$ because the largest real growth rate is σ . The polynomials have the claimed parity, finite sectors, zero mass, support, and rotation covariance. The gauge at lines 5534–5592 is a valid local inverse on two real variables; boundedness of A on X carries state error into coordinate error. Assuming the inherited orbit has the displayed jets, it matches the full protected coordinate with ancient-state error $O(e+\delta^{(M+1)})$.

The dependence of the intrinsic output q -exponent on the transient is explicit: $\exp(-dS_q)=q^{(dr/\sigma)}$ times a fixed logarithmic factor. The viscosity, remainder, and comparison errors supply other positive exponents. Thus an output exponent can be chosen below their minimum independently of the incoming reset exponent, while the incoming reset still must be good enough to justify source-reference history convergence.

Classification: conditional valid. The ancient-jet assertion at lines 5558–5563 is expressly an input; this lemma does not construct the ancient orbit.

A.6.6 5. Protected cap: necessary extra inputs and counterexamples to overreading

Lines 5622–5639 correctly say, “A weighted time norm alone does not supply a crossing slope.” The three actual-field bounds at lines 5626–5630 are separate instantaneous assumptions. Together with the global adjoint identity they give

$$A'=\lambda A+O(\delta^2)+\rho_q\delta+O(\kappa(1+H)\delta).$$

In the level band this implies the stated lower and upper modulus slopes. The envelope excludes earlier caps. The late polynomial coordinate straddles δ , and the positive band slope excludes a downward passage and any second crossing in that protected window. The quantitative early margin in lines 5697–5700 is valid: outside the band one is below $3\delta/4$; inside the band,

integration of the positive slope provides the specified margin. Cap time and circle-valued phase are continuous under uniform C_t X solution continuity.

Two counterexample mechanisms were actively checked:

1. **Cancellation without a transmission input.** A real divergence-free fourth-harmonic smooth stream field has a nonzero projected forcing (the imported gamma coefficient gives one). Its two pattern phases span the complex growing coordinate. Choose a compact time pulse and its amplitude/phases so that the discounted integral cancels the seed exactly. It has a finite weighted input norm but $C_q=0$. Hence the weighted input/residual/memory lemmas cannot imply a cap. Lines 5712–5718 correctly avoid this implication.
2. **A spike without pointwise protection.** Put a real fourth-harmonic stream pulse in the late cap window. Give it fixed small weighted integral η , but time width ℓ tending to zero. Its instantaneous projected forcing is of order η/ℓ , while its normalized state change is only $O(\eta)$. It can overwhelm the modulus derivative and cause a tiny downward excursion even though the weighted remainder is small and the state reset is accurate. Therefore a slope theorem from weighted time control alone is false. The explicit pointwise criterion excludes this construction.

Scope correction: §18 says at lines 5825–5828, “At the cap, and throughout the level interval ...” without a time restriction. Its own proof at lines 6129–6130 says, “no later Euler or Navier–Stokes return to the level is excluded.” The slope and uniqueness are established only in the protected late window. Read globally, the statement is unsupported; restrict it in the proposition as in the companion patch. This is a scope defect, not a demonstrated later-return counterexample to the intended local result.

A.6.7 6. Actual receiving stage: hypotheses, scale conversion, and stopping order

The child preparation, exact-cap reset, clock, and positive-envelope hypotheses are explicit at lines 5761–5784. The strengthened finite-order condition at lines 5776–5780 is needed because cap-to-reference comparison amplifies errors over $O(\log L)$ and the discounted history introduces $\delta^{-1/\beta}$. This is a genuine hypothesis, not a conclusion of the receiver proposition.

The source scale conversion is correct:

- $\kappa_i = \beta q^2 \kappa_c \leq \beta q^{7/2} \leq q^3$ for fixed β and sufficiently small q ;
- child time is $u = \beta(t - t_0)$;
- the projected fourth harmonic is $\beta q^6 \gamma \exp(4i(t - t_0)/5) B_c(u)$;
- changing variables in its discounted integral cancels β and gives $\mu_\beta = [\sigma + i(\tau - 4/5)]/\beta$.

The reference-phase formula $\phi_c = \theta_c + 4t_0/5$ at lines 5802–5808 agrees with parent core angular speed $-1/5$ in receiver time. It avoids double counting the reference rotation.

The full input powers, including the previously easy-to-miss recentering loss $q^{(-\epsilon_0)}$, are:

- **Own initial seed. Power:** $K - 6\epsilon_0 > 0$.
- **Descendants below the immediate child. Power:** $2\epsilon_0 > 3/2$.
- **Higher inner harmonics. Power:** $4\epsilon_0 > 7/2$.

- **Initial-seed part of older harmonic field. Power:** $K-4-C2 \lambda_{B-\epsilon_0} > 3/2$.
- **Packet part of older harmonic field. Power:** $2-C2(\lambda_{P+C4/\beta})-\epsilon_0 > 1/2$.

The inequalities at lines 5791–5794 suffice. The inner moment comparison at lines 5921–5933 is compatible with the diffusion scale: $\sqrt{\kappa_i H}/q \leq C q^{3/4} \sqrt{L} \rightarrow 0$. The positive aggregate has finite L_1 and L_p even though its summed Linfinity amplitudes diverge; tail interpolation, rather than aggregate Linfinity, is the appropriate leak argument. The first omitted exterior harmonic has moment degree $4(J+1)$, giving q^{4J+6} as claimed.

I checked the stopping order for circularity. Before t_0 the unconditional inactive theorem can be used with the child as current index; its x_i is the **full X difference from the radial heat base** (lines 3215–3222), so it really rules out the provisional receiver X stop. After t_0 , the receiver stop supplies the core protection needed for post-reference source moment control. Only an upper weighted input norm is used on a possibly truncated interval. The finite comparison then excludes the receiver stop through the deterministic H_+ . Only after that extension does the proof use two-sided history convergence and infer $|Cq| \geq |C^*|/2$. Thus no signed lower coefficient is used to reach the interval on which that coefficient is proved nonzero.

The source-history estimate remains conditional on actual parent/larger core and density control. The sharper stopped density $C_{\text{stop}} \delta$ gives $H \sqrt{C_{\text{stop}} \delta} \rightarrow 0$; fixed C_{stop} merely changes constants. For an explicit constant selection, absorb $\exp(C C_{\text{stop}} \delta H)$ into a factor 2 after choosing q small. This permits input constants to be fixed before selecting C_{stop} ; the stopping argument need not be circular in constants. In particular the heading “before the parent cap” in the imported source-history proposition is not a stopping hypothesis: its operative displayed assumptions are the actual larger-density/core bounds throughout its following interval (lines 4951–4957). Those remain valid past a hypothetical coordinate cap while the current receiver X stop has not been hit. Applying the moment proof on the full provisional interval to H_+ is therefore a valid completion and does not assume that the parent’s cap has already been delayed until H_+ .

At the late cap, the instantaneous base forcing is $O(q^6 L^D)$ plus the older strict-power field, hence $o(\delta)$. The global weighted adjoint field controls the exact cross-advection pairing by $\chi_q \|f\|_1 = o(\delta)$; a pointwise field estimate only on the collars would not suffice for that global pairing. The smooth radial heat correction is $O(\kappa_i H) = o(\delta)$. These independently establish all three instantaneous cap conditions. Here cap protection does follow, conditional on the actual-field lemmas.

The clock closes because

$$6/\sigma + C_0/\beta < C_0$$

is equivalent to $C_0 > 6\beta/[\sigma(\beta-1)]$. The extra reference advance is only $O(\log L)$, so strict clock slack absorbs it. The lower bound $t_i \geq (6/\sigma)L - O(\log L)$ supplies any fixed common $c_0 < 6/\sigma$. §18 does not explicitly restrict its input c_0 that way, so the sentence at line 6149 claiming the *same* lower bound “persists” needs this choice when used for induction. The written proposition does not otherwise assert an output lower c_0 -clock.

A.6.8 7. Remaining dependencies and limits

The following are substantive imported dependencies, not consequences of §§17–18 alone:

- the finite-sector spectral pair and semigroup;

- the global smooth adjoint test and exact scalar identity;
- an actual older-core/inactive-density theorem with constants uniform in depth;
- preparation and parent-radius Gaussian leakage/localization;
- the actual post-reference source positive-envelope moment theorem under a stopped parent environment;
- the common canonical ancient orbit, all fixed finite jets, and a nonzero bilateral transform at the selected beta;
- a deepest stage that satisfies the full inductive child hypothesis.

Sections 17–18 deliberately separate these inputs, rather than proving the entire coupled cascade. In particular their final scope remark at lines 6154–6162 is accurate. Certifying the global theorem requires §19 and the independent audits of these imported statements. Subject to those inputs and the local scope/notation repairs above, the receiver module passes this audit.

A.7 Section 19: independent global stress test

Audited directly from paper-snapshot.tex, lines 6166–6486. I reread the receiving implication (Section 18), the finite-forcing construction (Section 17), the stopped-parent moment assumptions (Section 15), the selected-source statement and reference advance (Section 16), and the statements of the passive-color and fixed-data compactness results. I did not read previous audits, and did not modify the snapshot. No nested agent was used for this additional assignment.

The substantive conclusion of this audit is conditional and precise: **the constant-selection, descending-clock, phase-degree and sparse-phase arguments in Section 19 are valid provided the quantitative selected-source, finite-forcing, actual-core, and actual receiving estimates used below are correct.** This does not independently certify their local proofs. The global proof cannot survive a failure of those inputs. This report supplies the global checks in detail and identifies a stopping-interval clarification needed in the written receiving proof.

A.7.1 Exact findings / qualifications

G1. Extend the stopped-parent future estimate through the full prospective interval

Location: Section 18, lines 5879–5895, feeding Section 19 lines 6280–6295 and 6330–6342.

Exact text at lines 5879–5883:

The actual postcap moment lemma gives, before the parent receiver cap, $|B_c(u)| \leq C(1+u)^4 e^{C\delta u/\beta} + o(q^P)$, $u \geq 0$, for every fixed P .

Exact later text at lines 5893–5894:

uniformly in depth on the full prospective interval reached below; the same argument gives convergence in the weighted

The deterministic prospective end in lines 6036–6048 is $H_{-+}=t_0+S_q+L_0$, strictly beyond the anticipated parent cap. A bound literally valid only before that cap is insufficient to exclude a prior density stop or to form the integral through H_{-+} .

Classification: repairable proof-scope omission, not a demonstrated failure of the required estimate. The stated stopped-parent proposition itself assumes only radial core/drift bounds and positive-envelope bounds on its whole interval; it does not require the parent to remain below its

cap coordinate. The current density stop in lines 5843–5850 supplies those core bounds before the density stop, including after an already attained small cap. The receiving proof should explicitly say that it applies the moment theorem up to $\min(H_+, \text{density stop})$ rather than only up to the parent cap.

Complete stop-extension argument: let T_{stop} be the first current density stop $\|w_i - W_{\text{kappa}_i}\|_X = C_{\text{stop}} \delta$, and set $T = \min(H_+, T_{\text{stop}})$. The unconditional inactive theorem at current index i controls every older color on $[0, T]$, because $T \leq C2L$. The current core bootstrap holds there since C_{stop} is a fixed finite constant and $C2L \sqrt{C_{\text{stop}} \delta} \rightarrow 0$. In child units, throughout its reference-to- T interval, the older radial drift is bounded by $C C_{\text{stop}} \delta |y|/\beta$ within a fixed fraction of the parent's core, whereas own and descendant velocities have a fixed uniform bound. The global older radial drift remains $C|y|/\beta$. The reference positive envelope has fixed moments and the parent-radius Gaussian tail. Thus the stopped-parent moment proposition applies to the entire source interval ending at T , including the portion after any current small cap. It gives the future polynomial/discounter bounds needed for the upper input norm on every shorter stop. Finite forcing comparison then gives $\|w_i - W_{\text{kappa}_i}\|_X \leq C a(t)$ up to T , with $a(H_+) = \delta e^{\{\sigma L_0\}/|C_*|}$. Choosing $C_{\text{stop}} > 2C e^{\{\sigma L_0\}/|C_*|}$ excludes $T_{\text{stop}} \leq H_+$. Only after this exclusion may one use the full prospective history, its two endpoints tending to infinities, and the signed lower bound on C_q .

The stop extension uses no signed growing lower bound. It makes explicit the argument the later text intends. If a prior actual-core lemma cannot supply its claimed radial bound at the density stop $C_{\text{stop}} \delta$, $G1$ becomes an unresolved substantive gap. I have not independently re-proved that core lemma in this additional audit.

G2. Keep the input bound independent of the auxiliary density-stop constant Location: Section 19 lines 6283–6289 and 6307–6321; Section 18 lines 6038–6047.

Exact text:

$$M_{\text{input}} \leq M_{\text{ref}} + CC_{\text{past}} \delta^{1-1/\beta} + o(1).$$

and

Compute finite-expansion bounds using the fixed input ball $M_{\text{input}} \leq M_{\text{ref}} + 1$.

The input norm uses a future source moment under the current density stop, while that stop is later chosen from the output density constant. This is a possible constant circle unless the future input majorant is explicitly uniform in each fixed trial stop constant.

Classification: repairable constant-selection clarification. The estimates written in the paper permit the needed uniform majorant; no recursive multiplication of a stop constant is necessary.

Resolution: for a fixed finite trial C_{stop} , the unexited source trajectories satisfy

$$\int |y|^4 P(u, y) dy \leq CM(1 + u + \sqrt{\kappa u})^4 e^{CC_{\text{stop}} \delta u / \beta} + \mathcal{E}.$$

The prefactor C in this form depends on the reference moment and the bounded own/descendant speed; the stop constant is needed in the displayed exponential, not in that prefactor. On $u \leq C2\beta L$, its exponent is at most $C C_{\text{stop}} \delta L$, which tends to zero. After choosing q small depending on the fixed trial constant, replace that exponential by two. The exited contribution is superpolynomially small and its stop-dependent fixed constants are absorbed by a later q threshold. This produces a future weighted input bound independent of the

trial stop constant, just as the preparation localization produces one independent of the trial past constant. Compute the expansion/output constant with that input majorant, choose C_stop , and finally decrease q to make the majorant valid. Thus G2 is not an unavoidable circularity.

The text should mention this uniformity along with the explicit trial C_past/C_reset argument. It must not merely take the unspecified constant from a conditional moment proposition whose stated dependence includes the local drift prefactor.

A.7.2 Checks of the invariant class

Parameter order and existence Lines 6209–6236 fix K , the profile constants, the reference, clocks and amplitude ratio before q . The conditions are simultaneously feasible:

- For fixed $K > 6$, $K > 6 + K/(\beta - 1)$ holds for all sufficiently large β .
- $6\beta/[\sigma(\beta - 1)] \rightarrow 6/\sigma < K/\sigma$, so fixed clock constants can be chosen with $C_0 > K/\sigma$ and a positive margin above the receiving fixed point.
- $\epsilon_0 = C_2\sigma/\beta$ tends to zero; the receiving condition $\epsilon_0 < K - 6$ can be attained simultaneously.
- The inactive rates are assumed to tend to zero in the stated order β , then finite inactive weight, then q . Once that prerequisite is granted, the displayed products with C_2 are strictly small.
- The additional selected-source condition $C_2 > K/\sigma + 1$ and its $e_P < \min(1, 8 - K)$ can be imposed too: $8 - K > 1$ for the allowed $K < 6.43$, so the latter is already implied by $e_P < 1$. The written “including” list should not be read as discarding the stronger selected-source clock margin.
- Analyticity/nonidentity of the bilateral transform, if established by the preceding transform theorem, permits a nonzero value in any open interval of allowed large β . Its zero set cannot contain that entire interval. Selecting β this way precedes the finite input constants; a small $|C_*|$ increases those constants but does not alter the possibility of a final small q .

I have verified the logical order here. I have not independently established the numerical selected-source exponent 6.43 or the smallness of the inactive rates from their local proofs.

Preparation localization and the history norm Lines 6241–6259 may be made uniform in trial C_past because its only additional preparation displacement is $O(C_past \delta)$; descendants add $O(\beta q L)$, and older errors tend to zero. For each fixed trial constant those displacements fit inside the same predetermined radial margin after decreasing q . The envelope mass is fixed by contraction, so fixed-ball localization gives depth-independent fixed moments. At parent radius c/q , the bounded source/descendant drift has deterministic displacement $O(L)$, and the parent-core radial coefficient has exponential $e^{\{O(\delta L)\}}$. Both are much smaller than $q - 1$; global moment growth contributes only fixed powers, dominated by the Gaussian $\exp(-c q^2/(\kappa L))$.

The small-cap-to-reference time has length

$$-s_q = \sigma^{-1} \log(a_{\text{ref}}/\delta) + O(\delta) = O(\log L).$$

The error amplification is at most $C\delta^{\{-C_W/\sigma\}}$. If the incoming reset is $C_reset \delta^{\{M+1\}} + q^{\chi} L^D$, the advance error is at most

$$CC_{\text{reset}} \delta^{M+1-C_W/\sigma} + Cq^{\chi} L^{D+4C_W/\sigma} + \text{actual foreign/viscous advance errors.}$$

The strengthened weighted requirement multiplies this by $\delta^{-1/\beta}(1+|s_q|)$. The selected inequality $M+1 > C_W/\sigma+2$ leaves a strictly positive power of δ even after this multiplication, because $1/\beta < 1$. A fixed positive χ survives every logarithmic loss. The foreign/viscous advance errors must have the positive power bounds established in the local source/receiver proofs; a mere qualitative convergence claim for those terms would be insufficient for the uniform class.

For the pre-small-cap history, put $v=u-s_q \leq 0$. Its discounted bound is exactly

$$\int_{-\infty}^{s_q} e^{-\sigma u/\beta} C_{\text{past}} \delta e^{\sigma(u-s_q)} du = \frac{C_{\text{past}} \delta e^{-\sigma s_q/\beta}}{\sigma(1-1/\beta)} \leq C C_{\text{past}} \delta^{1-1/\beta}.$$

This verifies eq:next-invariant-input, including its sign and exponent. It also explains why a large trial past constant can be chosen after the fixed reference input majorant. The mid-history is controlled by the fixed ancient segment and the advance error; the future is controlled by the stopped-parent estimate through G1. Every retained differentiated harmonic uses a fixed stream basis, so its spatial derivative norm is a fixed multiple of its actual moment coefficient. Increasing the finite derivative/order reserve changes the finite majorant, not the asymptotic vanishing of the trial contributions.

A noncircular choice of all class constants The following order makes lines 6297–6321 concrete:

1. Fix profile, reference, K, β, C_0, C_2 , inactive weights and strict spectral/clock margins.
2. Fix the finite expansion/harmonic/viscosity orders. Form the reference input majorant in the needed finite C^R norms using the universal localization and future bounds. The majorant is independent of fixed trial past/reset/stop constants after their later q thresholds.
3. Compute all finite coefficient/output bounds in the fixed input ball $M_{\text{ref}}+1$. This includes the cap-window density coefficient, memory/reset coefficient, and the selected-source bounds for the chosen order.
4. Choose C_{past} above twice the cap-relative output constant (using the fixed lower $|C_*|/2$) and the selected-source past constant. Choose C_{stop} above twice the whole prospective-window density constant. Choose C_{reset} above twice the intrinsic reset and selected-source reset coefficients.
5. Choose common positive power χ strictly below the fixed base/receiver/memory/residual powers; choose a common finite logarithmic exponent. Decrease q to meet every trial-class localization, input, inherited advance, stop, power and crossing threshold.

There are only finitely many constraints in this order. There is no dependence on N ; no infinite sequence of decreasing q choices is required. The “fixed-profile” comparison exponent $C_1 = C_p(|\text{grad} W|X+1)$ genuinely precedes the input constants because the finite perturbation gradient is made at most one by a later cap/viscosity threshold. Actual foreign velocity remains in divergence-free error transport and does not enlarge that exponent.

Intrinsic reset exponent Lines 6314–6325 require the normalized memory theorem, not ordinary continuous dependence on an incoming canonical state. With its stated hypotheses, the use is logically correct. The actual waveform determines the full first coefficient C_q . The future main field has a polynomial bound with an exponential rate strictly below σ ; after normalization its

direct forcing decays. Quadratic lower-order differences and the complementary propagator then decay at a fixed rate $d > 0$. At the settling time

$$S_q = \frac{6}{\sigma}L + O(\log L),$$

the memory transient is $e^{\{-dS_q\}} = q^{\{6d/\sigma\}}$ times a fixed logarithmic power. The inherited canonical error is used only to ensure C_q is close enough to the fixed nonzero C_* ; it is not used to set the memory transient exponent. Exact matching of the full protected coordinate removes the waveform-dependent time/amplitude and phase parameter. Thus taking a common χ below the base and intrinsic receiving powers does not multiply or shrink it at each descending stage.

This support is conditional on the memory and exact-gauge lemmas actually holding with the stated finite input ball. If their coefficient reserve or residual estimate fails, the assertion “No input reset exponent is successively lost” does not repair that failure.

Descending clocks and prospective ends The physical age of the child expressed in receiver time is τ_{j+1}/β ; the additional reference advance is $O(\log L)/\beta$. The receiving growth interval adds $(6/\sigma)L + O(\log L)$. Thus the recurrence in lines 6332–6335 is correct.

Let the uniform additive error be at most $C \log L$. The inequality $C_0 > 6\beta/[\sigma(\beta-1)]$ provides a fixed margin

$$C_0(1 - 1/\beta) - 6/\sigma > 0.$$

For sufficiently large L , this margin exceeds $C \log L/L$; therefore $\tau_{\text{child}} \leq C_0L$ implies $\tau_{\text{parent}} \leq C_0L$. The base clock $K L/\sigma + O(\log L)$ also fits by $C_0 > K/\sigma$. A fixed added cap window and the reference advance fit below C_2L because $C_2 - C_0 > 0$. The positive growth interval gives a common lower bound, for example any $c_0 < 3/\sigma$ after decreasing q , since every receiving clock is at least $(6/\sigma)L - O(\log L)$ and the base has $K > 6$.

No depth-dependent geometric sum of error constants is needed: the fixed one-step clock margin closes the invariant interval directly. On the complete prospective receiving interval, larger colors are controlled by the unconditional inactive theorem at current index j ; current density is stopped as in G1; all younger velocities and packet envelopes use convergent unrestricted sums. The viscosity relation is

$$\kappa_j = q^{3/2}(\beta q^2)^{N-j},$$

so every current stage satisfies the inactive upper viscosity threshold and every proper receiver satisfies the stronger receiving threshold. The infinite younger packet does not require a summed Linfinity density bound.

A.7.3 Phase-degree and fixed-data selection

prop:next-phase-degree, **lines 6354–6376** **Status:** verified conditional on the common protected actual caps.

At homotopy parameter zero all seeds except the selected seed vanish. The radial infinite stack is rotation invariant. With the pattern rotation defined earlier, $\Phi_{\text{theta}} = T_{\text{theta}} \Phi_0$, so scalar/velocity covariance gives exactly $w_1^{\text{theta}} = T_{\text{theta}} w_1$, and the fourth protected test gives $A(w_1^{\text{theta}}) = e^{\{i\text{theta}\}} A(w_1)$. Its modulus and first cap time are unchanged; its cap phase

has degree one. The end of the phase circle corresponds to a physical quarter-turn, under which all colors agree with their initial symmetry, so this is a well-defined circle map.

The infinite initial stack is continuous in the homotopy parameter in X by absolute convergence, and fixed-positive-viscosity solution continuity plus the uniform protected crossing supplies a continuous circle-valued homotopy. Its degree remains one; a circle map missing a point has degree zero. The proof does not require a differentiable phase map or a scalar transmission formula with vanishing error as $N \rightarrow \infty$.

thm:next-fixed-data-nonunique, lines 6389–6448 Status: the sparse-phase construction is verified conditional on phase surjectivity and protected caps. It truly produces one viscosity-independent datum.

The variable tail bound after index m is

$$\sup_{\theta, \tilde{\theta}} \|\omega_0^\theta - \omega_0^{\tilde{\theta}}\|_X \leq C\alpha \sum_{j \geq m} A_j(R_j^2 + R_j^{2/p}) \rightarrow 0.$$

All earlier assigned phases and the outer phase remain fixed. At step k , choose $N_k \geq 2$, tune its phase by exact surjectivity, then select a fixed-viscosity continuity radius r_k for the protected cap phase. Choosing $N_{\{k+1\}}$ so the entire freely variable tail has norm below $\min_{i \leq k} r_i$ controls **all** later changes collectively relative to each earlier assigned datum. There is no need for a continuity estimate uniform as viscosity goes to zero, and no accumulation of stepwise perturbation errors. The final infinite datum is within each such radius by the full tail bound, giving cap phase error at most 2^{-k} .

At fixed viscosity, the required continuous-dependence estimate follows directly from the vorticity mild equation and the kernel split. On a bounded initial X ball,

$$E(t) \leq E(0) + C\nu^{-1/2} \int_0^t (t-s)^{-1/2} E(s) ds,$$

for the X difference of two total solutions. Fractional Volterra comparison gives $\sup_{t \leq T} E(t) \leq C_{\{\nu, T\}} E(0)$. Hence the velocities converge uniformly. For the fixed outer color, the difference inequality displayed at lines 6432–6434 and the fixed $L1, L2$ contractions give color continuity in $C_t(L1 \text{ intersection } L2)$. Pairing with the bounded protected test gives uniform observable continuity; the common early margin and lower slope convert it to first-cap phase continuity. No gradient of the actual color is needed.

All final seeds have the same fixed amplitude α , and the selected viscosities are $q^{\{3/2\}}(\beta q^2)^{N_k-1} \rightarrow 0$, since the stack condition implies $\beta q^2 < 1$. Alternating target signs converge exactly to $+1$ and -1 on the two subsequences despite a fixed, nonzero q : exact surjectivity and the diagonal tail tolerances are what ensure this, not the $o(1)$ stage phase formula.

Limit separation, lines 6449–6467 Status: verified conditional on cp :compact and ph :colour with their stated regularity/convergence.

The outer color initial datum is fixed, smooth, compactly supported and bounded. Strong total-path convergence in $C_t X$ gives uniform velocity convergence by the Biot–Savart split. The stated passive-color result identifies its transport limit uniquely from the total Euler path and implies uniform observable convergence. The cap times lie in one fixed compact subset of $(0, T)$.

If $t_n \rightarrow t^*$ are protected cap times and the observables converge uniformly, then $|a(t^*)| = \delta$ and the limit cap coordinate is the limiting target $\pm \delta$. For every $d > 0$ within the common protection window, the explicit precap estimate from the first-cap lemma

gives a strict uniform margin up to t_{n-d} ; passing to the limit excludes every earlier crossing. The integrated positive slope excludes a second candidate first crossing inside the late window. Therefore t_* is the first cap of the limiting observable. If the two total Euler paths were equal, passive-color uniqueness would give the same observable and hence the same first-cap coordinate, contradicting $+\delta$ versus $-\delta$.

The time symbol T is also used for the protected test $T(x)$ in lines 6395–6409. This is a minor notation collision only; it can be removed by naming the final horizon T_{final} .

A.7.4 Result-by-result disposition and limits

- prop:next-uniform-induction. **Independent global disposition:** Its class/clock closure is logically supported by the explicit parameter order and quantitative estimates above, with $G1/G2$ clarified. Full validity remains contingent on the local selected-source, actual-core, finite-forcing and receiving estimates..
- eq:next-invariant-input. **Independent global disposition:** Verified directly, including exponent $1-1/\beta$..
- prop:next-phase-degree. **Independent global disposition:** Verified from rotational covariance, protected caps and fixed-viscosity continuity..
- thm:next-fixed-data-nonunique. **Independent global disposition:** Sparse selection and passive-observable separation are verified as implications. Actual nonuniqueness is not certified without all the prerequisite estimates and compactness theorem..

The following have not been re-proved in this additional audit: the selected-source cold-tail estimate and numerical upper exponent 6.43; the actual radial core theorem for a stopped density C_{stop} δ ; every residual/index inequality in Section 17; every finite harmonic leakage estimate in Section 18; and the strong compactness theorem. A failure of any of these can invalidate the global invariant stage class. This report is not a substitute for those independent local audits.

A.8 Complete original formal-statement index

The source contains 65 formal environments, including remarks. Every original section was assigned a direct source audit. Numbers below are the original numbers; unlabelled remarks are identified by their source line.

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