1 Fixed seeds: separate initial and forced prefix contributions

The following scalar comparison separates the common initial seeds from the forced response. Its coupling inequality is a hypothesis.

Lemma 1.1 (Two prefix norms)**.** *Let $\beta \geq 4$, $0 < q < 1$, $S \geq 0$, $b \geq 0$, and $A_i = \beta^{i-1}$. Fix a receiving index $j \geq 2$. Suppose nonnegative continuous functions x_i, $1 \leq i < j$, satisfy, on $0 \leq \theta \leq S$,*

$$x_i(\theta) \leq Mbe^{\lambda_0 \theta} + M \int_0^\theta e^{\lambda_0(\theta - s)} \left(\sum_{l<j} K_{il} x_l(s) + f_i(s) \right) \mathrm{d}s, \qquad \lambda_0 = \frac{\sigma + \eta}{\beta}, \tag{1}$$

where $M, C, \sigma > 0$, $\eta, r_* > 0$, $f_i \geq 0$ are integrable, and the nonnegative matrix K obeys

$$K_{ii} \leq C r_* / \beta, \qquad K_{il} \leq \begin{cases} C\beta^{l-j} q^{2(i-l)}, & l < i, \\ C\beta^{l-j} q^{6(l-i)}, & i < l < j. \end{cases}$$

Assume $\beta q^6 < 1$. Define

$$r_B = \frac{C}{\beta} \left(r_* + \frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right), \qquad \lambda_B = \lambda_0 + M r_B.$$

Then $x_i \leq h_i + p_i$, where the initial contribution satisfies

$$\max_{i<j} h_i(\theta) \leq Mbe^{\lambda_B \theta}. \tag{2}$$

For $D > \beta$ with $Dq^8 < \beta$, put $w_i = (Dq^6)^{j-i}$ and

$$r_P = \frac{C}{\beta} \left(r_* + \frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right), \qquad \lambda_P = \lambda_0 + M r_P.$$

The forced contribution satisfies

$$P(\theta) := \max_{i<j} \frac{p_i(\theta)}{w_i} \leq M \int_0^\theta e^{\lambda_P(\theta - s)} \max_{i<j} \frac{f_i(s)}{w_i} \, \mathrm{d}s. \tag{3}$$

All constants are independent of j. One can choose η, r_* small, then D large and q small, so both λ_B and λ_P are strictly smaller than $\sigma/2$.

Proof. The row sums of K are bounded by r_B: for $l = i + d$ the inner coefficient is at most $C\beta^{i-j}(\beta q^6)^d$, and for $l = i - d$ the outer coefficient is at most $C\beta^{i-j}(q^2/\beta)^d$. Use $\beta^{i-j} \leq 1/\beta$ and sum the geometric series.

Solve the positive comparison system

$$Y' = (\lambda_0 I + MK)Y + Mf, \qquad Y(0) = Mb\mathbf{1}.$$

Iteration of (**??**), or its integral comparison principle, gives $x \leq Y$. The variation-of-constants formula splits $Y = h + p$, with

$$h = e^{(\lambda_0 I + MK)\theta} Mb\mathbf{1}, \qquad p = M \int_0^\theta e^{(\lambda_0 I + MK)(\theta - s)} f(s) \, \mathrm{d}s.$$

The unweighted maximum norm gives (**??**). For the weighted maximum norm, the inner row acquires $(\beta/D)^d$ and the outer row acquires $(Dq^8/\beta)^d$. Their geometric sums give r_P and hence (**??**). Since λ_0 is close to $\sigma/\beta \le \sigma/4$, first make the diagonal term small. Increasing D makes the remaining q-independent term in r_P small; subsequently decreasing q makes all other terms small. □

Corollary 1.2 (The fixed-seed baseline is small in its actual field)**.** *Assume the hypotheses of Lemma* **??**, *with* $\lambda_B \le \lambda_* < \sigma/2$, *in the clock* $\theta = A_j t$. *Suppose* $H \ge 0$ *and the completed cap clocks satisfy*

$$T_{j,N} \le S/A_N + H \sum_{m=j}^{N-1} A_m^{-1}, \qquad N \ge j, \qquad S \ge H\beta/(\beta-1).$$

Let $0 \le \delta \le \delta_*$, *where* $\delta_* < \infty$ *is fixed, and put* $b = \delta e^{-\sigma S}$. *Then through* $T_{j,N}$ *the initial contribution is bounded uniformly by*

$$B = M\delta e^{-(\sigma-\lambda_*)S}.$$

Its normalized inward feedback on the receiver j *is at most*

$$C\sum_{i<j} \beta^{i-j} q^{2(j-i)} h_i \le CB\frac{q^2/\beta}{1-q^2/\beta}. \tag{4}$$

The forced contribution has feedback at most

$$CP\frac{Dq^8/\beta}{1-Dq^8/\beta}.$$

In particular, for a fixed $h_0 \ge 0$ *and* $H = h_0 \log(1/q)$ *one can take* $S = C_0 \log(1/q)$ *so that* $B = O(q^6)$ *and its feedback is* $O(q^8)$, *with constants independent of depth and uniform for* $0 \le \delta \le \delta_*$.

Proof. Writing $a = \beta^{j-N} \in (0,1]$ gives

$$A_j T_{j,N} \le Sa + \frac{H\beta}{\beta-1}(1-a) \le S.$$

Apply (**??**). The actual inward row in (**??**) is a geometric series with ratio q^2/β. For $p_i \le w_i P$ its ratio is Dq^8/β. Choose a fixed $C_0 \ge h_0\beta/(\beta-1)$ with $C_0(\sigma-\lambda_*) > 6$. Then $B \le M\delta_* q^{C_0(\sigma-\lambda_*)} = O(q^6)$, and its receiving-field row adds a factor $O(q^2)$. If $Dq^6 < 1$, the local chart bound follows from $x_i \le B + w_i P \le B + Dq^6 P$ whenever the latter quantity is below r_*. □

For the coloured Navier–Stokes evolution, the spatial q^2 and q^6 rows and the forced term must be estimated separately; see Proposition **??** and Section **??**.

Remark 1.3 (A generic smooth-profile self rate). The same positive-matrix proof works with $\lambda_0 = \gamma/\beta$ for any fixed $\gamma > 0$. Choose $\beta > 4\gamma/\sigma$, then choose the foreign-row parameters so Mr_B and Mr_P are less than $\sigma/4 - \gamma/\beta$. For a differential inequality whose diagonal rate is at most $\gamma\beta^{i-j}$, use γ/β as its common upper bound. Thus a fixed-profile Kato rate can be used independently of the unstable eigenvalue. The transform selection in Lemma **??** permits choosing a good β in any fixed sufficiently large interval, so this parameter change causes no conflict with the isolated source transform.

2 Actual inactive densities and a large amplitude ratio

Fix the smooth radial reference W and $p > 2$. The inactive comparison uses the actual coloured Navier–Stokes equation without differentiating an inactive colour. Its fixed-profile transport-stability rate is made small in the receiver clock by choosing the amplitude ratio sufficiently large. The first two closure corollaries assume an explicit packet bound; Section **??** derives a bound from the actual packet.

Let P denote angular averaging. Use

$$\begin{aligned} A_l &= A_1\beta^{l-1}, & R_l &= R_1 q^{l-1}, \\ G_l(t,x) &= A_l\left[e^{\nu t R_l^{-2}\Delta}W\right](x/R_l), & & \\ \zeta_l &= w_l - G_l, & h_l &= (I-P)w_l = (I-P)\zeta_l. \end{aligned}$$

For any density v put

$$\|v\|_{X_l} = \frac{\|v\|_1}{A_l R_l^2} + \frac{\|v\|_p}{A_l R_l^{2/p}}, \qquad x_l(t) = \|\zeta_l(t)\|_{X_l}.$$

Assume the initial colours are real, fourfold symmetric, have zero circulation, are supported in $B_{c_0 R_l}$, equal a constant in $B_{a_0 R_l}$, and satisfy

$$\|w_l(0)\|_1 \le C_0 A_l R_l^2, \qquad \|w_l(0)\|_\infty \le C_0 A_l.$$

Here a_0, c_0, C_0 are fixed. For the smooth seeded family these facts follow from the fixed compact derivative collars. The series of colours is absolutely convergent in $L^1 \cap L^p$, and $\beta q < 1$, so its common velocity and coloured evolution are well defined as before.

Choose a fixed dimensionless annulus $\mathcal{A}$ and $d_* > 0$ such that

$$\operatorname{dist}(\operatorname{supp}\nabla W, \mathbb{R}^2 \setminus \mathcal{A}) \ge d_*.$$

Set $\mathcal{A}_l = R_l\mathcal{A}$. Choose q afterwards so that these annuli lie inside the older constant-core probes and are separated from the small receiver truncation balls used below.

Proposition 2.1 (Stopped inactive row with explicit parabolic errors)**.** *There are constants $\gamma, C, c > 0$, depending only on the fixed smooth profile, p, and the displayed initial prefactors, with the following property. Fix $j \ge 2$ and*

$$T \le S/A_j, \qquad \kappa_j = \frac{\nu}{A_j R_j^2} \le q^{3/2}, \qquad 1 \le S \le C_1 \log(1/q).$$

Stop at the first time any $i < j$ reaches $x_i = \delta$. Assume $\beta \ge 4$, $0 < q < 1$, $\beta q \le 1/2$, $0 < \delta \le 1$, $x_i(0) < \delta$ for $i < j$, q sufficiently small, and

$$CS(\sqrt{\delta} + \beta q) \le c. \tag{5}$$

In the clock $\theta = A_j t$, every inactive $i < j$ obeys

$$\frac{\mathrm{d}^+ x_i}{\mathrm{d}\theta} \le \gamma\beta^{i-j}x_i + C\sum_{l<i}\beta^{l-j}q^{2(i-l)}x_l + C\sum_{i<l<j}\beta^{l-j}q^{6(l-i)}x_l + f_i(\theta) + E_{j-i}, \tag{6}$$

where

$$f_i(\theta) = \frac{1}{A_j} \left\| \mathbf{1}_{\mathcal{A}_i} \left(\sum_{l \geq j} K * h_l \right) \cdot \nabla G_i \right\|_{X_i} \qquad \text{at } t = \theta / A_j$$

is the actual current/deep nonradial forcing. With

$$e_m = C e^{CS} \exp\left[-\frac{c q^{-2m}}{\kappa_j S} \right], \qquad m \geq 1,$$

one can take

$$E_k = C\Big[\beta^{-k} e_k + \sum_{r>k} \beta^{-r} e_r + \sum_{1 \leq r < k} \beta^{-r} q^{6(k-r)} e_r + e_k \sum_{1 \leq r < k} \beta^{-r} q^{k-r} \Big]. \tag{7}$$

The sums may be enlarged to infinite sums of nonnegative terms. No small nonradial-density assumption is imposed on colours $l \geq j$.

Proof. The exact radial heat profile is an isolated Navier–Stokes solution. For the actual common velocity u, subtraction gives

$$(\partial_t + u \cdot \nabla - \nu \Delta) \zeta_i = - \left[K * \zeta_i + \sum_{l \neq i} K * w_l \right] \cdot \nabla G_i. \tag{8}$$

Every radial foreign velocity is tangential and annihilates ∇G_i, so the foreign fields on the right can be replaced by $K * h_l$. In particular diffusion produces no separate angular source. If every initial colour equals its radial reference $G_l(0)$, then $\zeta_i = h_i = 0$ solves the comparison system with the prescribed initial data. For arbitrary radial initial colours the common velocity remains tangential, and

$$h_i(t) = 0, \qquad \zeta_i(t) = e^{\nu t \Delta}\big(w_i(0) - G_i(0)\big).$$

This difference is radial and need not vanish. In the all-radial zero-seed member of the reference family the colours equal their references.

The scaled Biot–Savart split gives $\|K * \zeta_i\|_\infty \leq C_p A_i R_i x_i$. Heat contraction of the fixed reference gradient in L^1 and L^p, and the scalar Kato inequalities, bound the self term by $\gamma A_i x_i$. For example γ can be chosen as a constant multiple of $C_p(\|\nabla W\|_1 + \|\nabla W\|_p)$. No derivative of ζ_i enters this estimate; the actual nonlinear transport term disappears from the norm inequalities.

We verify the localized foreign-field estimates under the stop. For any fixed index l, the global radial component obeys

$$|u(t,x) \cdot e_r| \leq C A_l |x| + C A_l R_l \beta q. \tag{9}$$

Indeed every radial average has tangential velocity. For $k \leq l$, the fourfold origin-growth bound gives $|K * h_k(x)| \leq C \|h_k\|_\infty |x| \leq C A_k |x|$, and $\sum_{k \leq l} A_k \leq C A_l$. For $k > l$ use $\|K * w_k\|_\infty \leq C A_k R_k$ and $\sum_{k>l} A_k R_k \leq C A_l R_l \beta q$. The same estimate holds for only the foreign radial field. The constants in these sums are uniform for $\beta \geq 4$ and $\beta q \leq 1/2$.

Apply the actual outer-core bootstrap to the stopped finite prefix $l \leq j-1$. Its nonradial-density hypothesis follows from $\|h_l\|_1 \leq 2 A_l R_l^2 x_l \leq 2\delta A_l R_l^2$. All colours $k \geq j$ enter that bootstrap through the unrestricted, convergent velocity sum $\sum_{k>l} A_k R_k$. The parameter margin (**??**) and small q therefore give the individual core estimate

$$\|w_l(t) - c_l\|_{L^\infty(B_{aR_l})} \leq C A_l e^{-c R_l^2/(\nu t)}, \qquad a = a_0/2. \tag{10}$$

The bootstrap error threshold is of order S^{-1}; its improvement follows from $\kappa_j S \ll 1$ just as in the outer-core proof. Retaining the individual radius in that proof, rather than its maximum over l, gives (**??**).

Fix $c_2 > c_0$. On $B_{c_2 R_l}$ the own nonradial velocity is bounded by $C\sqrt{\delta} A_l R_l$. Older constant cores contribute only rotation; their harmonic remainders contribute $C\delta A_l R_l q^2/\beta$ and their core leaks are exponentially small. All inner velocities contribute at most $C A_l R_l \beta q$. Choose q so that this ball is strictly inside every older core probe disk. By (**??**), the integrated radial drift is less than a fixed fraction of $(c_2 - c_0)R_l$. The radial exit estimate for the positive scalar evolution from $|w_l(0)|$ gives

$$\int_{|x|>c_2 R_l} |w_l(t,x)| \, \mathrm{d}x \le C A_l R_l^2 e^{-cR_l^2/(\nu t)}. \tag{11}$$

For its post-exit fourth moment use $\phi(x) = (R_l^2 + |x|^2)^2$. Equation (**??**) gives $u \cdot \nabla \phi \le C A_l \phi$, while $\nu \Delta \phi \le C\nu R_l^{-2} \phi$. Conditioning at exit from $B_{c_2 R_l}$ consequently gives $\mathbb{E}[\phi(X_t) \mid \mathcal{F}_{\text{exit}}] \le C R_l^4 e^{C(A_l + \nu/R_l^2)T} \le C R_l^4 e^{CS}$. Multiplication by the exit probability proves

$$\int_{|x|>c_2 R_l} |x|^4 |w_l(t,x)| \, \mathrm{d}x \le C A_l R_l^6 e^{CS} e^{-cR_l^2/(\nu t)}. \tag{12}$$

Angular averaging preserves radial integrals. Inside $B_{c_2 R_l}$ use $\|h_l\|_1 \le 2 A_l R_l^2 x_l$; outside use (**??**). It follows that

$$\int |x|^4 |h_l(t,x)| \, \mathrm{d}x \le C A_l R_l^6 [x_l(t) + e_{j-l}]. \tag{13}$$

For an inner colour $i < l < j$, take a fixed small $\eta > 0$ such that $B_{\eta R_i}$ is inside the initial constant disk of colour i and $c_0 q < \eta/4$. In this ball the radial velocity estimate based on index i has the same small integrated bound: the own nonradial term is $C\sqrt{\delta} A_i R_i$, older remainders are controlled by (**??**), and all inner colours enter by $C A_i R_i \beta q$. Starting the lth positive colour in $B_{c_0 R_l}$, radial exit from $B_{\eta R_i}$ therefore gives

$$\int_{|x|>\eta R_i} |w_l(t,x)| \, \mathrm{d}x \le C A_l R_l^2 e^{-cR_i^2/(\nu t)}. \tag{14}$$

This receiver-radius estimate is essential.

On $\mathcal{A}_i$, for $l < i$, split h_l into its core and exterior. Equation (**??**) bounds the core density by $C A_l e^{-cR_l^2/(\nu t)}$; the origin-growth estimate bounds its velocity by $C A_l |x| e^{-cR_l^2/(\nu t)}$. The exterior part has mass at most $2 A_l R_l^2 x_l$ and a fourfold inverse-moment expansion starting at degree three. Thus

$$\frac{|K * h_l(x) \cdot e_r|}{R_i} \le C A_l [q^{2(i-l)} x_l + e_{j-l}]. \tag{15}$$

For $i < l < j$, truncate h_l at radius ηR_i. Every radial truncation has zero circulation because $P h_l = 0$, and fourfold symmetry cancels harmonic moments of degrees one to three. Its exterior field is bounded by $C R_i^{-5} \int |y|^4 |h_l(y)| \, \mathrm{d}y$. For the remaining density use (**??**), $\|h_l\|_\infty \le C A_l$, and Biot–Savart interpolation. This gives

$$\frac{|K * h_l(x) \cdot e_r|}{R_i} \le C A_l \left[q^{6(l-i)} (x_l + e_{j-l}) + q^{l-i} e_{j-i} \right]. \tag{16}$$

The fixed exponent c in e_m has been reduced to absorb square roots and polynomial factors.

Away from $\mathcal{A}_i$, ∇G_i is the Gaussian convolution of the compactly supported reference gradient. The fixed margin $d_* R_i$ gives, with $\tau_i = \nu t / R_i^2$, scaled L^1 and L^p tails after multiplication by

$|x| + R_i$ bounded by $Ce^{-cd_*^2/\tau_i}$. The Gaussian kernel has this decay whenever its argument is at least d_*R_i; integration and fixed polynomial weights are absorbed by decreasing c. The foreign version of (**??**) consequently bounds the total omitted forcing by CA_ie_{j-i}. Combining this with the self estimate, (**??**), (**??**), and the explicitly retained f_i proves (**??**)–(**??**). □

Corollary 2.2 (Inactive density closure under packet history)**.** *Let $\sigma > 0$ be the growth rate of the fixed smooth reference. Choose β sufficiently large that $\gamma/\beta < \sigma/4$, then a fixed $D > \beta$ sufficiently large, then q sufficiently small. Set $w_i = (Dq^6)^{j-i}$ and $b = \delta e^{-\sigma S}$. Assume $x_i(0) \le Mb$, and the packet-history bound*

$$\sup_{\theta \le A_j T} \int_0^\theta e^{\lambda_P(\theta - s)} \max_{i<j} \frac{f_i(s)}{w_i} \, \mathrm{d}s \le C_*\delta, \tag{17}$$

where $\lambda_P < \sigma/2$ is the weighted comparison rate below. Take $S = C_2 \log(1/q)$, with C_2 sufficiently large and fixed. Fix a finite $L_\delta > 0$ and choose the cap so that

$$q^{L_\delta} \le \delta(q) \le 1, \qquad CS(\sqrt{\delta(q)} + \beta q) \le c. \tag{18}$$

All constants and exponents are fixed before q is decreased. Then the stopped solution remains strictly below its stopping threshold:

$$x_i(t) \le B + w_i \left[CC_*\delta + o(q^K) \right] < \delta/4, \qquad B \le M\delta e^{-(\sigma - \lambda_B)S},$$

*for every $i < j$ and $t \le T$. Here K is any fixed prescribed power, the error in the forced weighted norm is also $o(\delta q^6)$, and the constants are independent of depth. Alternatively, the lower bound in (**??**) may be replaced by the direct relative-error condition*

$$\eta(q) S e^{\lambda_P S} = o(\delta(q) q^6), \qquad \eta(q) = \sup_{m \ge 1} \frac{e_m}{(\beta D q^6)^m}. \tag{19}$$

*The conclusion is conditional on (**??**).*

Proof. Bound each self coefficient in (**??**) by $\lambda_0 = \gamma/\beta$. This replaces, rather than identifies with, the earlier sharp spectral rate $(\sigma + \varepsilon)/\beta$. The unweighted off-diagonal row sum is at most

$$r_B = \frac{C}{\beta} \left[\frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right].$$

The row sum in the forced weights w_i is at most

$$r_P = \frac{C}{\beta} \left[\frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right].$$

Thus $\lambda_B = \lambda_0 + r_B$ and $\lambda_P = \lambda_0 + r_P$ are both below $\sigma/2$ in the prescribed parameter order. The positive comparison equation splits exactly into initial, packet, and parabolic-error contributions. Its initial part is bounded by $Mbe^{\lambda_B \theta}$, giving the stated B.

For the explicit error put

$$\eta(q) = \sup_{m \ge 1} \frac{e_m}{(\beta D q^6)^m}.$$

This is smaller than every power of q, uniformly when $\kappa_j \le q^{3/2}$ and $S = O(\log(1/q))$. Indeed successive ratios in this supremum are less than $1/2$ for small q, and its first term has Gaussian exponent of order $q^{-7/2}/S$. A direct summation of the four terms of (**??**) gives

$$\sup_{k\ge 1} \frac{E_k}{(Dq^6)^k} \le C\eta(q)\left[1 + \frac{Dq^6}{1-Dq^6} + \frac{1}{D-1} + \frac{\beta q}{1-\beta q}\right].$$

For example the third sum becomes $D^{-k}\sum_{r<k}\beta^{-r}q^{-6r}e_r \le \eta(q)D^{-k}\sum_{r<k}D^r$. The fourth uses $\sum_{r<k}\beta^{-r}q^{k-r} \le \beta^{-k}\beta q/(1-\beta q)$. Its weighted convolution is therefore at most

$$H_G(q) := C\eta(q)Se^{\lambda_P S} = o(q^L) \quad \text{for every fixed } L > 0.$$

Choose $L > L_\delta + 6$. Then (**??**) gives

$$\frac{H_G(q)}{\delta(q)q^6} \le \frac{o(q^L)}{q^{L_\delta+6}} \longrightarrow 0.$$

Thus the error is $o(\delta q^6)$ in the forced weighted norm. Under the alternative (**??**), this relative bound holds directly.

The positive-matrix variation of constants and (**??**) yield the asserted bound. Choose C_2 so $C_2(\sigma - \lambda_B) > 6$; then $B \ll \delta q^6$. Finally $\max_{i<j} w_i = Dq^6$, so small q makes the complete right side less than $\delta/4$, contradicting a first stopping time. The estimate bounds the comparison error; radial data have no angular forcing by (**??**). □

Corollary 2.3 (A static packet bound suffices for inactive density)**.** *For the inactive-density conclusion alone, the packet-history hypothesis* (**??**) *may be replaced by*

$$\sup_{\theta\le A_jT}\max_{i<j}\frac{f_i(\theta)}{w_i} \le C_*\delta.$$

Fix $C_2 > 6/\sigma$. Choose β, D, and then q, so that

$$C_2(\sigma - \lambda_B) > 6, \qquad C_2\lambda_P < 1.$$

With $S = C_2\log(1/q)$ and the cap condition (**??**)*, or the alternative relative condition* (**??**) *together with the radial margin, the actual inactive densities remain below $\delta/4$. Their normalized inward nonradial velocity on the current receiver's fixed collars is*

$$O\Big(\delta S q^{8-C_2\lambda_P}\Big) + o(\delta q^6) = o(\delta q^6).$$

The static packet bound is an explicit input. The packet estimate in Section **??** *closes the inactive stop directly, without requiring this static input to be $O(\delta)$.*

Proof. The stated parameter choices are possible since λ_B, λ_P can be made arbitrarily small by taking the fixed β, D large and then q small. The packet contribution to the forced weighted norm is at most $CC_*\delta Se^{\lambda_P S}$. Thus

$$x_i \le B + (Dq^6)^{j-i}\left[CC_*\delta S q^{-C_2\lambda_P} + o(q^K)\right].$$

The baseline $B = o(\delta q^6)$ and the nearest forced bound $C\delta S q^{6-C_2\lambda_P}$ are both below the stopping threshold. The parabolic errors are $o(\delta q^6)$ in the forced weighted norm by the preceding relative-error calculation, so they are below the stopping threshold as well. For the inward field use the actual older-core multipole estimate:

$$\sum_{i<j} \beta^{i-j} q^{2(j-i)} x_i \le B\frac{q^2}{\beta - q^2} + CC_*\delta S q^{-C_2\lambda_P}\frac{Dq^8}{\beta - Dq^8} + o(q^K).$$

The error field is at most $O(q^8)H_G = o(\delta q^6)$, and $C_2\lambda_P < 1$ makes the principal packet term $o(\delta q^6)$. □

3 An actual packet bound closes the inactive density stop

All colours in this section are the actual scalar colours of the common unforced Navier–Stokes velocity. Use the notation and fixed profile hypotheses of Proposition **??**. In addition choose q so small that $\beta q^{2/p} \le 1/2$.

Let $\mathcal{S}(t,s)$ be the positive scalar transport–diffusion propagator for that common velocity and put

$$P_j(t) = \mathcal{S}(t,0)\Big(\sum_{l\ge j} |w_l(0)|\Big), \qquad v_j = \sum_{l\ge j} w_l.$$

The symbol P_j here denotes a positive envelope, not angular averaging. The initial sum is in $L^1 \cap L^p$ and is supported in $B_{c_0 R_j}$. Positivity and monotone convergence imply

$$\sum_{l\ge j} |w_l(t)| \le P_j(t), \qquad \|P_j\|_1 \le CA_jR_j^2, \qquad \|P_j\|_p \le CA_jR_j^{2/p}.$$

The positive aggregate is used only in $L^1 \cap L^p$. The actual packet velocity is controlled by the individual colour contractions.

Lemma 3.1 (Actual packet moments)**.** *For every fixed integer $n \ge 2$, with $\theta = A_j t$ and $\kappa_j = \nu/(A_jR_j^2)$, one has*

$$\int (R_j^2 + |x|^2)^{n/2} P_j(t,x)\,\mathrm{d}x \le C_n A_j R_j^{n+2}(1 + \theta + \sqrt{\kappa_j\theta})^n e^{C_n\theta/\beta}. \tag{20}$$

Constants are independent of j, depth, viscosity and all seed phases. The moment bound holds without the inactive stop.

Proof. Write $u = u_{<j} + u_{\ge j}$. Individual colour contractions and Biot–Savart interpolation give

$$\|u_{\ge j}\|_\infty \le C\sum_{l\ge j} A_l R_l \le CA_jR_j.$$

The fourfold origin-growth inequality applied to each older colour yields $|u_{<j}(x)| \le CA_j|x|/\beta$ for $\beta \ge 4$. The constants in these two sums are uniform for $\beta \ge 4$ and $\beta q \le 1/2$. Thus the actual radial drift is bounded by $L|x| + V$, with $L = CA_j/\beta$ and $V = CA_jR_j$. Tangential rotation does not affect radial weights.

Let $M = \int P_j$ and $Q_n = \int (R_j^2 + |x|^2)^{n/2} P_j$. Differentiation of the weight, positivity, and Hölder with respect to $P_j\,\mathrm{d}x$ give

$$Q_n' \le C_n L Q_n + C_n V M^{1/n} Q_n^{(n-1)/n} + C_n \nu M^{2/n} Q_n^{(n-2)/n}.$$

If $M = 0$ there is nothing to prove. Otherwise $Y = (Q_n/M)^{1/n}$ satisfies $Y \geq R_j$ and

$$Y' \leq C_n L Y + C_n V + C_n \nu / Y.$$

A supersolution is $e^{C_n L t}(C_n R_j + C_n V t + C_n \sqrt{\nu t})$, after enlarging C_n: the derivative of the square-root term controls $C_n \nu / Y$ for $t > 0$. The initial support bounds $Y(0)$ by $C_n R_j$, proving the claim. To justify the moment calculation without assuming moments in advance, start with a finite truncation of the positive initial aggregate. The common velocity is bounded on the physical window, so its positive diffusion representation gives finite moments. Apply the displayed uniform calculation and pass monotonically to the full positive datum. Equivalently one may use truncated radial weights. The constants in the refined estimate use only L, V above, not the crude global drift bound. □

Proposition 3.2 (Actual force on a stopped inactive prefix)**.** *Under the stopped hypotheses and parameter margins of Proposition* **??**, *set*

$$\mathcal{M}(S) = (1 + S + \sqrt{\kappa_j S})^4 e^{C_4 S/\beta}.$$

For every $i < j$, $k = j - i$, and $0 \leq \theta \leq A_j T \leq S$, the actual packet force in that proposition obeys

$$f_i(\theta) \leq C q^{6k} \mathcal{M}(S) + C q^k \exp\left[-\frac{c_p q^{-2k}}{\kappa_j S} \right]. \tag{21}$$

In particular, for the weights $w_i = (D q^6)^{j-i}$ with fixed $D > 1$,

$$\sup_{\theta \leq A_j T} \max_{i<j} \frac{f_i(\theta)}{w_i} \leq C \mathcal{M}(S) + \epsilon(q),$$

where $\epsilon(q)$ is smaller than every fixed power of q, uniformly for $\kappa_j \leq q^{3/2}$ and $S = O(\log(1/q))$.

Proof. Let $h_{\geq j} = (I - P) v_j$, where P again denotes angular averaging. Its fourth absolute moment is bounded by twice that of P_j. Choose a fixed $\eta > 0$ so that $B_{\eta R_i}$ is inside the constant core probe of colour i, and is separated from $\mathcal{A}_i$. On this ball the own radial field is bounded by $C\sqrt{\delta} A_i R_i$, the older fields by their actual core/harmonic bounds, and all inner fields by $C A_i R_i \beta q$. The stop and its radial margin therefore give a displacement strictly smaller than $\eta R_i / 2$ up to T. Every positive packet constituent starts in $B_{c_0 R_j} \subset B_{\eta R_i / 4}$. The radial exit estimate consequently gives

$$\int_{|x| > \eta R_i} P_j(t, x) \, \mathrm{d}x \leq C A_j R_j^2 e^{-c R_i^2/(\nu t)}.$$

This estimate uses the unrestricted convergent inner velocity sum, not a small-density assumption on the packet.

Radial truncation of $h_{\geq j}$ has zero circulation and vanishing harmonic moments of degrees one to three. Its field on $\mathcal{A}_i$ is therefore bounded by $C R_i^{-5} \int |x|^4 |h_{\geq j}|$. For the exterior leakage use the last exit estimate and the L^p contraction of P_j. The elementary kernel split gives, for $p > 2$,

$$\|K * f\|_\infty \leq C_p \|f\|_1^{\frac{p-2}{2(p-1)}} \|f\|_p^{\frac{p}{2(p-1)}}.$$

Indeed the near integral is at most $C_p d^{1-2/p} \|f\|_p$, and the far integral at most $C \|f\|_1 / d$; optimize d. Angular averaging preserves radial tail masses and contracts L^p. Thus the leaked nonradial

packet has velocity at most $CA_jR_je^{-c_pR_i^2/(\nu t)}$. Combining this with the fourth moment and the uniform scaled heat-gradient bound on G_i proves (**??**): the force normalization is velocity divided by A_jR_i.

After division by w_i, the first term is at most $CD^{-k}\mathcal{M}(S)$. The second is at most

$$CD^{-k}q^{-5k}e^{-c_pq^{-2k}/(\kappa_jS)}.$$

For small q successive such terms decrease, and the first is smaller than every q-power because $\kappa_jS \le q^{3/2}C\log(1/q)$. This proves the weighted assertion. □

Corollary 3.3 (Unconditional stopped inactive closure)**.** *Fix $K > 6$ and any fixed $C_2 > 6/\sigma$. Suppose $x_i(0) \le Mq^K$ for every $i < j$. Choose β sufficiently large, then D sufficiently large, and then q sufficiently small so that*

$$C_2\lambda_B < K - 6, \qquad C_2(\lambda_P + C_4/\beta) < 1, \qquad \beta q^{2/p} \le 1/2.$$

Set $S = C_2\log(1/q)$ and $\delta = (\log(1/q))^{-4}$. For every $T \le S/A_j$ with $\kappa_j \le q^{3/2}$ the actual older densities satisfy

$$x_i(t) \le Mq^{K-C_2\lambda_B} + C(Dq^6)^{j-i}S(1+S)^4e^{(\lambda_P+C_4/\beta)S} + o(q^K) < \delta/4. \tag{22}$$

*The core persistence conclusion and the harmonic older-field estimates hold on this window, with the packet force supplied by Proposition **??**. On the current colour's fixed collars, the normalized older inward nonradial field is*

$$O(q^{K+2-C_2\lambda_B}) + O\left(S(1+S)^4q^{8-C_2(\lambda_P+C_4/\beta)}\right) + o(q^K).$$

These statements control magnitudes; they give no nonzero signed impulse.

Proof. Use the positive comparison matrix from the proof of Corollary **??**. Its initial unweighted part is bounded by $Mq^Ke^{\lambda_BS}$. Its packet part is bounded in the forced weights by $CS\mathcal{M}(S)e^{\lambda_PS} + o(q^K)$, using Proposition **??**. The parabolic error sum is the same sum as in that corollary. The cap $\delta = (\log(1/q))^{-4}$ satisfies $q^{L_\delta} \le \delta$ for every fixed $L_\delta > 0$ once q is small, so its error is also $o(\delta q^6)$. Since $\kappa_jS \ll 1$, this proves (**??**). Write $a = C_2(\lambda_P + C_4/\beta) < 1$. The nearest forced density is bounded by $CDq^{6-a}S(1+S)^4$, and the initial term has exponent $K - C_2\lambda_B > 6$. Their sum and the Gaussian error are $o(\delta)$, giving the strict $\delta/4$ improvement. The radial margin holds since $S\sqrt{\delta} = O((\log(1/q))^{-1})$ and $S\beta q \to 0$. A first stopping time is thus impossible. Finally sum the actual older core harmonic row $\sum_{i<j}\beta^{i-j}q^{2(j-i)}x_i$; the initial and packet parts give the two displayed field powers. The choices of β, D are possible because $\lambda_B, \lambda_P, C_4/\beta$ can be made arbitrarily small in this parameter order. This closes the density stop and the older-field magnitude estimates. A signed transfer or protected active cap requires additional information. □