

A uniform nonlinear annular cascade: a conditional reconstruction

Independent reconstruction from first principles

7 October 2026

Scope. This text proves the cascade implication from explicitly listed analytic, spectral, multipole, and transport inputs. Those inputs are not consequences of an informal contour calculation. In particular, the full material contour space, a twice differentiable invariant graph, the full contour norm interaction estimates, and a locally Lipschitz contour operator on each compact regular canonical trajectory must be supplied separately. The proof below does not require regularity estimates for spent packet boundaries. It uses the whole actual discounted pulse, including the actual past, and never assumes the recipient's future cap.

1 Precise input contract and what is being proved

Let $E = c^{1,\gamma}(S^1; \mathbb{R}^2)^2$ be the real little Hölder space of two material contours with fourfold symmetry, including their tangential components. Put

$$r_1 = \rho, \quad r_2 = 1, \quad \rho^2 = 3/5, \quad \sigma = \rho^5/2 > 0, \quad \tau = 3/10, \quad X_i^0(\theta) = r_i e^{i\theta}.$$

Every domain has its reference area, and a colour has signed vorticity $-\mathbf{1}_{D_1} + \rho^2 \mathbf{1}_{D_2}$ in normalized units. Its circulation is zero. The constraints and symmetry are preserved by the common transport flow.

The following are *inputs*, with constants fixed independently of the finite number of colours and of q .

(H1) In a fixed reference neighbourhood the stationary material gauge has the mild equation

$$h_t = Ah + N(h) + f(t), \quad G(t) = e^{tA}, \quad A = -\text{diag}(-1/5, 0)\partial_\theta.$$

G is a strongly continuous isometric group on E ; $N(0) = 0$ and N is C^2 (analytic is more than sufficient), with bounded derivatives on a slightly larger neighbourhood. The bounded perturbation $A + DN(0)$ generates a group $S(t)$. A commuting projection P_c has complementary group $S_c(t)$ satisfying

$$\|S_c(t)\| \leq C(1+t), \quad \|S(t)\| \leq M_\eta e^{(\sigma+\eta)t}, \quad t \geq 0.$$

The unstable real plane is identified with $\mathbb{C}$ and has exponent $\lambda_0 = \sigma + i\tau$; its growing relative-normal fourth-mode vector is $(1, -i\rho)$. The complementary projection is the spectral complement of that plane. All these bounds concern the full E space.

(H2) The relabeling-invariant moment coordinate is

$$Q_i(X) = \frac{1}{2\pi r_i^6} \int_{D_i(X)} \bar{z}^4 dz, \quad a(X) = \ell Q(X), \quad \ell = (1/2, i/(2\rho)).$$

It is C^2 ; its derivative at X^0 is the unstable left dual, and $(a, b) = (a(X^0 + h), P_c h)$ is a C^2 local chart with inverse $H(a, b)$. There is a C^2 invariant unstable graph $b = g(a)$,

$g(0) = Dg(0) = 0$, with $\|g(a)\| \leq C_g|a|^2$. The graph is invariant for every isolated bounded-time flow, not only for its period map. The reference-fixing rotation

$$(\mathcal{R}_\phi X)_i(\theta) = e^{i\phi} X_i(\theta - \phi)$$

is an isometry, commutes with the isolated equation, and obeys $a(\mathcal{R}_\phi X) = e^{-4i\phi}a(X)$, $P_c\mathcal{R}_\phi = \mathcal{R}_\phi P_c$, and $g(e^{-4i\phi}a) = \mathcal{R}_\phi g(a)$. The isolated differentiated moment has the exact local form

$$\dot{a} = \lambda_0 a + \mathcal{N}(a, b), \quad |\mathcal{N}(a, b)| \leq C_a(|a|^2 + |a| \|b\| + \|b\|^2).$$

Here and below $\|b\|^2$ means $(\|b\|)^2$.

- (H3) A nonconstant ancient orbit on that graph converges to X^0 as $t \rightarrow -\infty$. Its physical colour continues for every finite positive time, with regular separated chord-arc interfaces. On every fixed compact segment of this orbit the material contour operator is locally Lipschitz in E on a fixed tube. The colour preserves its mass and L^∞ bound, so that its normalized support radius is at most $C(1+t)$ for $t \geq 0$. This last support bound may equivalently be supplied by the transport velocity estimate in (H5).
- (H4) For local source and receiver contours, after removal of the exact constant solid rotations, the physical external fields satisfy the full contour norm estimates

$$l > i : CA_l q^{6(l-i)} x_l, \quad l < i : CA_l q^{2(i-l)} x_l, \quad x_l = \|h_l\|_E.$$

The constrained torus image fields gain a factor at most q^8 ; the local self-image term is at most $CA_i q^8 x_i$. For a fourfold zero-circulation source interior to circular receiving interfaces, with physical fourth moment $B = \int \bar{x}^4 w(x) dx$, the unstable projection of its degree-four velocity is exactly

$$D_4 B / R_i^6, \quad D_4 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0.$$

The same coefficient evaluated on a perturbed receiver differs by at most C times its full leading field norm times x_i . The exterior remainder starts at degree eight: its normalized field is bounded by the source mass times its support radius to the eighth power, divided by the receiving radius to the tenth power. Thus a source at scale qR_i and with radius $Cq(1+L)R_i$ produces a remainder at most $CA_{i+1} q^{10}(1+L)^8$. These last two assertions, including their E norm, are part of the exact multipole input, rather than a consequence of the q^6 upper bound alone.

- (H5) Every transported colour has $\|w_l\|_1 \leq CA_l R_l^2$, $\|w_l\|_\infty \leq CA_l$, and velocity at most $CA_l R_l$. For an aggregate packet $l \geq k$, use the material envelope

$$\mathcal{P}_k(t) = \bigcup_{l \geq k} \overline{D_{l,2}(t)}, \quad \mathcal{P}_{>k}(t) = \bigcup_{l > k} \overline{D_{l,2}(t)}.$$

These are unions of physical outer domains, not the support of a signed sum, which could be smaller by cancellation. Put $d(s) = \text{rad}(\mathcal{P}_k(t))/R_k$, $s = A_k(t - T_k)$. While the larger colours are local and the packet is in their central constant-vorticity regions,

$$D^+ d(s) \leq C_v + C_r q^2 d(s)^3.$$

Its initial radius is at most a fixed d_0 . If $d(s) \leq D_0(1+s)$, its fourth moment is at most $CA_k R_k^6 (1+s)^4$ and its full field on receiver $j = k-1$, divided by A_j , is at most $C_p q^6 (1+s)^4$. The same weighted bound on the inactive system is at most $C_p q^6 (1+s)^4 / D$. Here C_p depends on the fixed D_0 , not on the depth. The moment, exterior-remainder, and image bounds also apply to arbitrary spent packets. No boundary norm of a spent packet is asserted.

(H6) For each finite stack a common bounded-vorticity transport solution exists. The initially nested domains retain their inclusions and fourfold symmetry. Each currently followed contour is an E mild solution whenever its self interfaces remain in the local reference chart or the compact canonical tube and the other colours are separated from that contour. Local continuation holds under a strict tube and separation margin. Larger local colours contain a fixed central disk $B_{r_{\text{pl}} R_i}$. In that disk, after the constant rotation is subtracted, their harmonic velocities are cubic with bounded coefficients. The same cubic assertion with a constant C_S applies inside the central hole of a compact regular canonical source tube.

The strength C^2 in (H2) cannot be replaced here by continuity and $g(a) = O(a^2)$. Those two facts alone do not prove the flattened map estimate used below. The contour comparison assumption on compact trajectories in (H3) likewise cannot be inferred from analyticity only near circles.

Set

$$A_i = A_1 \beta^{i-1}, \quad R_i = q^i, \quad 4 \leq \beta \leq 5, \quad L = \log(1/q) \geq 1, \quad \varepsilon = q^6.$$

The deepest colour starts on one fixed ancient orbit at normalized time $-S_0$; every other colour starts circular. Physical time starts at $t = 0$. Its initialization is not an $O(\varepsilon)$ perturbation.

Theorem 1. *Under inputs (H1)–(H6), one can fix a good $\beta \in [4, 5]$, constants D, D_0, K , a cap $\delta > 0$, windows S, S_0 , and then $q > 0$, all independently of the finite depth N , so that the deepest source reaches its first cap, and thereafter every receiver $j = N - 1, \dots, 1$ reaches its first cap $|a_j| = \delta$. If T_k is the completed incoming cap and $\theta = A_j(t - T_k)$, the recipient's first q^3 crossing lies in $[3L/\sigma - C, 3L/\sigma + C]$, and its cap occurs by $7L/\sigma + C < 9L/\sigma$. Throughout that stage its weighted inactive prefix is at most $K\delta/2$ and its packet radius is at most $D_0(1 + \beta\theta)/2$. At its cap the flattened error satisfies*

$$\|b_j - g(a_j)\| \leq Cq^{6-8\eta/\sigma}(1+L)^6 = o(1), \quad 0 < \eta < \sigma/32.$$

For each colour used as an incoming source, $k \geq 2$, every fixed preceding and following source window converges to a rotated canonical source window. The actual signed recipient seed has size at least $c_s\varepsilon$ after the fixed incoming window and before the first q^3 crossing. All constants are uniform in j, N .

The proof is organized so that the signed estimate is used only after all non-amplitude exits in the recipient's first phase are excluded. The source's preceding cap is already completed by descending induction.

2 A causal bounded-time forced map lemma

Choose nested local neighbourhoods $U_0 \Subset U_1$ so every isolated orbit from U_0 exists in U_1 for $|r| \leq P$, where $P = 10\pi$. This is possible by the mild local existence theorem and bounded-time Grönwall around the stationary state. All chart derivatives and Lipschitz constants below are bounded on U_1 .

Lemma 1. *Let h, u solve respectively the forced and unforced mild equations on $[t, t+r]$, with $0 \leq r \leq P$, in U_1 . For strongly measurable $f \in L^1([t, t+r]; E)$,*

$$\|h(t+r) - u(t+r)\| \leq e^{K_0 P} \left(\|h(t) - u(t)\| + \int_t^{t+r} \|f(v)\| dv \right).$$

The same assertion holds backwards on an existing interval. If Φ_r is the isolated endpoint map and χ is either the raw chart (a, b) or the flattened chart $(a, y) = (a, b - g(a))$, then

$$\chi(h(t+r)) = (\chi\Phi_r\chi^{-1})(\chi(h(t))) + \xi_{t,r}, \quad \|\xi_{t,r}\| \leq C \int_t^{t+r} \|f\| dv.$$

These estimates involve only forcing on the interval actually completed.

Proof. Subtract the two mild equations. Since G is an isometry and N has Lipschitz constant K_0 on U_1 ,

$$\|h(t+v) - u(t+v)\| \leq \|h(t) - u(t)\| + \int_0^v \{K_0 \|h(t+w) - u(t+w)\| + \|f(t+w)\|\} dw.$$

Integral Grönwall gives the displayed bound. Reversing time uses the same isometric group $G(-v)$. Applying the Lipschitz chart only to the endpoints proves the second assertion. It includes the change of $g(a)$ caused by the force; no derivative of the unbounded label generator through that graph is used. $\square$

Differentiating the mild fixed point with respect to its initial state once and twice gives uniformly bounded derivatives of Φ_r for $|r| \leq P$: the variational equations have bounded coefficients $DN(h)$ and $D^2N(h)$ and the isometric group, so Grönwall bounds both variations. Thus Φ_r is uniformly C^2 . At the origin its derivative is $S(r)$. Since P_c commutes with this group, the complementary derivative has no unstable-to-complementary term.

Lemma 2. *For every $0 \leq r \leq P$, uniformly on the chosen local chart,*

$$\begin{aligned} b(t+r) &= S_c(r)b(t) + R_{b,r}(a(t), b(t)) + \xi_{b,t,r}, \\ \|R_{b,r}(a, b)\| &\leq C(|a|^2 + |a| \|b\| + \|b\|^2), \\ y(t+r) &= S_c(r)y(t) + R_{y,r}(a(t), y(t)) + \xi_{y,t,r}, \\ \|R_{y,r}(a, y)\| &\leq C(|a| + \|y\|) \|y\|, \quad \|\xi_{b,t,r}\| + \|\xi_{y,t,r}\| \leq C \int_t^{t+r} \|f\| dv. \end{aligned}$$

Moreover $\|y(t+r)\| \leq C \|y(t)\| + C \int_t^{t+r} \|f\|$.

Proof. The raw remainder is the second-order Taylor remainder of the C^2 endpoint map in the raw chart. The flattened endpoint fixes $y = 0$ because the isolated graph is continuously invariant. If its complementary component is $\Psi_r(a, y)$, then $\Psi_r(a, 0) = 0$ and $D_y \Psi_r(0, 0) = S_c(r)$. Integrating $D_y \Psi_r$ along the segment from 0 to y and using the bounded second derivatives proves the flattened remainder. Apply Lemma 1 for the forcing error. For the last assertion, compare with the isolated graph trajectory starting at $H(a(t), g(a(t)))$. Its initial distance from the actual state is at most $C \|y(t)\|$; it remains on the graph, and the flattened chart is Lipschitz. $\square$

At complete periods $M_c = S_c(P)$ satisfies $\|M_c^n\| \leq C(1+n)$. Iterating the raw or flattened recurrence is therefore an ordinary discrete Duhamel estimate with a genuine $L_t^1 E$ forcing error. Lemma 2 handles the last incomplete period and supplies causal interpolation without a spurious linear $|a|$ contribution to b .

The differentiated moment is legitimate for mild material solutions. Undo the label translations. In the physical material equation, the embedding is absolutely continuous in E on compact intervals and its velocity is $V(X) + f$. Differentiate the C^1 moment functional there and restore the labels. Relabeling invariance gives

$$\dot{a} = Da(X)[V(X) + f] \quad \text{almost everywhere.}$$

This does not assert that AX is an E -valued vector. Spatial rotating-frame terms are computed by $a \mapsto e^{-4i\phi}a$. The same proof of Lemma 1, on finitely many local Lipschitz neighbourhoods, compares a forced trajectory to any fixed compact regular canonical trajectory, with constants depending on that trajectory and window.

3 Frames and the canonical bilateral seed

The larger-colour reference rotation for receiver j is $\Omega_j^{\text{ext}} = -(1/5) \sum_{i < j} A_i$. For $k = j + 1$, $\Omega_k^{\text{ext}} = \Omega_j^{\text{ext}} - A_j/5$. Passing from receiver native axes to source native axes adds $\mathcal{R}_{\theta/5}$ in receiver time θ and changes its moment exponent to

$$\lambda = \sigma + i\tau - 4i/5 = \sigma - i/2.$$

The complementary norm is unchanged. Because $P/5 = 2\pi$, this additional action is the identity over a receiver period; the same period map estimates apply. For a partial period it is an isometry. More precisely the incoming-frame subperiod derivative is $S_c^{\text{in}}(r) = \mathcal{R}_{r/5} S_c(r)$, and its raw and flattened maps are the maps of Lemma 2 followed by $\mathcal{R}_{r/5}$. Equivariance of g makes their remainders and force-error norms unchanged. Thus one need not incorrectly identify the two subperiod maps themselves. In source time $s = \beta\theta$ the exponent is $z_\beta = \lambda/\beta$. The force is divided by β under this time change, so its integrated norm is preserved. The torus image is included in the forcing, rather than treated as rotation equivariant.

Fix a small canonical cap $|a| = \delta$, with cap time zero. On the graph, $\dot{a} = \lambda_0 a + O(|a|^2)$. For sufficiently small reference caps,

$$\sigma_0 |a| \leq \frac{d}{dt} |a| \leq (\sigma + C\delta) |a|, \quad \sigma_0 = 3\sigma/4,$$

throughout its local ancient part. In particular $|a(t)| \leq \delta e^{\sigma_0 t}$ for $t \leq 0$. It never vanishes. The scalar coefficient $\dot{a}/a - \lambda_0 = O(|a|)$ is integrable backwards; integrating it shows

$$a(t) = a_- e^{\lambda_0 t} (1 + O(e^{\sigma_0 t})), \quad a_- \neq 0.$$

The nonzero constant follows explicitly by exponentiating the convergent integral from any fixed nonzero $a(t_1)$.

The physical fourth moment is

$$B_*(t) = 2\pi[-\rho^6 Q_1(t) + \rho^2 Q_2(t)] = b_- e^{\lambda_0 t} + O(e^{(\sigma+\sigma_0)t}),$$

where $b_- = 2\pi(-\rho^6 - i\rho^3)a_- \neq 0$ because the growing normal vector is $(1, -i\rho)$. This vector and its left dual are part of the full spectral input. On positive times, $|B_*(t)| \leq C(1+t)^4$ by mass and support.

Lemma 3. *The bilateral transform*

$$F(z) = \int_{\mathbb{R}} e^{-zt} B_*(t) dt, \quad 0 < \text{Re } z < \sigma,$$

is holomorphic and not identically zero. There is $\beta \in [4, 5]$ for which $F(z_\beta) \neq 0$. Shrinking the cap by ancient time translation preserves this nonvanishing, and $|F_\delta(z_\beta)| \asymp \delta^{1/\beta}$.

Proof. On every compact substrip the negative exponential and positive polynomial bounds provide an integrable majorant also after any fixed number of z derivatives. They prove holomorphy by differentiation under the integral. For $0 < c < \sigma$, $e^{-ct} B_*(t)$ is a nonzero $L^1(\mathbb{R})$ function. If F were identically zero, its Fourier transform on the line $c + i\mathbb{R}$ would vanish. Fourier uniqueness can be seen without an inversion hypothesis: convolve the function with Gaussians, whose Fourier inversion is valid and whose transforms never vanish; each convolution is zero, and the Gaussian approximate identity converges in L^1 . This contradicts the nonzero ancient asymptotic. The points z_β form a nonconstant line segment inside the strip. If all were zeros, the identity theorem would make F identically zero.

Let $t_\delta \rightarrow -\infty$ be the old orbit time at the smaller cap. The new orbit is $B_\delta(t) = B_*(t + t_\delta)$, so a change of variable gives $F_\delta(z) = e^{zt_\delta} F(z)$. The ancient asymptotic gives $\delta \asymp e^{\sigma t_\delta}$, proving the magnitude assertion. $\square$

Fix such a β and, after the final cap has been chosen, define

$$c_0 = |D_4 F(z_\beta)| > 0.$$

The whole bilateral integral is used. $B_*(t)$ need not have any pointwise sign or phase condition for positive time.

4 The depth-independent inactive estimate

In a stage with receiver j and source $k = j + 1$, put

$$\theta = A_j(t - T_k), \quad s = \beta\theta, \quad H = 9L/\sigma, \quad w_i = (D\varepsilon)^{j-i}, \quad I_j(\theta) = \max_{i < j} x_i(\theta)/w_i.$$

For $j = 1$, $I_j = 0$. Fix $D > \beta$ and eventually $D\varepsilon < 1$. At the completed incoming cap the induction data are

$$I_j(0) \leq DK\delta\varepsilon, \quad x_j(0) \leq DK\delta\varepsilon, \quad d(0) \leq d_0.$$

All preceding stage clocks and pointwise prefixes are also part of the induction data; norms at time zero alone do not determine the signed seed.

Lemma 4. *Choose $0 < \eta < \sigma/32$. One can choose D , a fixed $\alpha_* < \sigma/2$, and then sufficiently small q , so that on a stopped interval with $I_j \leq 2K\delta$, all inactive colours local, and the packet radius and separation barriers,*

$$I_j(\theta) \leq C_w e^{\alpha\theta} I_j(0) + C_w \int_0^\theta e^{\alpha(\theta-v)} [x_j(v) + \varepsilon(1+v)^4] dv, \quad 0 < \nu \leq \alpha \leq \alpha_*.$$

Here $\nu = (\sigma + \eta)/\beta$. The current normalized feedback and its discounted integral satisfy

$$\begin{aligned} \|F_{\text{out}}(\theta)\| &\leq CDq^8 I_j(\theta), \\ q^8 \int_0^\theta e^{-\sigma v} I_j(v) dv &\leq Cq^8 [I_j(0) + \varepsilon] + Cq^8 \int_0^\theta e^{-\sigma v} x_j(v) dv. \end{aligned}$$

All constants are independent of the number of inactive rows.

Proof. For an inactive receiver $i < j$ and inner inactive source $l = i + d < j$, its coefficient in receiver time θ , after weights, is

$$\frac{A_l}{A_j} q^{6d} \frac{w_l}{w_i} = \frac{A_i}{A_j} (\beta/D)^d.$$

For an outer source $l = i - d$ it is

$$\frac{A_l}{A_j} q^{2d} \frac{w_l}{w_i} = \frac{A_i}{A_j} (Dq^8/\beta)^d.$$

The weighted current-source coefficient is $D^{-(j-i)} \leq 1/D$. Since $A_i/A_j \leq 1/\beta$ and $w_i \leq D\varepsilon$, the isolated quadratic remainder divided by w_i is at most $2CDK\delta\varepsilon I_j/\beta$. Add image rows and the packet input. The maximum of the finite mild inequalities therefore obeys

$$\begin{aligned} I_j(\theta) &\leq M_\eta e^{\nu\theta} I_j(0) \\ &\quad + M_\eta \int_0^\theta e^{\nu(\theta-v)} \{r(D, q) I_j(v) + C[x_j(v) + \varepsilon(1+v)^4]/D\} dv, \\ r(D, q) &\leq \frac{C}{\beta} \left[\frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} + q^8 + 2DK\delta\varepsilon \right]. \end{aligned}$$

This takes a maximum of inequalities; it does not sum their number. Multiply by $e^{-\nu\theta}$ and apply scalar integral Grönwall. The result has $\alpha = \nu + M_\eta r(D, q)$. Choose D sufficiently large to make $\nu + M_\eta r(D, 0) < \alpha_* < \sigma/2$, then choose q with $Dq^8 < \beta/2$ and $\alpha \leq \alpha_*$.

Feedback from $i = j - d$ has coefficient $(A_i/A_j)q^{2d}w_i = (Dq^8/\beta)^d$. Summing this geometric row proves its bound. Insert the Volterra estimate into its discounted integral. Tonelli applied to nonnegative majorants gives, for $v \leq \theta$,

$$\int_v^\theta e^{-\sigma u} e^{\alpha(u-v)} du \leq e^{-\sigma v} / (\sigma - \alpha_*).$$

The discounted polynomial is integrable, proving the final estimate. $\square$

Lemma 5. *Suppose at a stopped current time $\theta \leq H$, for every $v \leq \theta$,*

$$x_j(v) \leq C_x |a(\theta)| e^{-\sigma_0(\theta-v)} + C_n \varepsilon (1+L)^{12}, \quad \sigma_0 = 3\sigma/4.$$

Then

$$I_j(\theta) \leq C_{\text{pr}} |a(\theta)| + C_{\text{rem}} \varepsilon (1+L)^{12} e^{\alpha\theta}.$$

C_{pr} depends only on the fixed $C_x, C_w, \sigma_0, \alpha_*$, not on K, C_n or later window constants. If $K \geq 4C_{\text{pr}}$ and $|a| \leq \delta$, sufficiently small q improves the prefix to $I_j \leq K\delta/2$.

Proof. Insert the displayed bound into Lemma 4. The above-noise integral is bounded by $C_w C_x |a(\theta)| / (\sigma_0 - \alpha_*)$. The initial term and the noise term have the claimed form. The packet term has this form because $\alpha \geq \nu > 0$ and $\int_0^\infty e^{-\nu v} (1+v)^4 dv < \infty$. Finally

$$\varepsilon (1+L)^{12} e^{\alpha H} \leq q^{6-9\alpha_*/\sigma} (1+L)^{12} \rightarrow 0,$$

since $\alpha_* < \sigma/2$. Choose q so its fixed multiple is at most $K\delta/4$. The amplitude term is at most the other $K\delta/4$. $\square$

5 Before q^3 : raw centre, upper amplitude, and actual errors

Choose $D_0 \geq 4(d_0 + C_v)$ before choosing K . On a packet stop $d(s) \leq D_0(1+s)$ and $s \leq \beta H$, the Dini inequality integrates to

$$d(s) \leq d_0 + C_v s + \frac{C_r q^2 D_0^3}{4} [(1+s)^4 - 1].$$

For small q , division by $1+s$ improves its bound to $D_0/2$, because $q^2(1+\beta H)^3 \rightarrow 0$. Choosing also $qD_0(1+\beta H) < r_{\text{pl}}/2$ improves separation. This argument is independent of a signed seed.

While $I_j \leq 2K\delta$, x_j is local, and these packet barriers hold, the full force in receiver time is

$$\|F_j(\theta)\| \leq C_p \varepsilon (1+\theta)^4 + CDq^8 I_j(\theta) + Cq^8 x_j(\theta) \leq F_* \varepsilon (1+L)^4. \quad (1)$$

F_* is fixed after the displayed constants, because $q^8/\varepsilon = q^2$.

Stop the first phase at the first $|a| = q^3$, $\|b\| = 2B\varepsilon(1+L)^6$, $I_j = 2K\delta$, packet or separation departure, local chart departure, or $\theta = H$. Before this first event $|a|^2 \leq \varepsilon$. Iterate Lemma 2 over at most CL complete periods and use $\sum_{m \leq CL} (1+m) \leq C(1+L)^2$. The last incomplete period is handled by the same lemma. It gives

$$\|b\| \leq C_b \varepsilon (1+L)^6 + C_b q^3 (1+L)^2 (2B\varepsilon(1+L)^6) + C_b (1+L)^2 (2B\varepsilon(1+L)^6)^2.$$

The first term includes the inherited $O(\varepsilon)$ centre, the a^2 remainder, and the integrated force. Choose $B \geq 4C_b$; then small q makes the other two terms at most $B\varepsilon(1+L)^6/2$. Consequently

$$\|b(\theta)\| \leq B\varepsilon(1+L)^6. \quad (2)$$

This proves the centre bound on the current stopped interval without a source sign or a future recipient cap.

In incoming axes the exact moment equation is

$$a_\theta = \lambda a + \mathcal{N}(a, b) + F_a(\theta), \quad \lambda = \sigma - i/2.$$

The current circular leading packet force is bounded by a discounted polynomial. Moving the receiver contours introduces at most $C\varepsilon(1+\theta)^4(|a| + \|b\|)$, plus the smaller multipoles. Use the weighted feedback estimate with $x_j \leq C_{\text{ch}}(|a| + \|b\|)$ and put $U(\theta) = e^{-\sigma\theta}|a(\theta)|$. The centre square is an additive term at most $C\varepsilon^2(1+L)^{12}$, and

$$q^8 \int_0^\theta e^{-\sigma v} I_j(v) dv \leq Cq^8\varepsilon(1+L)^6 + Cq^8 \int_0^\theta U(v) dv.$$

Scalar Duhamel therefore gives

$$U(\theta) \leq C_u\varepsilon + C[q^3 + B\varepsilon(1+L)^6 + q^8 + \varepsilon(1+L)^4] \int_0^\theta U(v) dv.$$

The coefficient times H tends to zero. Grönwall proves

$$|a(\theta)| \leq C_u\varepsilon e^{\sigma\theta}. \quad (3)$$

In particular the first q^3 crossing cannot occur before $3L/\sigma - C$. Fixed outgoing source windows will thus be available once non-amplitude recipient exits are excluded.

Choose $a_* = C_*\varepsilon(1+L)^{12}$ with fixed $C_* \geq 1$. When $a_* \leq |a| \leq q^3$, equations (1) and (2) give

$$\frac{|\mathcal{N}|}{|a|} \leq C[q^3 + B\varepsilon(1+L)^6 + B^2\varepsilon/C_*], \quad \frac{|F_a|}{|a|} \leq (F_*/C_*)(1+L)^{-8}.$$

They tend to zero. Thus, for small q ,

$$\frac{d}{d\theta}|a| \geq \sigma_0|a|, \quad a_* \leq |a| \leq q^3. \quad (4)$$

The crossing of a_* is one-way: a first downward return would contradict this strict lower derivative near that level. If the current amplitude is below a_* , no earlier amplitude exceeded it. If it is above, every preceding above-noise amplitude is at most its current value times $e^{-\sigma_0(\theta-v)}$. Below noise the contour norm is $C_n\varepsilon(1+L)^{12}$; above noise $\|b\|/|a| \leq (B/C_*)(1+L)^{-6}$, so $x_j \leq C_x|a|$ for a fixed C_x chosen before B . This proves the hypothesis of Lemma 5 at every current time.

Choose $K \geq 4C_{\text{pr}}$. That lemma strictly improves the prefix; the packet bound already improves; and $x_j \leq C_{\text{ch}}(q^3 + B\varepsilon(1+L)^6) < r_*/2$ for small q , while $x_i \leq K\delta(D\varepsilon)^{j-i}/2 < r_*/2$ for inactive colours. All non-amplitude first-phase exits are excluded. This is the point at which a fixed outgoing source window may be invoked.

For later use, keep the complex interaction-picture coefficient. Before the first q^3 crossing, current recipient remainders satisfy the following explicit discounted bounds:

$$\begin{aligned} \int_0^\theta e^{-\sigma v} |a|^2 dv &\leq C\varepsilon q^3(1+L), \\ \int_0^\theta e^{-\sigma v} (|a| \|b\| + \|b\|^2) dv &\leq C\varepsilon^2(1+L)^{13}, \\ \int_0^\theta e^{-\sigma v} \|F_{\text{out}}\| dv &\leq C\varepsilon q^8(1+L)^7, \\ \int_0^\theta e^{-\sigma v} [\varepsilon(1+v)^4 x_j(v)] dv &\leq C[\varepsilon q^3 + \varepsilon^2](1+L)^{12}, \\ \int_0^\theta e^{-\sigma v} \|F_{\text{image}}\| dv &\leq C\varepsilon q^8(1+L)^{12}. \end{aligned} \quad (5)$$

For the first line use $|a| \leq q^3$ and (3), rather than integrating the exponentially growing upper bound squared up to an unspecified time. For the second use (2) and (3); the centre square has a bounded discounted integral. For feedback use Lemma 4. The remaining lines follow from the moving-contour and image inputs. Each is $o(\varepsilon)$; their sum is at most $C(q^9 + q^{12} + q^{14})(1 + L)^{20}$.

6 The entire actual source pulse and its negative history

We now prove the signed seed; we do not impose it as induction data. Fix incoming $k = j + 1$ with its already completed cap T_k . Every earlier recipient stage $m \geq k$ has a completed clock at most $9L/(\sigma A_m)$, and the base initialization lasts $(S_0 + O(q^8))/A_N$. Thus

$$A_k T_k \leq \frac{9\beta}{\sigma(\beta - 1)} L + C_{S_0}, \quad A_j T_k \leq \frac{9}{\sigma(\beta - 1)} L + C_{S_0}.$$

Define

$$g_\beta = 9/(\beta - 1) \leq 3.$$

On the actual past, the recipient discount is at most Cq^{-g_β} , and the past has length at most $C(1 + L)$ in recipient units. This bound is uniform in depth. It is essential that the gain is $q^{-9/(\beta-1)}$, not an unbounded gain depending on N .

Throughout that past the recipient j was inactive. The completed prefix estimates give $x_j \leq CDK\delta\varepsilon$. Every colour $i < j$ has the stronger bound $x_i \leq CK\delta(D\varepsilon)^{k-i}$. Consequently the recipient's own nonlinear past contributes at most $Cq^{12-g_\beta}(1 + L)$ to its discounted coefficient. Its outer feedback and self-image contribute at most $Cq^{14-g_\beta}(1 + L)^{12}$: the nearest outer contour has size $O(\varepsilon^2)$ and its outer interaction has an extra q^2 . The past moving-receiver correction is at most $Cq^{12-g_\beta}(1 + L)^{12}$ by $x_j = O(\varepsilon)$ and the leading packet force $O(\varepsilon(1 + L)^4)$. These quantities are $o(\varepsilon)$. The past inner packet's torus image field is $O(\varepsilon q^8(1 + L)^4)$ before discount, and hence contributes at most $Cq^{14-g_\beta}(1 + L)^{12}$ after the actual-history gain. It is included in the same $o(\varepsilon)$ image remainder.

There is a depth-uniform bound on the support of the deeper packet $l > k$ before T_k : in source units it is at most $Cq(1 + L)$. Indeed at a preceding stage with current index $m \geq k + 1$, its local current source has radius at most CR_m and its aggregate packet has radius at most $C(1 + L)R_{m+1}$. Since $R_m \leq R_{k+1}$, their union has radius at most $C(1 + L)R_{k+1}$. The same statement holds at base initialization. After the cap of $k + 1$ and during receipt by k , it is the confined packet. This covers the whole past interval, without adding one support constant per stage. Its normalized fourth moment is therefore at most $Cq^6(1 + L)^4$, by its mass $CA_{k+1}R_{k+1}^2$.

For a non-base source $k < N$, the first-phase noise estimate and the post- q^3 estimate proved in the next section concatenate to give, relative to its completed cap at source time $s = 0$,

$$x_k(s) \leq C\delta e^{\sigma_0 s} + C\varepsilon(1 + L)^{12}, \quad s \leq 0. \quad (6)$$

To avoid a logical cycle, that post- q^3 estimate is applied only to the *preceding completed source stage*, not to the current recipient. Here is the partition of its whole actual past. On the completed receiving stage for k , below-noise times have $x_k \leq C\varepsilon(1 + L)^{12}$. Above-noise first-phase times and all second-phase times satisfy $x_k \leq C_x|a_k|$, and their concatenated radial rate is at least σ_0 . Integrating that rate backwards from the *completed* amplitude δ gives $|a_k(t)| \leq \delta e^{\sigma_0 A_k(t-T_k)}$ at every such time. Before that receiving stage, the actual history is the base interval and the finitely many completed stages with current index $m \geq k + 1$. Their pointwise prefixes give $x_k(t) \leq K\delta(D\varepsilon)^{m-k} \leq DK\delta\varepsilon$; the base prefix gives the identical assertion with $m = N$. This covers every physical time $0 \leq t \leq T_k$ and proves (6) on the entire actual negative interval, including the times before k first began to receive a pulse.

The source's below-noise term and the deeper packet thus contribute at most

$$Cq^{6-g_\beta}(1 + L)^{13} = o(1) \quad (7)$$

to the source transform before $s = 0$. Its above-noise negative tail before $-S$ contributes at most

$$C\delta \int_{-\infty}^{-S} e^{(\sigma_0 - \sigma/\beta)s} ds \leq C\delta e^{-(\sigma_0 - \sigma/\beta)S}.$$

This does not assume a cap yet to be reached: T_k is completed.

For the actual positive pulse, aggregate support gives, for every available stopped recipient time with $s \geq S$,

$$\int_S^{\beta\theta} e^{-\sigma s/\beta} \frac{|B_{\geq k}(s)|}{A_k R_k^6} ds \leq C \int_S^\infty e^{-\sigma s/\beta} (1+s)^4 ds \leq C e^{-\sigma S/\beta} (1+S)^4. \quad (8)$$

The last constant is finite and uniform for the fixed β . It follows, for example, by substituting $s = S + u$ and expanding $(1 + S + u)^4$. This is a bound for the actual full aggregate pulse; it does not compare it to the canonical trajectory at large times.

Lemma 6. *For each completed non-base source cap, and on the next recipient's first-phase interval, every fixed window $[-S, S]$ is available and*

$$\sup_{|s| \leq S} \left| \frac{B_{\geq k}(T_k + s/A_k)}{A_k R_k^6} - e^{i\phi_k} B_*(s) \right| \leq r(q), \quad r(q) \leq C_S q^{6-8\eta/\sigma} (1+L)^{20} \rightarrow 0.$$

Moments here are taken in the source's native rotating axes, whose angle is fixed at its actual cap. For the base source the same assertion has error $C_{S,S_0,\delta} q^8$.

Proof for a non-base source. Its completed cap has flattened error at most $Cq^{6-8\eta/\sigma}(1+L)^6$. The inverse chart puts it that close in the full material norm to the graph point with the same a . $\text{SO}(2)$ equivariance identifies that graph point with a rotated canonical cap, because the group acts transitively on $|a| = \delta$. The upper radial bound in the completed source stage puts its fixed negative window after its q^3 crossing for small q . Its forcing on that window is $O(\varepsilon(1+L)^4)$. Backwards compact comparison from the completed cap proves the negative-window assertion.

For the positive half, (3) and (2) give $x_j \leq C\varepsilon(1+L)^6$ on $0 \leq s \leq S$, so the current receiver's nonradial field on source k is $O(q^8(1+L)^6)$; inactive rows are smaller. The regular torus image on a compact source tube is $O(q^8)$, using constrained mass and moments. A separate support stop is needed for the deeper packet since source k may leave its near-circle chart. Write

$$e(s) = \text{rad}(\mathcal{P}_{>k}(T_k + s/A_k))/R_k, \quad e(0) \leq C_d q(1+L).$$

The regular canonical trajectory on $[0, S]$ has a positive central hole and a compact chord-arc tube with the same property. To justify this geometric fact, its common fourfold transport homeomorphism fixes the origin, which started strictly inside both domains; it therefore keeps the origin strictly inside them. Every regular interface on this compact interval has positive origin distance. Joint continuity of the embeddings in time and label, and compactness of the interval and label circle, make the minimum distance positive. A sufficiently small E tube retains half that distance and positive chord-arc and cross-interface separation margins. Inside this hole the constant rotation is removed; the remaining field is cubic with coefficient C_S . Larger local colours satisfy the same cubic bound. Summed self velocities of the deeper colours give $C\beta q$. Hence, on that tube and before any coupled exit,

$$D^+ e(s) \leq C\beta q + C_S e(s)^3.$$

Stop with $e(s) \leq 4C_d q(1+L+\beta s)$, enlarging fixed C_d once to include the linear velocity. Its cubic increment is at most $C_S q^3(1+L+\beta S)^3 S = o(q(1+L))$. It improves the bound to $2C_d q(1+L+\beta s)$, which is inside one quarter of the canonical hole for small q . Deep mass $O(A_k R_k^2 \beta q^2)$ now gives moment and full contour field $O(q^6(1+L)^4)$ on the source.

Start a simultaneous stop at compact source-tube departure, central hole loss, the displayed deep-radius barrier, or $s = S$. The forcing just bounded, together with the completed cap error, tends to zero. Compact bounded-time comparison improves the source tube and hole margins strictly. The deep support improvement was independent of that comparison. Thus none of these stops can fail first. The moment is Lipschitz on the compact tube and the deep packet moment is $O(q^6(1+L)^4)$, yielding the asserted error. The current recipient cannot reach q^3 before $3L/\sigma - C > S/\beta$, by (3); all its other first-phase exits were already excluded. $\square$

Choose S once the final cap and c_0 are fixed, so the actual and canonical positive tails and the above-noise negative tails are less than $c_0/(100(1+|D_4|))$. For the base source choose in addition $S_0 \geq S + 1$ so its omitted canonical past before $-S_0$ has that same bound. The window lemma and (7) then give, for every available $\beta\theta \geq S$ before the recipient reaches q^3 ,

$$\left| D_4 \int_{-A_k T_k}^{\beta\theta} e^{-z_\beta s} \mathcal{B}_{\geq k}(s) ds - e^{i\phi_k} D_4 F(z_\beta) \right| \leq c_0/10 + C_S r(q) + C q^{6-g_\beta} (1+L)^{13}. \quad (9)$$

Here $\mathcal{B}_{\geq k}$ is the actual aggregate moment in the source native rotating axes, divided by $A_k R_k^6$. The proof partitions the actual integral into its actual past tail, its fixed window, and its actual future tail. It neither extends an actual trajectory to an unknown future cap nor replaces its whole trajectory by the canonical orbit.

The recipient's exact initial moment is zero at physical time zero, because it was initialized circular. Integrating its moment equation from that actual initial time, in incoming axes, gives

$$e^{-\lambda\theta} a(\theta) = \varepsilon D_4 \int_{-A_k T_k}^{\beta\theta} e^{-z_\beta s} \mathcal{B}_{\geq k}(s) ds + \mathcal{E}_j(\theta). \quad (10)$$

The factor β in the receiver-time leading force $\beta \varepsilon D_4 \mathcal{B}_{\geq k}(\beta\theta)$ cancels in the change of integration variable. The inherited $a(0)$ at the source cap is included in this exact past integral. Merely assuming $a(0) = O(\varepsilon)$ with no history would not justify a lower bound; that free complex number could cancel the seed.

The higher exterior harmonics contribute at most $C q^{10-g_\beta} (1+L)^{13}$ on the actual negative history and $C q^{10} (1+L)^8$ on the positive side, by the exact degree-eight remainder input. Their fourth-moment projection on imperfect contours is bounded by the full field, so this estimate is safely crude. Combine these with the recipient's past remainder estimates and (5). Uniformly before the first q^3 crossing,

$$\begin{aligned} |\mathcal{E}_j(\theta)| &\leq C \{q^{12-g_\beta} + q^{14-g_\beta}\} (1+L)^{13} \\ &\quad + C q^{10-g_\beta} (1+L)^{13} + C (q^9 + q^{12} + q^{14}) (1+L)^{20} \\ &\leq C (q^7 + q^9 + q^{12} + q^{14}) (1+L)^{20} = o(\varepsilon). \end{aligned}$$

In the last line use $g_\beta \leq 3$. The additional source-past noise error in (9), after multiplication by ε , is $O(q^{12-g_\beta} (1+L)^{13})$ and belongs to this same bound.

Equations (9)–(10) therefore give, for small q and all such current times,

$$|e^{-\lambda\theta} a(\theta)| \geq (c_0/2)\varepsilon =: c_s \varepsilon, \quad \theta \geq \Theta_w = S/\beta. \quad (11)$$

The phase $e^{i\phi_k}$ is arbitrary and harmless, because it is a single phase multiplying the whole nonzero bilateral transform.

7 The post- q^3 reset and the ordered first-exit argument

The first-phase strict improvements and (11) already imply existence of a first q^3 crossing $\theta = \tau$: if it had not occurred by $[3L + \log(2/c_s)]/\sigma < H$, the signed inequality would give $|a| \geq 2q^3$, a

contradiction. Thus

$$3L/\sigma - C \leq \tau \leq 3L/\sigma + C.$$

At that crossing $\|y(\tau)\| \leq (B + C_g)\varepsilon(1 + L)^6$. This establishes the first-phase endpoint before the second phase is introduced.

Put $u = \theta - \tau$ and $\underline{\sigma} = 2\sigma/3$. In phase two retain the prefix, packet, separation and local barriers. Add the stops

$$|a| = \delta, \quad \|y\| = 2Y\varepsilon(1 + L)^6 e^{2\eta u}, \quad |a| < q^3 e^{\underline{\sigma} u}, \quad \theta = H.$$

The growth stop means a first *downward* violation for $u > 0$; equality at the restart does not stop the phase. We prove a strict radial improvement at that restart and thereafter.

Choose q with $2Y\varepsilon(1 + L)^6 e^{2\eta H} < \delta$. On the stopped phase, $|a|, \|y\| \leq \delta$. At periods counted from τ , the flattened recurrence has remainder at most $2C_E\delta\|y\|$, and forcing error at most $PC_E F_*\varepsilon(1 + L)^4$. Use $\|M_c^n\| \leq C_\eta e^{\eta n P}$, multiply the Duhamel sum by $e^{-\eta n P}$, and sum the forcing geometric series. This yields

$$e^{-\eta n P} \|y_n\| \leq C_\eta [\|y_0\| + \varepsilon(1 + L)^4] + \kappa_\eta \delta \sum_{r < n} e^{-\eta r P} \|y_r\|.$$

Choose the cap earlier with $\kappa_\eta \delta \leq \eta P/2$. The elementary discrete Grönwall estimate follows by induction from $Z_n \leq A + k \sum_{r < n} Z_r$, which implies $Z_n \leq A(1 + k)^n \leq A e^{kn}$. Together with the uniform last-subperiod comparison it gives

$$\|y(u)\| \leq C_Y \varepsilon(1 + L)^6 e^{3\eta u/2}.$$

Choose $Y \geq 2C_Y$, including the initial value and the interpolation constant. Then

$$\|y(u)\| \leq Y\varepsilon(1 + L)^6 e^{2\eta u}, \tag{12}$$

a strict half-size improvement of the stated stop.

On the weaker growth barrier,

$$\frac{\|y\|}{|a|} \leq 2Yq^3(1 + L)^6 e^{-(\underline{\sigma} - 2\eta)u}, \quad \frac{|F_a|}{|a|} \leq F_*q^3(1 + L)^4.$$

Since $b = g(a) + y$, the exact moment remainder obeys $|\mathcal{N}|/|a| \leq C\delta + C_Y q^3(1 + L)^6$. Choose the cap with $C\delta < \sigma/8$, then small q so that the total remainder and force ratio is below $\sigma/4$. At every current time in this stopped phase,

$$\frac{d}{du}|a| \geq \sigma_0|a| > \underline{\sigma}|a|, \quad \frac{d}{du} \log |a| \leq \sigma + C_{\text{rad}}\delta + \zeta(q), \quad \zeta(q) \leq Cq^3(1 + L)^6 \rightarrow 0. \tag{13}$$

These are almost-everywhere inequalities for an absolutely continuous amplitude. Integrating the first proves a nonempty initial growth interval and excludes every later downward growth exit.

Below noise in phase one the norm is $C_n\varepsilon(1 + L)^{12}$; above noise it is $C_x|a|$. In phase two, graph quadraticity and the small y/a ratio give the same $C_x|a|$, after the cap was chosen small. The radial rates concatenate. Therefore at each stopped current time in phase two, every earlier above-noise amplitude is bounded by its current amplitude times $e^{-\sigma_0(\theta - v)}$, and the below-noise contribution has the stated uniform noise bound. Lemma 5 strictly improves the prefix again. The independent packet improvement persists. The local norms are strictly smaller than their local margin because

$$C_{\text{ch}}(\delta + C_g\delta^2) + C_{\text{ch}}2Y\varepsilon(1 + L)^6 e^{2\eta H} < r_*/2,$$

and inactive contours have the additional weight $D\varepsilon$. All non-amplitude second-phase exits are now excluded.

Integrating the lower radial inequality shows that the cap must occur within

$$u_{\text{cap}} \leq \sigma_0^{-1} \log(\delta/q^3) \leq 4L/\sigma + C$$

of τ . Hence it occurs by $7L/\sigma + C < H$, excluding the deadline. Only *after* existence of that cap has been established do we integrate the upper radial bound to obtain

$$u_{\text{cap}} \geq \frac{3L + \log \delta}{\sigma + C_{\text{rad}}\delta + \zeta(q)} \geq cL.$$

Thus every fixed backward cap window is contained in this completed second phase for small q . At the actual cap, (12) gives

$$\|y_{\text{cap}}\| \leq Cq^6(1+L)^6 e^{2\eta(4L/\sigma+C)} \leq Cq^{6-8\eta/\sigma}(1+L)^6 = o(1).$$

This is the weak reset needed for the next fixed-window comparison; no long-time canonical shadowing has been asserted.

8 The deepest fixed ancient source: a separate base

The deepest source starts at a fixed ancient time $-S_0$ on the canonical orbit. As $q \downarrow 0$ its nonzero initial amplitude is fixed, and can exceed q^3 . Neither the raw-centre bootstrap nor the above-noise first phase is applied to this source.

Choose a fixed small forward time $r_b > 0$, independent of the final cap and windows, on which the graph orbit remains local and its radial derivative is at least $\sigma_0|a|$. The graph and inverse chart yield a cap-independent constant C_{anc} such that

$$x_*(s) \leq C_{\text{anc}}\delta e^{\sigma_0 s}, \quad s \leq r_b,$$

with the fixed forward factor included in C_{anc} . This constant is distinct from the later raw-centre bootstrap constant. Use deepest time $s = A_N t - S_0$, and weights $w_i = (D\varepsilon)^{N-i}$ for $i < N$. All outer colours start circular, so $I_N(-S_0) = 0$. The same weighted row computation as before gives

$$I_N(s) \leq \frac{C}{D} \int_{-S_0}^s e^{\alpha_b(s-v)} x_N(v) dv, \quad 0 < \alpha_b < \sigma/2, \quad (14)$$

on its stopped local prefix, with constants independent of S_0, N . Feedback into the deepest source, including its self-image, is

$$\|f_N(s)\| \leq Cq^8[x_N(s) + DI_N(s)]. \quad (15)$$

Choose K before δ, S, S_0 , large enough for the recipient prefix bound and for four times $C(C_{\text{anc}} + 1)e^{\sigma_0 r_b}/[D(\sigma_0 - \sigma/2)]$. Stop the base at first source-local-tube departure, $I_N = 2K\delta$, first cap, or $s = r_b$. Its force is $O(q^8\delta)$. Compare from the identical initial state at $-S_0$ over the fixed interval using Lemma 1. It gives

$$\sup \|X_N(s) - X_*(s)\| \leq C_{S_0, \delta} \delta q^8.$$

The possibly large constant constrains q , not K . Choose q so this error is less than the fixed tube margin and than $\delta e^{-\sigma_0 S_0}$. Then the causal envelope $x_N(s) \leq (C_{\text{anc}} + 1)\delta e^{\sigma_0 s}$ holds on the stop. Insert it in (14); since $\sigma_0 - \alpha_b \geq \sigma_0 - \sigma/2 > 0$, it improves the prefix to $K\delta/2$. The source local tube also improves by comparison.

The canonical cap is transverse: its radial derivative at zero is at least $\sigma_0\delta$. The exact bounded moment functional and the comparison imply convergence of the actual moment and its moment derivative near zero, by (15). Choose a fixed short interval around zero where the canonical radial derivative is $\geq c\delta$. The actual derivative is $\geq c\delta/2$ for small q . On the earlier compact interval $[-S_0, -r]$, ancient monotonicity gives a positive fixed gap below the cap. Thus the intermediate value theorem and the derivative lower bound establish its unique first cap at

$$T_N = (S_0 + \zeta_N)/A_N, \quad |\zeta_N| \leq C_{S_0, \delta} q^8, \quad \|y_N(T_N)\| \leq C_{S_0, \delta} \delta q^8.$$

The source's first cap precedes $s = r_b$; no stopped exit failed first. The time estimate follows by dividing its $O(\delta q^8)$ amplitude error by the fixed $c\delta$ radial derivative.

We next prove its outgoing fixed source window directly. Recenter at the completed cap and choose the compact canonical tube on $[0, S + 1]$. Its source support is $C_S R_N$. Far-field mass and support bounds give a weighted source term C_S/D in the outer equations, even if the source is no longer close to a circle. The causal estimate is

$$I_N(T_N + s/A_N) \leq C e^{\alpha_b s} K \delta + \frac{C_S}{D} \int_0^s e^{\alpha_b(s-v)} dv.$$

Let M_S be twice a fixed upper bound for this expression on $[0, S + 1]$, allowing $\alpha_b \leq \sigma/2$. Stop at $I_N = 2M_S$, an outer local departure, source compact-tube departure, or $s = S + 1$. Choose q after M_S so that $D\varepsilon M_S$ is in the outer chart and $C_S q < r_{pl}/2$. Weighted nonlinear rows are then small. The preceding bound improves the prefix to M_S , and source feedback is $\|f\|_N \leq C_S q^8(1 + DM_S)$. Compact comparison from the completed cap strictly improves its source tube. Thus the outgoing interval exists, independently of any later recipient cap, and its full source error is $O_{S,S_0,\delta}(q^8)$.

The negative window is already present because $S_0 \geq S + 1$ and $|\zeta_N| = O(q^8)$. Backwards compact comparison from the completed cap gives the same error there. Moment Lipschitz continuity proves the base case of Lemma 6. In particular,

$$\int_{-S_0-\zeta_N}^S e^{-z\beta s} \mathcal{B}_N(s) ds = e^{i\phi_N} \int_{-S_0}^S e^{-z\beta s} B_*(s) ds + O_{S,S_0,\delta}(q^8).$$

The omitted canonical past is controlled by the fixed choice of S_0 , and the seed comparison error is $O(q^6 q^8) = O(q^{14})$.

M_S is deliberately different from $K\delta$. No statement says the spent deepest source retains the universal cap prefix throughout its outgoing window. At its cap the stronger universal prefix holds; through the outgoing window the reindexed next recipient still has norm $O(D\varepsilon M_S) = O(\varepsilon)$. One must not enlarge K afterwards to dominate M_S/δ : doing so would destroy the stated choice order and can create a cap-window circularity.

9 Reindexing, finite choices, and closure of the induction

At a completed ordinary cap T_j , the new weights for receiver $j - 1$ differ by exactly $D\varepsilon$. Hence

$$I_{j-1}(T_j) \leq D\varepsilon I_j(T_j) \leq DK\delta\varepsilon/2, \quad x_{j-1}(T_j) \leq DK\delta\varepsilon/2.$$

The already spent packet has radius $O(q(1 + L)R_j)$. Adding newly capped colour j produces a packet of radius at most a fixed $d_0 R_j$, for sufficiently small q . The new recipient therefore has inherited norm $O(\varepsilon)$ with no growing power of L . The reset supplies a fresh fixed-window error; it does not multiply the previous source's window error by a stage-dependent constant. Its proof used only current mass, support, weighted prefix, and forcing bounds.

The descending induction now has an ordered logical structure:

1. The base comparison proves the deepest cap and its outgoing window, with universal cap prefix and separate outgoing constant.
2. The inherited local data permit the next recipient's stopped first phase. Raw centre, upper amplitude, current-time growth above noise, prefix improvement, packet and separation improvements exclude all non-amplitude exits independently of the signed seed.
3. The source cap is completed. Its past envelope, reset, and outgoing compact-tube comparison provide the actual whole pulse. The exact initialization at physical time zero gives the seed identity, and the entire discounted remainder is $o(\varepsilon)$. This proves the signed seed and forces the first q^3 crossing.

4. The flattened second-phase recurrence, with the weaker radial barrier, gives the stronger radial derivative, improved prefix and separation, and then the recipient cap. Only after that cap exists is its backward fixed window located and its new reset used.
5. Reindexing reproduces the next local data and packet radius.

At no point is the recipient's future cap an input to an earlier prefix or source comparison. This completes Theorem 1.

The first-exit statements use continuous quantities: the finite maximum I_j , the contour norm and moment, and the support radius under the continuous transport homeomorphism. The support radius is locally Lipschitz in time whenever the physical velocity is bounded; the supplied upper Dini inequality therefore integrates in the usual scalar comparison sense. At a putative endpoint of a strict tube and separation improvement, local contour continuation from (H6) extends the currently followed contours, while the globally existing bounded-vorticity transport continues the other colours. Thus an endpoint cannot be a loss of existence hidden behind the listed barriers. For each fixed N only finitely many descending stages occur. The uniform estimates concern their constants, rather than an unproved infinite-stage contour continuation theorem.

For clarity, the finite parameter order is as follows.

1. Fix cap-independent local analytic and chart constants, a small reference unstable orbit, and its bilateral transform. Select a good $\beta \in [4, 5]$ by Lemma 3.
2. Choose $0 < \eta < \sigma/32$, then D and $\alpha_* < \sigma/2$. Fix the packet constant D_0 from d_0, C_v . These precede K because packet forcing constants can depend on D_0 .
3. Fix C_x from the inverse chart and graph, uniformly for all sufficiently small caps. Choose K from the prefix lemma and the cap-independent ancient envelope needed for the base.
4. Shrink δ by translating the same ancient orbit so that its local chart, radial, flattened-map, and cap-tube inequalities hold. The transform remains nonzero. Fix $c_0 > 0$ for this final cap.
5. Choose S , then $S_0 \geq S + 1$, to bound actual and canonical tails relative to c_0 . Fix compact-tube and deep-radius constants. Choose M_S for the base outgoing window separately from $K\delta$. Fix B, C_*, Y and their finite forcing and remainder constants.
6. Choose q last to satisfy all strict inequalities.

There are finitely many fixed inequalities. Their vanishing factors are positive powers of q times fixed powers of $1 + L$:

$$q^2(1+L)^3, \quad q^3(1+L)^{12}, \quad q^{6-9\alpha_*/\sigma}(1+L)^{12}, \quad q^{6-18\eta/\sigma}(1+L)^6, \\ q^{6-8\eta/\sigma}(1+L)^{20}, \quad q^{6-g_\beta}(1+L)^{13}, \quad q^{10-g_\beta-6}(1+L)^{20}.$$

All exponents are positive because $\alpha_* < \sigma/2$, $\eta < \sigma/32$, and $g_\beta \leq 3$. Any additional fixed requirement $\beta q < 1$ or $\beta q^{2/p} < 1$ can be imposed at this final step. The bounded-time constants may grow with the fixed windows; they only constrain this one final choice of q . There is no selection of q depending on j or N .

10 Limits of the result and the hypothesis ledger

This reconstruction proves a conditional cascade, not the underlying analytic contour theorem, full material spectrum, or eventual Euler compactness argument. The exact signed multipole coefficient, degree-eight remainder, constrained torus image cancellation, and packet Dini estimate

are explicit required inputs. A normal-only linear calculation cannot replace (H1); a bounded force in the normal coordinate cannot replace the full E force used in Lemma 1. Analyticity near circles cannot replace the compact-trajectory Lipschitz assumption used for the positive source window. Conversely, this proof does not require any uniform norm of spent contours or a global regularity assertion for every finite torus interface.

The uniform signed bound cannot follow from inherited norm bounds alone. An arbitrary admissible $O(\varepsilon)$ complex inherited moment can cancel the canonical bilateral seed. The exact initialization and actual negative history in (10) prevent that free cancellation and are therefore part of the construction.

The separate source and hypothesis ledger identifies the supporting modules needed to turn this conditional result into the proposed Euler construction. Every estimate proved in this text is a deduction from that ledger; no prior audit or review verdict is used as evidence.