Unforced Euler on the torus: verified reductions and an annular cascade candidate

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Abstract

No nonuniqueness theorem is proved in this note. We establish a strong-compactness reduction, an initial-time obstruction, and an explicit unbounded finite-L^p stationary torus background. We compute a growing planar vortex-interface mode, prove persistence for one sufficiently small torus vortex, and compute a nonzero planar coupling between two scales. These calculations suggest a coupled annular cascade, but do not establish the nonlinear, uniform-in-scale transfer needed for nonuniqueness. The unproved step is stated explicitly. The standard renormalization and existence inputs are distinguished from the calculations supplied here.

1 Precise target and the admissibility reduction

Fix $2 < p < \infty$, $T > 0$, and the square torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. Vorticities have zero spatial mean and velocities have zero spatial mean. Set

$$u = \mathcal{K}\omega := \nabla^\perp \Delta^{-1}\omega, \qquad \nabla^\perp = (-\partial_2, \partial_1).$$

The desired theorem would exhibit $\omega_0 \in L^p(\mathbb{T}^2)$ and two distinct unforced solutions with the same initial trace,

$$\partial_t \omega + \operatorname{div}(u\omega) = 0, \qquad \partial_t \beta(\omega) + \operatorname{div}(u\beta(\omega)) = 0,$$

the second identity holding for every smooth compactly supported β. The initial condition is part of both formulations. For this p, the flux is integrable and DiPerna–Lions theory implies renormalization; see [2, 3]. Consequently every integrable Casimir, the vorticity distribution, and the L^p norm are conserved.

Proposition 1 (Strong compactness of exact approximations)**.** *Let ω_n be smooth exact unforced Euler solutions on $[0,T]$, with $\omega_n(0) \to \omega_0$ strongly in $L^p(\mathbb{T}^2)$ and fixed zero velocity mean. Then a subsequence converges strongly in $C([0,T]; L^p)$ to a renormalized Euler solution with datum ω_0. The same conclusion holds for bounded-vorticity Yudovich approximants with their continuous L^p representatives.*

Proof. Transport gives a uniform bound M for $\|\omega_n\|_{L^\infty_t L^p_x}$. Periodic elliptic estimates and $p > 2$ yield

$$\|u_n\|_{L^\infty_t W^{1,p}_x} + \|u_n\|_{L^\infty_{t,x}} \le C_p M.$$

Moreover

$$\partial_t \omega_n = -\operatorname{div}(u_n \omega_n), \qquad \partial_t u_n = -\mathcal{K}\operatorname{div}(u_n \omega_n).$$

The first expression is bounded in $L^\infty_t W^{-1,p}_x$. Since $\mathcal{K}\,\mathrm{div}$ is an order-zero periodic singular integral, the second is bounded in $L^\infty_t L^p_x$. Weak-pairing equicontinuity and a countable dense set of tests give uniform weak-L^p convergence of ω_n. Compactness of $W^{1,p}$ in C^γ, $0 < \gamma < 1 - 2/p$, combined with time equicontinuity in L^p, gives

$$u_n \longrightarrow u \quad \text{in } C([0,T]; C^\gamma)$$

after extraction, with a smaller γ if necessary. Thus the flux passes to the limit. The initial boundary term supplies the initial trace.

Renormalization applies to the limiting transport equation: both ω and ∇u are in $L^\infty_t L^p_x$, and $2/p < 1$. Its representative is strongly continuous in L^p and satisfies

$$\|\omega(t)\|_p = \|\omega_0\|_p .$$

At every fixed time the weak convergence and $\|\omega_n(t)\|_p = \|\omega_n(0)\|_p \to \|\omega_0\|_p$ give strong convergence by uniform convexity. For uniform-in-time convergence, take any hypothetical bad sequence $t_n \to t_*$. Uniform weak-pairing convergence implies $\omega_n(t_n) \rightharpoonup \omega(t_*)$, and the same norm argument implies strong convergence. Strong continuity of ω contradicts the bad sequence. □

Corollary 2 (Sufficient separation criterion)**.** *Suppose two approximation families as above share the same strong initial limit. If, for some $c > 0$ and times $t_n \in [\tau, T]$, $\tau > 0$,*

$$\left\|\omega_n^{(1)}(t_n) - \omega_n^{(2)}(t_n)\right\|_p \geq c,$$

then a joint subsequence produces two distinct renormalized solutions from that datum.

Proof. Take a joint strongly convergent subsequence and $t_n \to t_* \geq \tau$. The inequality passes to the limits by Proposition 1. □

This criterion does not construct separation. It says precisely what an instability argument must prove. Recent primary literature explicitly leaves uniqueness of finite-L^p approximation solutions open [4].

2 Two rigorous constraints on a cap construction

Lemma 3 (A uniform initial-time modulus)**.** *Under Proposition 1, for every smooth g,*

$$\|\omega_n(t) - \omega_0\|_p \leq \|\omega_n(0) - \omega_0\|_p + 2\,\|\omega_0 - g\|_p + C_p M t\,\|\nabla g\|_p .$$

In particular, fixed physical L^p separation cannot occur at times $t_n \downarrow 0$.

Proof. Let X_n be the measure-preserving flow. Compose the norm with $X_n(t)$ and use

$$\omega_n(t, X_n(t, a)) = \omega_n(0, a).$$

Insert ω_0 and g. The two $\omega_0 - g$ terms have equal norms by measure preservation. Finally,

$$g(X_n(t, a)) - g(a) = \int_0^t \nabla g(X_n(s, a)) \cdot u_n(s, X_n(s, a))\,\mathrm{d}s.$$

Minkowski, measure preservation, and the velocity bound give the stated estimate. First approximate ω_0 by a fixed g, then send t to zero. □

Lemma 4 (Shrinking-support separation disappears)**.** *For renormalized branches from one L^p datum, a difference supported on sets whose measure tends to zero has L^p norm tending to zero. The assertion is uniform for exact approximations with strongly convergent initial data.*

Proof. Equimeasurability gives, for any measurable E and $L > 0$,

$$\int_E |\omega(t)|^p \leq L^p |E| + \int_{\{|\omega_0| > L\}} |\omega_0|^p.$$

Use $|a - b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, then first take $|E| \to 0$ and next $L \to \infty$. For approximations, strong initial L^p convergence supplies uniform integrability of their pth powers, and transport preserves it. □

Thus a rescaled core can become visibly deformed while its physical norm remains negligible. A successful construction must propagate its effect to pre-existing vorticity carrying a nonvanishing norm.

3 An exact unbounded stationary background on the torus

Set $m = 4$, $\rho = \sqrt{3/5}$ and define the mean-zero radial profile

$$W(r) = \begin{cases} -(1-\rho^2), & 0 \le r < \rho, \\ \rho^2, & \rho < r < 1, \\ 0, & r > 1. \end{cases}$$

Its integral on $\mathbb{R}^2$ is zero. Choose $R_1 < 1/8$, $0 < q < \rho$, $A_1 > 0$ and $0 < \alpha < 2/p$. Put

$$R_j = R_1 q^{j-1}, \qquad A_j = A_1 q^{-\alpha(j-1)}, \qquad \bar{\omega} = \sum_{j=1}^{\infty} A_j W(|x|/R_j).$$

The disk is embedded around the origin of the torus. Values at the origin and interfaces are immaterial.

Proposition 5 (Admissible stationary datum)**.** *The series converges in $L^p(\mathbb{T}^2)$ to an unbounded mean-zero vorticity. Its zero-mean velocity is supported in the disk of radius R_1. It is an exact stationary renormalized unforced Euler solution.*

Proof. For a fixed constant C_W,

$$\|A_j W(|x|/R_j)\|_p = C_W A_j R_j^{2/p}, \qquad \sum_j A_j R_j^{2/p} < \infty.$$

Each summand has zero circulation. Its planar radial velocity is tangential, vanishes for $r > R_j$, and has supremum bounded by CA_jR_j. Since $\alpha < 2/p < 1$, these velocities converge uniformly. Each has a streamfunction constant outside its disk; subtract that constant and extend periodically. It therefore also gives exactly the zero-mean periodic Biot–Savart velocity. The limiting velocity is bounded, continuous and in $W^{1,p}$.

On $R_{j+1} < r < \rho R_j$, the terms with index greater than j vanish and the terms with index at most j take their negative core values. Consequently

$$\bar{\omega}(r) = -(1-\rho^2) \sum_{k \le j} A_k,$$

which is unbounded as $j \to \infty$ on annuli of positive area.

Write $\bar{u} = v(r)e_\theta$. Then $\operatorname{div}(\bar{u}\beta(\bar{\omega})) = 0$ distributionally for every bounded smooth β: the scalar is radial and the velocity tangential, with no radial flux at interfaces or at the origin. This proves the stationary renormalized vorticity identities. To check the full velocity equation, take

$$\bar{P}(r) = -\int_r^{R_1} \frac{v(s)^2}{s}\,\mathrm{d}s \quad (r < R_1), \qquad \bar{P} = 0 \quad (r \ge R_1).$$

The series bounds imply $|v(r)| \le Cr^{1-\alpha}$ near zero. Since $\alpha < 1$, the integral converges. Now $\bar{P}' = v^2/r$ and $(\bar{u}\cdot\nabla)\bar{u} = -v^2 e_r/r$, proving unforced velocity Euler as well. □

The finite partial sums are bounded-vorticity stationary solutions. Radial smoothing also gives smooth stationary approximations. This construction supplies a permissible datum; it supplies no nonstationary branch.

4 Exact planar interface computations

Scope. The following matrix and coupling coefficients use the planar logarithmic Green function. They are exact first variations of circular material interfaces in that model. They are not eigenvalues of the full nested torus operator. Interface first variations are vorticity sheets, not L^p perturbations; finite displacements, however, are admissible bounded patch vorticities.

For general integer $m \geq 2$, put $\rho^2 = (m-1)/(m+1)$ and consider one profile $AW(r/R)$. Displace its two interfaces by the real part of $h_i(t)e^{im\theta}$, at $r_1 = \rho R$ and $r_2 = R$, and write $z_i = h_i/r_i$. The outward vorticity jumps are $d_1 = A$ and $d_2 = -A\rho^2$. The vorticity first variation is

$$\delta\omega = -\sum_i d_i h_i \delta(r - r_i) e^{im\theta}.$$

The radial Green solution gives

$$\delta\psi(r) = \sum_i \frac{d_i h_i r_i}{2m} \left(\frac{\min(r, r_i)}{\max(r, r_i)} \right)^m e^{im\theta}.$$

Inserting $\delta u_r = -\partial_\theta \delta\psi / r$ into the material-interface equation yields

$$z' = \frac{iA}{2} \begin{pmatrix} \rho^2 & \rho^m \\ -\rho^{m+2} & \rho^2 \end{pmatrix} z. \tag{1}$$

There is a growing mode

$$\lambda_+ = \frac{A\rho^{m+1}}{2} + i\frac{A\rho^2}{2}, \qquad v_+ = \begin{pmatrix} 1 \\ -i\rho \end{pmatrix}, \qquad \ell_+ = \begin{pmatrix} 1/2 & i/(2\rho) \end{pmatrix}.$$

Direct multiplication verifies the right and left eigenrelations and $\ell_+ v_+ = 1$. For $m = 4$, the growth is approximately $0.1394274A$ and the frequency is $0.3A$.

For two nested copies with radius ratio $q < \rho$, project their interface perturbations onto these local growing vectors. The coefficient from the inner copy to the outer copy is

$$C_m A_{\text{in}} q^{m+2}, \qquad C_m = \frac{\rho^{1-m} + \rho^{m+1}}{4} - \frac{i(1-\rho^2)}{4}. \tag{2}$$

Indeed the inner growing vector produces the exterior moment

$$S_m = A_{\text{in}} (qR)^{m+2} (\rho^{m+2} + i\rho^3).$$

Its relative radial forcing at the outer interfaces is $-iS_m(\rho^{-m-2}, 1)^{\mathsf{T}}/(2R^{m+2})$. Applying ℓ_+ gives (2). The reverse projected coefficient is $C_m A_{\text{out}} q^{m-2}$. In particular

$$C_4 \simeq 0.6076281 - 0.1i \neq 0.$$

In the full concentric baseline, larger cores add solid-body rotation. The local diagonal frequency becomes

$$\nu_j = \frac{\rho^2 A_j}{2} + \frac{m(1-\rho^2)}{2} \sum_{k<j} A_k, \qquad \nu_{j+1} - \nu_j = \frac{\rho^2 A_{j+1}}{2} + \frac{A_j}{2}.$$

Couplings to all other interfaces and the other local mode are still present; projection does not close a scalar evolution.

If one artificially prescribes a single incoming linear burst $a_{j+1}(t) = \varepsilon e^{\lambda_{j+1} t}$, neglecting every other coupling, then

$$a_j(t) = C_m A_{j+1} q^{m+2} \varepsilon \frac{e^{\lambda_{j+1} t} - e^{\lambda_j t}}{\lambda_{j+1} - \lambda_j}.$$

After sufficiently many incoming e-folds, this is a nonzero seed of order q^{m+2} times the incoming cap. This statement concerns the displayed forced linear two-mode system only.

Proposition 6 (Growing modes of finite planar stacks)**.** *For fixed resonant $m \geq 2$ and $0 < \alpha < 2/p$, take sufficiently small fixed $q > 0$. Every finite planar concentric stack has a growing contour eigenvalue near the local growing mode at each scale. Its real part is at least $\sigma A_j/2$, uniformly in the number of levels, where $\sigma = \rho^{m+1}/2$.*

Proof. Work in the angular-m contour block, a finite $2N$ dimensional space. Diagonalize each two-interface diagonal block using the fixed matrix with columns v_+ and $v_- = (1, i\rho)^{\mathsf{T}}$. The condition number is independent of scale. Cross-block row sums are bounded by

$$\varepsilon(q)A_j, \qquad \varepsilon(q) = C_{m,\rho}\left[\frac{q^{m+2-\alpha}}{1-q^{m+2-\alpha}} + \frac{q^{m-2+\alpha}}{1-q^{m-2+\alpha}}\right] \longrightarrow 0.$$

The first term sums forcing from smaller cores; the second sums forcing from larger ones. The diagonal centers are $\pm\sigma A_j + i\nu_j$. Opposite-sign centers at different scales are separated by at least $\sigma(A_j + A_k)$, and same-sign centers by at least $\sigma|A_j - A_k|$. Thus their Gershgorin disks are pairwise disjoint whenever

$$\varepsilon(q) < \frac{\sigma}{2}\min\left\{1, \frac{q^{-\alpha}-1}{q^{-\alpha}+1}\right\}.$$

For small q this holds uniformly in N. The separated-disk counting theorem gives one eigenvalue in each disk. Growing disks have real part at least $(\sigma - \varepsilon(q))A_j \geq \sigma A_j/2$. □

This proves scale-local linear growth. A fastest growing eigenmode can reach a large relative deformation only in the last core while all physical differences still vanish. Proposition 6 therefore does not prove Corollary 2's separation hypothesis.

5 What the square torus changes

Fourfold, rather than threefold, symmetry is useful because it is an exact symmetry of the square torus. Near the origin its Green function can be written

$$G_{\mathbb{T}^2}(z) = \frac{1}{2\pi}\log|z| + c|z|^2 + \operatorname{Re}\sum_{n\geq 1} c_{4n} z^{4n} + c_0,$$

on a small fixed disk, identifying the plane with $\mathbb{C}$. This follows by removing the logarithmic singularity and the constant-Laplacian quadratic part; the remainder is analytic harmonic and invariant under rotation by $\pi/2$.

For a fourfold-symmetric mean-zero source f supported on that disk, its complex moments of degrees one, two and three vanish. The quadratic term gives zero velocity. Convolving the degree-four harmonic gives only a spatial constant, also giving zero velocity. The first possible image-velocity term comes from degree eight. Analytic-series bounds imply, on a smaller fixed disk,

$$|u_{\text{image}}(x)| \leq C|x|^3 \int |y|^4 |f(y)|\,\mathrm{d}y.$$

For amplitude scale A and radius R, the right side is bounded by $CAR^6|x|^3$. On its own interfaces the corresponding relative-displacement forcing is $O(AR^8)$. The same moment calculation applies to a pure interface variation using its weighted total variation.

For one vortex, this estimate does prove spectral persistence, as follows. It does not by itself control a complete stack or nonlinear evolution after the supports deform.

Proposition 7 (An isolated unstable torus contour mode)**.** *Fix $s \geq 0$, $\rho^2 = 3/5$ and $A > 0$. For sufficiently small $R > 0$, the stationary torus patch $AW(r/R)$ has an eigenvalue of its linearized material-interface operator*

$$\lambda_R = A\left(\frac{\rho^5}{2} + \frac{3i}{10}\right) + O_s(AR^8), \qquad \operatorname{Re}\lambda_R > \frac{A\rho^5}{4}.$$

The statement is on the complexified fourfold-symmetric H^s interface space, with each interface's angular mean zero. It is a linear contour statement, not a nonlinear instability theorem or an L^p vorticity eigenmode.

Proof. In the planar problem, the Fourier block for positive $n \in 4\mathbb{N}$ is

$$L_n = \frac{iA}{2}\begin{pmatrix} (2n/5) - 1 & \rho^n \\ -\rho^{n+2} & 3/5 \end{pmatrix};$$

negative modes are the conjugate blocks. The operator domain is $H^{s+1} \times H^s$: the inner interface has nonzero angular transport, while the outer interface has zero background angular speed. For $n = 4$ the displayed growing eigenvalue is simple. For $n \geq 8$ the matrix discriminant, divided by $A^2/4$, is

$$\frac{4}{25}(n-4)^2 - 4\left(\frac{3}{5}\right)^{n+1} > 0,$$

so these eigenvalues are imaginary. The outer branch accumulates at $3iA/10$ and the inner branch grows linearly up the imaginary axis. The conjugate growing eigenvalue is separate. Thus the chosen growing eigenvalue is isolated from all other spectrum.

Choose a circle of radius ηA around it, with fixed small $\eta > 0$ containing no other spectrum. The explicit finite blocks and large-n diagonal dominance show a uniform resolvent bound C_s/A on this circle in $H^s \times H^s$.

The periodic image contribution acts as a smoothing bounded operator on this space, with norm at most $C_s A R^8$. The moment bound above also bounds angular derivatives of its smooth kernel. The background transport is unchanged, since the stationary radial velocity is exactly the compactly supported planar velocity. A Neumann resolvent expansion now applies uniformly on the circle when R is small. The Riesz projection retains complex rank one, so an eigenvalue persists inside. The same simple-eigenvalue perturbation estimate gives the stated $O_s(AR^8)$ displacement. There is separately its conjugate eigenvalue; the corresponding real unstable space has dimension two. □

6 Smoothing the isolated building block

Proposition 8 (A smooth unstable stationary torus core)**.** *For each fixed finite $p > 2$, there are smooth compactly supported radial mean-zero stationary vorticities on the square torus whose vorticity linearization on L^p has an unstable eigenvalue. No uniform nonlinear estimates in a smoothing parameter are asserted.*

Proof. First normalize $A = R = 1$ in the plane. Let $\chi_{a,\epsilon}$ be a smooth monotone radial disk indicator, equal to one below $a - \epsilon$ and zero above $a + \epsilon$. Define

$$s_\epsilon = \frac{\int_0^\infty \chi_{\rho,\epsilon}(r) r \, \mathrm{d}r}{\int_0^\infty \chi_{1,\epsilon}(r) r \, \mathrm{d}r}, \qquad \omega_\epsilon = -\chi_{\rho,\epsilon} + s_\epsilon \chi_{1,\epsilon}.$$

The circulation is exactly zero and $s_\epsilon \to 3/5$. The angular velocity $\Omega_\epsilon(r) = r^{-2}\int_0^r \omega_\epsilon(t) t \, \mathrm{d}t$ converges uniformly to that of the patch. The derivatives ω_ϵ' have uniformly bounded total variation and converge as measures to its two jumps.

For mode $m = 4$ and $\operatorname{Re}\lambda > 0$, the eigenfunction stream amplitude solves

$$\phi'' + \frac{\phi'}{r} - \frac{m^2\phi}{r^2} - \frac{im\omega_\epsilon'(r)}{r(\lambda + im\Omega_\epsilon(r))}\phi = 0.$$

Start with the origin-regular harmonic solution at a fixed $a < \rho/2$, and match to the decaying harmonic solution beyond a fixed $b > 2$. On compact spectral sets in the right half-plane the

denominators are bounded away from zero. Uniform total-variation bounds give bounded first-order transfer matrices. Across each shrinking layer ϕ changes by $O(\epsilon)$ and its derivative converges to

$$\phi'(r_i+) - \phi'(r_i-) = \frac{imd_i}{r_i(\lambda + im\Omega(r_i))}\phi(r_i).$$

The Evans determinant $D_\epsilon(\lambda) = b\phi'(b,\lambda) + m\phi(b,\lambda)$ therefore converges uniformly holomorphically to the patch transmission determinant. Eliminating the interface values identifies its unstable zero with that of the two-by-two contour determinant, up to nonvanishing factors. The zero is simple. Rouché's theorem supplies $\lambda_\epsilon \to \lambda_+$ with positive real part. Its planar eigen-vorticity

$$\eta_\epsilon = \frac{im\omega_\epsilon'\phi}{r(\lambda_\epsilon + im\Omega_\epsilon)}e^{im\theta}$$

is smooth and supported in the transition annuli.

Fix one sufficiently small ϵ, then scale to amplitude A and radius R. On the fourfold, zero-mean subspace of $L^p(D_R)$, with $D_R = RD_1$ a disk containing the support, the planar linearization is

$$L = \mathcal{A} + B, \qquad \mathcal{A} = -\Omega(r)\partial_\theta, \qquad B\eta = -(K_{\mathbb{R}^2}\eta)\cdot\nabla\omega_\epsilon.$$

Radial transport generates an isometric C_0 group on L^p, with generator $\mathcal{A}$ and right-half-plane resolvent bound $1/\operatorname{Re}\lambda$. The operator B is compact: local Biot–Savart maps $L^p(D_R)$ into $W^{1,p}(D_R)$, Rellich is compact, and the fixed smooth gradient is bounded. Both operators preserve the stated subspace. Banach-space analytic Fredholm theory in $\operatorname{Re}\lambda > 0$ makes the unstable eigenvalue isolated with finite algebraic multiplicity.

The exact base velocity is unchanged on the torus. The image-kernel estimate gives

$$\|K_{\text{image}}\eta\|_{L^\infty(D_R)} \le CR^{9-2/p}\|\eta\|_p, \qquad \|\nabla\omega_\epsilon\|_p \le C_\epsilon AR^{2/p-1}.$$

Hence the change in B has norm at most $C_{\epsilon,p}AR^8$. After isometric spatial rescaling, the fixed planar Riesz-circle resolvent bound is $C_{\epsilon,p}/A$. For sufficiently small R a Neumann-series homotopy preserves its nonzero Riesz rank and an unstable torus eigenvalue. This last step does not require simplicity of the full-plane eigenvalue across all angular blocks. The constants and admissible radius depend on the fixed smoothing width. The torus eigen-vorticity is asserted to lie in L^p; no further regularity bootstrap is claimed. □

7 The missing nonlinear theorem

For $m = 4$, the formal adjacent outward forcing is of size $A_j q^{6-\alpha}$ times the inner amplitude. The local growth is proportional to A_j, so a seed of relative size $q^6\delta$ could reach relative size δ after a delay of order

$$\frac{\log(q^{-6})}{A_j}.$$

These delays are summable:

$$\sum_{j\ge 1} A_j^{-1} = \frac{A_1^{-1}}{1-q^\alpha} < \infty.$$

This is a scaling compatibility calculation, not a proof of an activation cascade.

A precise proposed construction is as follows. Let $\bar\omega_N$ be the finite stationary sum. Rearrange only its innermost copy by a small fourfold area-preserving map supported strictly inside the next larger interface. Its datum $\omega_{0,N}^{\mathrm{a}}$ can satisfy

$$\left\|\omega_{0,N}^{\mathrm{a}} - \bar\omega_N\right\|_p \le CA_N R_N^{2/p} \longrightarrow 0.$$

Here all larger summands are constant on the support of that map. Thus both the active data and stationary data converge strongly to $\bar{\omega}$. Evolve the active data by exact Yudovich Euler, which is well defined for every finite N.

Unproved transfer requirement. Establish constants $c > 0$, $0 < \tau < T < \infty$, independent of N, and times $t_N \in [\tau, T]$ such that

$$\|\omega_N^{\mathrm{a}}(t_N) - \bar{\omega}_N\|_p \geq c. \tag{3}$$

Corollary 2 would then give the requested nonuniqueness theorem, with one stationary branch and one active renormalized branch.

The linear computations do not establish (3). In particular:

1. The projected equations have complex coefficients and forcing from other modes. There is no proved positive invariant cone allowing a scalar lower comparison.
2. After the inner core reaches its cap, its later moment continues to force the outer mode. The full Duhamel integral may cancel the earlier seed. A nonzero early projection gives no lower bound on the complete integral.
3. A slower outer growth interval lasts many inner turnover times. Uniform nonlinear shadowing and backward feedback control on this interval have not been proved.
4. Colored components conserve their own norms, but their support and shape need not remain close to their initial annuli. Moment and separation bounds must account for this.
5. Repeating a two-core estimate across arbitrarily many levels requires uniform control of all previous cores, torus corrections and errors. A calculation for one pair is insufficient.
6. A cap must detect a difference in total physical vorticity at a fixed scale. A large interface coefficient confined to the last core disappears by Lemma 4.

Replacing a complex coefficient by its absolute value in a lower-bound argument would be invalid. Assuming that a deformed core becomes a permanent positive source would also introduce an unproved mechanism. These are substantive gaps, not omitted routine details.

8 Other routes checked

The direct hot/cold branching of the attached Boltzmann paper [1] has no Euler counterpart: colored vorticities are passively transported by their common total velocity, so a vanishing colored L^p norm stays vanishing. Active differences can grow, but only by changing arrangements of the background.

An exact radial unforced similarity profile obeys $\Omega + ar\Omega' = 0$, so it is a power law. Binz–Coiculescu prove stability of that power-law profile in specified weighted L^2 spaces, including m-fold subspaces with $m \geq 2$ and $0 < \alpha < 2$ [5]. This is not a universal nonlinear uniqueness theorem, but it blocks the straightforward radial unstable-profile argument in those spaces.

Nonradial singular steady saddles can be constructed. For example, solve $-\Delta w = w^{-\delta}$ on $(0, \pi)^2$, $w > 0$, $w = 0$ on the boundary, with $0 < \delta < 1/p$. Odd reflection across both boundaries gives a periodic streamfunction ψ with

$$\omega = -\operatorname{sgn}(\psi)|\psi|^{-\delta} \in L^p.$$

A subsolution $c \sin x \sin y$ bounds the singular source by an L^p function; regularized Dirichlet problems and periodic elliptic estimates give $\psi \in W^{2,p}$. The relation $\omega = F(\psi)$ makes the field stationary and renormalized. Pointwise trajectory branching along exceptional separatrices does not yield different Eulerian vorticities: the fixed Sobolev velocity has a unique regular Lagrangian flow.

Coiculescu's recent linear ill-posedness theorem for nonradial homogeneous steady states identifies a nondense maximal L^2 operator domain, rather than a growing eigenmode [6]. Near singular rays, finite displacements can be L^p-small although their formal vorticity derivatives are

outside L^p. Such examples suggest material-displacement coordinates, but no closed nonlinear branching equation was obtained.

Existing forced instability constructions retain essential forcing, and the known unforced integrable-vorticity convex-integration examples discussed in [2] are nonrenormalized. Neither supplies the missing transfer theorem on the torus.

Assessment

The strong-compactness criterion, admissible singular backgrounds, and isolated torus contour instability are established. The nonzero coupling between scales is established as a planar first-variation calculation. These leave a concrete annular-cascade research program. A uniform nonlinear estimate of the form (3) has not been proved, and no claim of renormalized unforced Euler nonuniqueness follows from this note.

References

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