

Nonuniqueness of renormalized solutions of unforced two-dimensional Euler on the square torus

Complete proof draft

8 October 2026

Abstract

For each finite $p > 2$, an infinite stack of signed, zero-circulation annuli supplies a stationary L^p vorticity and a distinct renormalized solution with the same initial datum. The proof uses the full material contour spectrum, an isolated strong unstable orbit, a nonzero bilateral fourth-moment seed, and a uniform finite-depth nonlinear induction. The periodic transport and strong limit arguments are included, and the external contour inputs are checked with their exact hypotheses.

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1 Statement and normalization

Write $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. A renormalized solution with exponent $p > 2$ means a real mean-zero function $\omega \in C([0, T]; L^p(\mathbb{T}^2))$, with $u = \nabla^\perp \Delta^{-1} \omega$, satisfying

$$\partial_t \omega + \operatorname{div}(u\omega) = 0$$

in distributions including its initial trace, and, for every $\beta \in C^1(\mathbb{R})$ with bounded derivative, satisfying

$$\partial_t \beta(\omega) + \operatorname{div}(u\beta(\omega)) = 0$$

with initial trace $\beta(\omega_0)$. This definition in particular implies conservation of the L^p norm by truncation, as proved below. The velocity has zero spatial mean. There is no external force.

Theorem 1.1. *For every finite $p > 2$ there exist $T > 0$ and a real mean-zero $\omega_0 \in L^p(\mathbb{T}^2)$ for which the unforced two-dimensional Euler equation has two distinct renormalized solutions in $C([0, T]; L^p(\mathbb{T}^2))$. One solution is stationary. Both solutions are invariant under the quarter turn about the origin.*

All scales in this proof lie in the central coordinate disk of the unit square torus. A colour is the complete signed annulus

$$W = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}, \quad \rho^2 = 3/5.$$

Both of its domain areas will be preserved separately. Its circulation is zero. After the finite parameters are selected, its scales and amplitudes are

$$R_j = q^j, \quad A_j = A_1 \beta^{j-1}, \quad 4 \leq \beta \leq 5, \quad A_1 = 1, \quad \varepsilon = q^6, \quad L = \log(1/q).$$

The proof constructs exact bounded Euler solutions with finitely many colours. An initial perturbation is placed only in the deepest colour. A uniform descending induction brings a fixed fourth-moment change to the outermost colour. Strong compactness then retains that change while the initial perturbation tends to zero in L^p .

Four external analytic inputs are used: the disk divergence inverse of Dacorogna–Moser, the complex disk-volume bound of Hernandez, the planar geometric continuation theorem of Chemin, and the local contour estimate on general chord-arc curves in Verdera’s contour-dynamics appendix. Their exact hypotheses and applications are checked where they enter. The periodic transport, renormalization, spectral decomposition, unstable graph, multipoles, nonlinear induction, and limiting argument are proved here.

2 Periodic transport: the analytic facts needed for the limit

Throughout, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ has volume one, $\nabla^\perp = (-\partial_2, \partial_1)$, and velocities have zero mean. For a mean-zero distribution w , put $\mathcal{K}w = \nabla^\perp \Delta^{-1}w$. The Fourier definition is

$$G(x) = - \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{2\pi i k \cdot x}}{4\pi^2 |k|^2}, \quad K = \nabla^\perp G, \quad \mathcal{K}w = K * w.$$

These equalities initially mean distributions. They give $\Delta G = \delta_0 - 1$, $\operatorname{div} \mathcal{K}w = 0$, and $\operatorname{curl} \mathcal{K}w = w$. Near zero, $G = (2\pi)^{-1} \log |x| + H$, where $\Delta H = -1$ and H is smooth. Indeed subtract a cut-off logarithm, solve the resulting smooth mean-zero Poisson equation by its rapidly decreasing Fourier coefficients, and use interior harmonic regularity. Harmonic regularity here also follows from the circle Poisson formula. In particular K is integrable, is $O(|x|^{-1})$ at zero, and is smooth away from zero.

Lemma 2.1 (Periodic velocity estimates). *For $1 < r < \infty$,*

$$\|\mathcal{K}w\|_{W^{1,r}} \leq C_r \|w\|_r.$$

For $r \geq 2$, its derivative satisfies the more precise bound

$$\|\nabla \mathcal{K}w\|_r \leq C_r \|w\|_r. \tag{F.1}$$

For bounded w , writing $m = \|w\|_1$ and $M = \|w\|_\infty$,

$$\|\mathcal{K}w\|_\infty \leq C\sqrt{mM} + Cm, \quad |\mathcal{K}w(x) - \mathcal{K}w(y)| \leq CMd(1 + |\log d|), \tag{F.2}$$

where $d = \operatorname{dist}_{\mathbb{T}^2}(x, y) \leq 1/2$. If $p > 2$, then also $\|\mathcal{K}w\|_\infty \leq C_p \|w\|_p$. All assertions hold componentwise without a mean-zero hypothesis, with the mean removed in the definition of Δ^{-1} .

Proof. Every component T of $\nabla \mathcal{K}$ is a Fourier multiplier bounded by one in absolute value. Its L^2 norm is at most one by Parseval. Away from zero its kernel obeys $|t(x)| \leq C|x|^{-2}$ and $|\nabla t(x)| \leq C|x|^{-3}$. The singular part is a second derivative of the logarithm, whose angular average is zero; a possible constant multiple of the identity does not affect the argument below.

Here are the details giving the dependence on r . Initially take $f \in L^1 \cap L^2$, so the multiplier and every term of its finite dyadic decompositions have their ordinary L^2 meaning. For $\lambda \geq \|f\|_1$, perform the dyadic decomposition on the unit square, with opposite edges identified, at maximal squares Q with average $|f|$ greater than λ . Write $f = g + \sum_Q b_Q$, where $b_Q = (f - f_Q)\mathbf{1}_Q$. Maximality gives $\|g\|_\infty \leq 4\lambda$, $\|g\|_1 \leq \|f\|_1$, $\sum |Q| \leq \|f\|_1/\lambda$, $\int b_Q = 0$, and $\sum \|b_Q\|_1 \leq 2\|f\|_1$. Parseval and Chebyshev give $|\{Tg > \lambda/2\}| \leq C\|f\|_1/\lambda$. Outside a fixed concentric enlargement of Q , subtract the kernel at its center c_Q and integrate the derivative bound:

$$\int_{\mathbb{T}^2 \setminus 5Q} |Tb_Q(x)| dx \leq \int_Q |b_Q(y)| \int_{\mathbb{T}^2 \setminus 5Q} |t(x-y) - t(x-c_Q)| dx dy \leq C\|b_Q\|_1.$$

The same estimate on squares crossing an edge uses their periodic lift. Enlarged squares have total measure at most $C\|f\|_1/\lambda$. Summing proves weak type $(1, 1)$. For $\lambda < \|f\|_1$ that conclusion follows from the total volume. The countable decomposition follows from finite truncations, using the L^2 convergence of the remainder for this initial class and the summable outside-enlargement estimates. The interpolation arguments below can first be performed on this dense class, since both pieces of a height truncation also belong to $L^1 \cap L^2$. Their strong L^r estimates then extend T by norm completion to every L^r , $1 < r < \infty$; testing against smooth functions verifies agreement with the distributional multiplier. Thus no pointwise meaning for an arbitrary L^1 multiplier was presumed in the decomposition.

For clarity, interpolation near one has quantitative norm $\|T\|_{L^s \rightarrow L^s} \leq C/(s-1)$ for $1 < s \leq 3/2$: split f at height proportional to the level λ , apply the weak $(1, 1)$ bound to its large part and the L^2 bound to its small part, and integrate $s \int_0^\infty \lambda^{s-1} |\{Tf > \lambda\}| d\lambda$. Fubini gives respectively factors $(s-1)^{-1}$ and $(2-s)^{-1}$. The latter is bounded on this interval. On $[3/2, 2]$ ordinary interpolation between the $3/2$ and 2 bounds is uniform. The adjoint has the same kernel and L^2 properties. Duality with $s = r/(r-1)$ proves (F.1). Integrability of K and Young's inequality give the remaining $W^{1,r}$ bound.

Split $K * w$ into $|z| \leq a$ and $|z| > a$. The bounds are CMa and $Cm/a + Cm$. Choosing $a = \min(1/4, \sqrt{m/M})$ proves the first part of (F.2), including $m = 0$. For the difference of two velocities, the integral on the two disks of radius $2d$ is at most CMd . Off those disks the mean value theorem gives a bound by $CMd \int_{2d}^{1/2} s^{-1} ds + CMd$. This proves the second part. Finally $K \in L^{p'}$ for $p' = p/(p-1) < 2$, so Hölder's inequality gives the L^p to L^∞ bound. $\square$

Proposition 2.2 (Bounded Euler and its common transport flow). *Every bounded mean-zero initial vorticity w_0 admits a unique bounded distributional solution of*

$$\partial_t w + \operatorname{div}(uw) = 0, \quad u = Kw \tag{F.3}$$

on every finite time interval. It has a unique measure-preserving homeomorphism flow $\Phi(t)$ and $w(t) = w_0 \circ \Phi(t)^{-1}$. It belongs to $C([0, T]; L^r)$ for every finite r , and preserves all finite L^r norms and the essential supremum. Each bounded initial colour can be transported by this same flow; its area, signed circulation, norm, and nesting relations are preserved. If w_0 is invariant under a quarter turn, so is $w(t)$, and the flow fixes the origin.

Proof. Let η_ν be periodic smooth nonnegative mollifiers invariant under quarter turns, and let $K_\nu = \eta_\nu * K$. Solve the ordinary differential equation on continuous periodic displacement maps

$$\dot{\Phi}_\nu(t, x) = \int_{\mathbb{T}^2} K_\nu(\Phi_\nu(t, x) - \Phi_\nu(t, y)) w_0(y) dy, \quad \Phi_\nu(0, x) = x. \tag{F.4}$$

Its right side is bounded and globally Lipschitz in the uniform norm, with Lipschitz constant at most $2\|DK_\nu\|_\infty\|w_0\|_1$. Successive approximation gives a global solution. The spatial field on the right is smooth and divergence free. Its classical flow is a diffeomorphism with Jacobian one, since the Jacobian solves $J' = J \operatorname{div} u_\nu = 0$. Thus, with $w_\nu = w_0 \circ \Phi_\nu^{-1}$, $u_\nu = K_\nu * w_\nu$ and w_ν solves its exact transport equation. In particular all its norms are preserved.

The estimates of Lemma 2.1 give a uniform spatial log-Lipschitz modulus for u_ν and a uniform supremum bound. Moreover

$$\partial_t u_\nu = - \sum_j (\eta_\nu * \partial_j K) * (u_{\nu,j} w_\nu)$$

is uniformly bounded in L^r for every fixed $1 < r < \infty$ by the same order-zero multiplier estimate. For $r > 2$, the uniform $W^{1,r}$ bound and this time estimate make u_ν precompact in $C([0, T] \times \mathbb{T}^2)$. To see the time equicontinuity directly, the L^r time difference is at most $C|t - s|$, while the spatial Hölder seminorm is uniformly bounded by Sobolev's inequality; averaging on a disk of radius a bounds its supremum by $Ca^{1-2/r} + Ca^{-2/r}|t - s|$, which tends to zero after optimizing a .

Osgood's inequality for the modulus $d(1 + |\log d|)$ gives uniform spatial continuity of Φ_ν and its inverse on $[0, T]$. For example a separation initially at most d_0 is at most $Cd_0^{\exp(-CMT)}$. Their time continuity follows from the uniform speed bound; for inverse maps combine this bound with the spatial estimate. Arzelà–Ascoli gives uniform limits Φ, Ψ and u along a subsequence, with $\Phi\Psi = \Psi\Phi = \text{Id}$. Passing to the integral flow equation identifies Φ with the flow of u . Its uniqueness follows from the same Osgood inequality. Testing continuous functions and passing to the limit proves that Φ preserves measure. Approximate w_0 by a smooth f in L^r ; then

$$\sup_t \|w_0 \circ \Phi_\nu^{-1} - w_0 \circ \Phi^{-1}\|_r \leq 2\|w_0 - f\|_r + \sup_t \|f \circ \Phi_\nu^{-1} - f \circ \Phi^{-1}\|_r \longrightarrow 0.$$

This proves strong convergence and continuity for finite r . Passing to the transport equation and to $u_\nu = K_\nu * w_\nu$ gives (F.3). The supremum and all norms are preserved by the limiting measure-preserving flow.

We include uniqueness because it is what transfers symmetries to the exact finite stacks. If u_1, u_2 have bounded curls by M , their velocity equations are $\partial_t u_i + \mathbb{P} \operatorname{div}(u_i \otimes u_i) = 0$, where $\mathbb{P}$ is the zero-mean divergence-free Fourier projection. The equivalence follows by taking curl, divergence, and mean; a zero-mean divergence-free curl-free field on the torus is zero by Fourier coefficients. For each fixed r , $u_i \in L^\infty W^{1,r}$ and $\partial_t u_i \in L^\infty L^r$. The latter follows from the transport equation and the multiplier estimates. Consequently the L^2 energy calculation is justified by time mollification. Put $v = u_1 - u_2$ and $Z = \|v\|_2^2$. The transport term with u_1 integrates to zero, and Hölder and interpolation give

$$|Z'| \leq C \|\nabla u_2\|_r \|v\|_{2r/(r-1)}^2 \leq CrM(CM)^{2/r} Z^{1-1/r}, \quad r \geq 2. \quad (\text{F.5})$$

Here the supremum bound on v follows from (F.2). If $M = 0$ the assertion is immediate. Otherwise divide Z by a fixed upper bound CM^2 and choose $r = \max(2, |\log(Z/(CM^2))|)$ when $Z > 0$. Then $Z' \leq C_M Z(1 + |\log(Z/(CM^2))|)$. Since the integral of the reciprocal of this modulus diverges at zero, separation from zero in finite time is impossible: integrate the inequality first with Z replaced by $Z + \epsilon$ and let $\epsilon \downarrow 0$. Hence equal initial data imply $Z = 0$. Renormalization and the initial trace for any bounded distributional solution follow from Proposition 2.3 below; that proposition also justifies all its norms and weak time traces used in (F.5). Thus uniqueness holds in the entire bounded class, not just among the constructed solutions.

Each colour is its initial function composed with Φ^{-1} ; composition preserves all the stated geometric and norm properties. Quarter-turn invariance of K and uniqueness imply the invariance of w and the equivariance of u and Φ . The origin is fixed, since equivariance forces $u(t, 0) = 0$. $\square$

Proposition 2.3 (Renormalization and strong time traces). *Let $p > 2$, $w \in L^\infty(0, T; L^p)$, and let $u \in L^\infty(0, T; W^{1,p})$ be divergence free. Suppose $\partial_t w + \operatorname{div}(uw) = 0$ with initial weak L^p trace w_0 . Then, for every $\beta \in C^1(\mathbb{R})$ with bounded derivative,*

$$\partial_t \beta(w) + \operatorname{div}(u\beta(w)) = 0, \quad \beta(w)|_{t=0} = \beta(w_0) \quad (\text{F.6})$$

in distributions including the initial boundary term. The solution has a representative in $C([0, T]; L^p)$ and $\|w(t)\|_p = \|w_0\|_p$ for every t .

Proof. Sobolev's inequality gives $u \in L_{t,x}^\infty$. The transport equation therefore gives a weakly continuous L^p representative: its pairings with smooth test functions are Lipschitz in time, and density and the uniform L^p bound extend these pairings continuously to every $L^{p'}$ test function. This representative exists at both endpoints and at every interior time, with the same L^p bound, by weak lower semicontinuity.

Spatially mollify w . The commutator in $\partial_t w_\epsilon + u \cdot \nabla w_\epsilon = r_\epsilon$ has, up to an irrelevant sign, the representation

$$r_\epsilon(x) = \int \nabla \eta_\epsilon(z) \cdot [u(x) - u(x-z)][w(x-z) - w(x)] dz. \quad (\text{F.7})$$

The subtracted $w(x)$ term is zero because $\operatorname{div} u = 0$; this follows by integration by parts in z . The Sobolev translation estimate and Hölder give

$$\|r_\epsilon(t)\|_{p/2} \leq C \|\nabla u(t)\|_p \sup_{|z| \leq \epsilon} \|w(t, \cdot - z) - w(t)\|_p.$$

Translation continuity in L^p and dominated convergence imply $r_\epsilon \rightarrow 0$ in $L_t^1 L_x^{p/2}$, hence in $L_{t,x}^1$. For fixed ϵ , the mollified equation is absolutely continuous in time with values in every fixed spatial smooth norm locally; the scalar chain rule gives (F.6) with error $\beta'(w_\epsilon)r_\epsilon$. This error tends to zero in L^1 , and Lipschitz continuity of β permits passage to the limit.

To check the trace instead of assuming a renormalized trace theorem, extend w to a short negative time interval as the constant w_0 and extend u there by zero. The weak trace makes the extended linear transport equation hold with no time Dirac mass. Apply the same commutator proof on the extended interval. It proves precisely the initial trace in (F.6). At any time τ , extend w after τ by its weak value $w(\tau)$ and extend u by zero. Again there is no time Dirac mass. This verifies the renormalized trace at every τ , including T .

Choose a smooth nonincreasing $\chi : [0, \infty) \rightarrow [0, 1]$, equal to one on $[0, 1]$ and zero on $[2, \infty)$, and put

$$\beta_R(s) = \int_0^{|s|} p a^{p-1} \chi(a/R) da.$$

It is C^1 with bounded derivative and increases to $|s|^p$ as $R \uparrow \infty$. Integrating its renormalized equation over the torus, using the verified traces, gives $\int \beta_R(w(\tau)) = \int \beta_R(w_0)$ at every time. Monotone convergence proves conservation of the L^p norm. For the final strong-continuity implication one may use the elementary L^p convexity inequality, valid here because $p > 2$,

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{\|f\|_p^p + \|g\|_p^p}{2}.$$

For its scalar proof, divide by the larger absolute value and set $0 \leq t \leq 1$ equal to their ratio. The function $(1+t)^p + (1-t)^p - 2^{p-1}(1+t^p)$ has nonnegative derivative: convexity of x^{p-1} gives $(1+t)^{p-1} - (1-t)^{p-1} \geq (2t)^{p-1}$. Its value at $t = 1$ is zero, so its value on $[0, 1]$ is nonpositive. Integrate this scalar inequality to obtain the display. If $f_j \rightharpoonup f$ and $\|f_j\|_p \rightarrow \|f\|_p$, weak lower semicontinuity applied to $(f_j + f)/2$ makes the second term tend to zero. Thus weak convergence plus convergence of norms implies strong convergence. Apply this criterion to $w(t_j)$ for every $t_j \rightarrow t$, using the verified all-time norm equality and weak continuity. This proves the claimed strong representative on the entire closed interval. $\square$

Proposition 2.4 (Strong compactness of exact transported stacks). *Let $p > 2$. Suppose w_n are bounded exact Euler solutions, $\sup_n \|w_n\|_{L_t^\infty L^p} < \infty$, and $w_n(0) \rightarrow w_0$ strongly in L^p . On each fixed finite interval a subsequence converges strongly in $C([0, T]; L^p)$ to a renormalized Euler solution with initial datum w_0 .*

Proof. Their velocities $u_n = \mathcal{K}w_n$ are uniformly bounded in $L^\infty W^{1,p}$ and $L_{t,x}^\infty$. The equation gives $\partial_t u_n = -\sum_j \partial_j K * (u_{n,j} w_n)$ uniformly in $L^\infty L^p$, by the verified multiplier estimate. The spatial

and temporal compactness argument in the proof of Proposition 2.2 therefore gives $u_n \rightarrow u$ in $C([0, T] \times \mathbb{T}^2)$. The vorticity equation bounds $\partial_t w_n$ in $L^\infty W^{-1,p}$. Diagonal Arzelà–Ascoli on the pairings with a countable dense set of smooth functions gives $w_n \rightarrow w$ in $C([0, T]; L^p_{\text{weak}})$. Uniform convergence of u_n and weak convergence of w_n pass the flux to the limit. Thus $u = \mathcal{K}w$ and w solves Euler with initial datum w_0 . Proposition 2.3 makes it renormalized and strongly continuous, with $\|w(t)\|_p = \|w_0\|_p$.

If $t_n \rightarrow t$, weak convergence in the preceding sense implies $w_n(t_n) \rightharpoonup w(t)$ in L^p , whereas exact transport gives $\|w_n(t_n)\|_p = \|w_n(0)\|_p \rightarrow \|w_0\|_p = \|w(t)\|_p$. The weak-convergence-plus-norm criterion proved above gives strong convergence at these varying times. If convergence were not uniform in time, choose times realizing a fixed positive distance and then a convergent subsequence of those times. The just proved assertion, together with strong continuity of w , is a contradiction. $\square$

3 Analytic contour equations and signed-patch continuation

We use two independent real coordinates for planar vectors. Their complexification has values in $\mathbb{C}^2$, with bilinear dot product $z \cdot w = z_1 w_1 + z_2 w_2$. In particular, the complexified denominator below is $z \cdot z$, not a Hermitian square. Set

$$K(z) = \frac{1}{2\pi} \frac{(-z_2, z_1)}{z \cdot z}, \quad z \cdot z \neq 0.$$

Fix $0 < \gamma < 1$. On a compact smooth parameter manifold, $c^{1,\gamma}$ denotes the closure of smooth functions in $C^{1,\gamma}$. All these spaces carry the norm consisting of the uniform norms of the function and its first derivatives and the γ -Hölder seminorm of those derivatives. Thus

$$E_\gamma = c^{1,\gamma}(S^1; \mathbb{R}^2)^2, \quad E_{\gamma, \mathbb{C}} = c^{1,\gamma}(S^1; \mathbb{C}^2)^2.$$

In all periodic statements $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, with area one and zero spatial mean for the Biot–Savart velocity. The entire pair of vector-valued contour functions is retained: normal, tangential, and independent material-label perturbations are allowed. The optional quarter-turn symmetry is the closed subspace defined by $Z_i(\theta + \pi/2) = QZ_i(\theta)$, where $Q(x_1, x_2) = (-x_2, x_1)$. Restriction to it is made only after the full-space construction.

Let $0 < r_1 < r_2$, $D_i = B_{r_i}$, and $Z_i^0(\theta) = r_i(\cos \theta, \sin \theta)$. For small real $h = (h_1, h_2) \in C^{1,\gamma}(S^1; \mathbb{R}^2)^2$, the curves $Z_i = Z_i^0 + h_i$ are positively oriented embedded Jordan curves; their bounded interiors Ω_i are nested and their boundaries are disjoint. The coefficients $a_1, a_2 \in \mathbb{R}$ are arbitrary, and the signed patch and its material velocity are

$$w_Z = a_1 \mathbf{1}_{\Omega_1} + a_2 \mathbf{1}_{\Omega_2}, \quad V_i(Z)(\theta) = (K * w_Z)(Z_i(\theta)). \quad (\text{C.1})$$

The planar velocity is continuous across each interface. The expression in (C.1) at a boundary point therefore has its usual absolutely convergent volume-integral interpretation.

3.1 A constant-Jacobian extension of every small boundary perturbation

Lemma 3.1. *For a disk $D = B_r$, there are neighborhoods of zero and a real analytic map*

$$h \mapsto T_h \in C^{1,\gamma}(\overline{D}; \mathbb{R}^2)$$

such that $T_0 = \text{Id}$, $T_h(re^{i\theta}) = re^{i\theta} + h(\theta)$, and

$$\det DT_h = J(h), \quad J(h) = \frac{1}{2|D|} \int_0^{2\pi} \det(Z^0 + h, \partial_\theta(Z^0 + h)) d\theta. \quad (\text{C.2})$$

For real small h , T_h is a diffeomorphism onto the interior bounded by this curve, and $J(h) = |\Omega_h|/|D| > 0$. Both maps have holomorphic extensions to the complexifications of their real Banach spaces.

Proof. A bounded linear boundary extension can be given explicitly. Choose a smooth radial cutoff χ which is zero for radius at most $r/2$ and one near r , and put $Hh(se^{i\theta}) = \chi(s)h(\theta)$. This is bounded from $C^{1,\gamma}(S^1; \mathbb{R}^2)$ to $C^{1,\gamma}(\overline{D}; \mathbb{R}^2)$; the angular coordinates are used only on an annulus bounded away from zero.

Write

$$Y = \{f \in C^\gamma(\overline{D}) : \int_D f = 0\}.$$

Theorem 2 and Remark (ii) of Dacorogna–Moser [1] give a fixed bounded linear operator

$$B : Y \longrightarrow C^{1,\gamma}(\overline{D}; \mathbb{R}^2), \quad \operatorname{div} Bf = f, \quad Bf|_{\partial D} = 0. \quad (\text{C.3})$$

Their boundary hypothesis is $C^{3,\gamma}$ when the input exponent is $C^{0,\gamma}$; a disk satisfies it. Extend B, H complex linearly when necessary. Define

$$\mathcal{F}(h, f) = \det D(\operatorname{Id} + Hh + Bf) - J(h).$$

For every $f \in Y$, the integral of this expression is zero. Indeed, the two-dimensional determinant identity and Green’s formula give

$$\int_D \det DT = \frac{1}{2} \int_{\partial D} (T_1 dT_2 - T_2 dT_1).$$

The boundary trace of $T = \operatorname{Id} + Hh + Bf$ is $Z^0 + h$. This proves the assertion, including for complex functions by bilinearity. Green’s formula here follows first for smooth functions and then by C^1 approximation. Thus $\mathcal{F}$ is a continuous polynomial into Y , $\mathcal{F}(0, 0) = 0$, and

$$D_f \mathcal{F}(0, 0)f = \operatorname{div} Bf = f.$$

The analytic implicit-function theorem supplies a unique analytic $f = f(h)$, and $T_h = \operatorname{Id} + Hh + Bf(h)$ satisfies (C.2). Conjugation and uniqueness show that real h gives real $f(h)$.

After shrinking the neighborhood, $\|DT_h - I\|_\infty < 1/2$. Convexity of D and integration along segments imply

$$|T_h(x) - T_h(y)| \geq \frac{1}{2}|x - y| \quad (x, y \in \overline{D}).$$

The inverse-function theorem and this global injectivity make T_h a diffeomorphism onto its image. Its boundary is precisely the given Jordan curve, and its image is that curve’s bounded interior. Real change of variables now identifies $J(h)$ with the ratio of areas. The complex statement uses the polynomial in (C.2); it makes no claim about a geometric area for complex-valued maps. $\square$

3.2 Norm-valued analyticity of the singular self interaction

For the next lemma the space is the big Hölder space $Y_D = C^{1,\gamma}(\overline{D}; \mathbb{C}^2)$. Hernandez’s [2] Definition 1.1 and Proposition 1.3 apply to an arbitrary complex map near the identity on a bounded convex C^2 domain. They provide $\delta, M > 0$ such that

$$F_D(X)(a) = \int_D K(X(a) - X(b)) db, \quad \sup_{\|X - \operatorname{Id}\|_{Y_D} < \delta} \|F_D(X)\|_{Y_D} \leq M. \quad (\text{C.4})$$

There is no determinant constraint in this imported bound.

Lemma 3.2. *The map F_D in (C.4) is Fréchet holomorphic on a smaller complex ball. The physical self-interface contribution is the real analytic map*

$$h \longmapsto J(h) \operatorname{Tr}_{\partial D} F_D(T_h). \quad (\text{C.5})$$

Proof. First establish holomorphy on every affine complex line inside the ball. If $X_\zeta = X_* + \zeta H$, the bilinear denominator of the kernel is bounded below in modulus by a positive constant times $|a - b|^2$, uniformly on any slightly smaller parameter disk. This follows by expanding $(X_\zeta(a) - X_\zeta(b)) \cdot (X_\zeta(a) - X_\zeta(b))$ around $|a - b|^2$. The Lipschitz perturbation is small. The integrand is holomorphic in ζ and bounded by $C|a - b|^{-1}$, which is integrable in two dimensions. Scalar dominated Cauchy integration therefore makes $u(\zeta, a) = F_D(X_\zeta)(a)$ holomorphic for each a . Here is a justification in the entire $C^{1,\gamma}$ norm. The bound (C.4) makes spatial first difference quotients converge to $D_a u(\zeta, a)$ uniformly in ζ , with error $O(|t|^\gamma)$ for spatial increment t . Their scalar components are holomorphic; their uniform limits are holomorphic too. For an interior parameter circle $|\zeta| = R$, define the pointwise Cauchy coefficients

$$U_n(a) = \frac{1}{2\pi i} \int_{|\zeta|=R} u(\zeta, a) \zeta^{-n-1} d\zeta.$$

Differentiation in a under this scalar integral is justified by the uniform difference-quotient convergence. Integrating the uniform and Hölder bounds in (C.4) proves $\|U_n\|_{C^{1,\gamma}} \leq CM R^{-n}$. Consequently $\sum_n U_n \zeta^n$ converges in Y_D for $|\zeta| < R$, and it equals $u(\zeta, \cdot)$ by pointwise scalar Cauchy's formula. This proves norm-valued holomorphy on each line, without presupposing a Bochner integral of an already norm-continuous function.

For completeness, the passage from lines to Fréchet holomorphy uses only this local norm bound. At a point X_* , linewise Cauchy coefficients define homogeneous maps $P_n(H)$ with $\|P_n(H)\| \leq CM(\|H\|/R)^n$ on a ball of radius R inside the domain. On every finite-dimensional span of directions, iterated scalar Cauchy formulas show that P_n is a homogeneous polynomial. Polarization gives a continuous symmetric multilinear form, with bound at most $CMn^n/(n!R^n)$, representing this polynomial. The diagonal bounds show that $\sum_n P_n(H)$ converges in norm for $\|H\| < R$ and equals $F_D(X_* + H)$. This is a convergent Banach-space analytic expansion.

For real h , real change of variables and (C.2) yield

$$\int_{\Omega_h} K(Z_h(\theta) - y) dy = J(h) \int_D K(T_h(re^{i\theta}) - T_h(b)) db.$$

Taking the boundary trace proves (C.5). The scalar factor $J(h)$ is part of the physical velocity; omitting it would give the wrong velocity whenever the boundary area changes. Composition of the holomorphic maps already constructed proves its asserted regularity. $\square$

3.3 The full two-contour vector field

Green's formula gives the following real contour representation:

$$V_i(Z)(\theta) = - \sum_{j=1}^2 \frac{a_j}{2\pi} \int_0^{2\pi} \log |Z_i(\theta) - Z_j(\phi)| Z_j'(\phi) d\phi. \quad (\text{C.6})$$

For the diagonal term the logarithm is locally integrable. The sign in (C.6) follows from $\nabla_x \log |x - y| = -\nabla_y \log |x - y|$ and, for positive orientation, $n^\perp d\sigma = dZ$.

Proposition 3.3. *On a fixed neighborhood of Z^0 in the full space E_γ , the material vector field (C.1) is bounded real analytic. It has a holomorphic extension to a complex neighborhood. On a smaller ball,*

$$\|V(Z^0 + h) - V(Z^0) - DV(Z^0)h\|_{E_\gamma} \leq C \|h\|_{E_\gamma}^2. \quad (\text{C.7})$$

The same statements hold for the flat square torus when the two contours and their interiors lie in a fixed coordinate disk whose diameter is less than one half of a period.

Proof. First work in the big $C^{1,\gamma}$ spaces. Treat each self term by (C.5), with its own reference disk. For $i \neq j$, the reference boundaries have positive separation. In a sufficiently small complex neighborhood,

$$(Z_i(\theta) - Z_j(\phi)) \cdot (Z_i(\theta) - Z_j(\phi))$$

stays uniformly away from zero and has the logarithm branch obtained by perturbing its positive real reference value. Replace the real logarithm in the cross term by one half of this complex logarithm. The logarithm's convergent algebra-valued power series, multiplication by Z'_j , and integration in ϕ give a holomorphic map into $C^{1,\gamma}$ in θ . This handles the entire cross term, including evaluation on the inner boundary inside the outer patch; one does not differentiate a singular cross volume integral.

For the torus write its zero-mean Green function near zero as $G_{\mathbb{T}}(x) = (2\pi)^{-1} \log |x| + g(x)$. The regular part is real analytic on the coordinate disk of differences: its Laplacian is constant, so subtracting a quadratic leaves a harmonic function. Set $R = \nabla^\perp g$. This contributes the bounded analytic term

$$\sum_j a_j J_j(h_j) \int_{D_j} R(Z_i(\theta) - T_{h_j}(b)) db. \quad (\text{C.8})$$

Real change of variables identifies it with the physical regular part. Its complex extension is obtained from the convergent local Taylor series of R . On the torus, (C.1) means the convolution with $K_{\mathbb{T}}$, and the vorticity's spatial mean can be subtracted: a constant vorticity has zero velocity under this convention.

It remains to pass to the little spaces. For a smooth embedded contour, the function

$$q(\theta, \phi) = \frac{|Z(\theta) - Z(\phi)|^2}{4 \sin^2((\theta - \phi)/2)}$$

extends smoothly across the diagonal and is positive there. The diagonal logarithmic kernel is the sum of $\log |2 \sin((\theta - \phi)/2)|$ and the smooth function $\frac{1}{2} \log q$. The former is an integrable convolution kernel; all derivatives can be placed on the smooth boundary derivative. Thus (C.6) maps smooth embeddings to smooth contour velocities. The cross terms and (C.8) also preserve smoothness.

Every real little-space embedding in this open neighborhood is a $C^{1,\gamma}$ limit of smooth embeddings in the same neighborhood. Continuity of the big-space analytic map and closedness of the little space show that its value belongs to E_γ . At a real little base point, repeated real finite differences imply that every Taylor multilinear coefficient, evaluated on real little directions, has values in this closed subspace. Complex linearity and the convergent big-space series put the restricted complex series in $E_{\gamma,\mathbb{C}}$. Its bounds are the same big-space bounds. This proves full-space analyticity and (C.7). Symmetry restrictions are valid since rotations and label changes preserve (C.1). $\square$

3.4 A stationary label gauge without derivative loss

Suppose the reference satisfies $V_i(Z^0) = \Omega_i \partial_\theta Z_i^0$ for constants Ω_i . Let

$$(S(s)h)_i(\theta) = h_i(\theta - \Omega_i s), \quad N(h) = V(Z^0 + h) - V(Z^0).$$

Translation is an isometry for the specified norm. Density of smooth functions proves that S is a strongly continuous group on E_γ . Its generator is $Ah = -\text{diag}(\Omega_i) \partial_\theta h$, on the domain for which this distributional derivative belongs to E_γ . No assertion that A is bounded on E_γ is used.

Proposition 3.4. *For every fixed $P < \infty$, a neighborhood of $h = 0$ has a unique mild flow on $[-P, P]$ for*

$$h_s = Ah + N(h), \quad h(s) = S(s)h(0) + \int_0^s S(s-r)N(h(r)) dr. \quad (\text{C.9})$$

Its fixed-time maps are C^2 , with uniformly bounded first and second derivatives on a smaller neighborhood. If $S(P) = I$, its time- P map equals the material time- P map in displacement coordinates.

Proof. Choose a ball on which N has bounded first and second derivatives, with $N(0) = 0$. Picard iteration in continuous E_γ -valued paths, over a sufficiently short interval, is a contraction. The same argument uses the group for negative times. If $\|DN\| \leq L$ and the analytic ball has radius b , Gronwall gives $\|h(s)\| \leq e^{L|s|} \|h(0)\|$. Initial norm less than $\frac{1}{2}be^{-LP}$ therefore yields existence through $[-P, P]$ by successively applying the same contraction.

The first derivative in initial direction k is the unique solution

$$y_k(s) = S(s)k + \int_0^s S(s-r)DN(h(r))y_k(r) dr.$$

The second derivative in directions k, ℓ solves the same linear Volterra equation with added integrand $D^2N(h(r))[y_k(r), y_\ell(r)]$ and zero initial value. If $\|D^2N\| \leq M_2$, their norms on the interval are at most $e^{LP} \|k\|$ and $M_2Pe^{2LP} \|k\| \|\ell\|$, respectively. Taylor's formula for N in the contraction equation proves that these are the actual Fréchet derivatives; their continuity follows from the same Volterra estimates. The unbounded label generator is never differentiated as an operator on the base space.

Put $Z_g(s) = Z^0 + h(s)$. The identity $AZ^0 + V(Z^0) = 0$ makes (C.9) equivalent to the mild equation

$$Z_g(s) = S(s)Z_g(0) + \int_0^s S(s-r)V(Z_g(r)) dr.$$

The physical field commutes with independent constant label shifts. Multiplying this identity by $S(-s)$ therefore shows that $Z_m(s) = S(-s)Z_g(s)$ satisfies

$$Z_m(s) = Z_g(0) + \int_0^s V(Z_m(r)) dr.$$

Here the contours of $Z_m(r)$ and $Z_g(r)$ have identical images, so the physical field along Z_m is defined throughout even when its labels are far from the selected reference labels. This is the material contour equation. When $S(P) = I$, their endpoints agree. $\square$

For the annulus used in the construction,

$$r_1^2 = \frac{3}{5}, \quad r_2 = 1, \quad a_1 = -1, \quad a_2 = \frac{3}{5}, \quad \Omega_1 = -\frac{1}{5}, \quad \Omega_2 = 0.$$

The values of Ω_i follow by integrating the radial circulation. Consequently $P = 10\pi$ satisfies $S(P) = I$. These statements are for the isolated planar annulus; a torus regular field is an additional bounded analytic term or forcing, rather than a claimed circular stationary reference.

3.5 Global planar continuation for finitely many signed interfaces

The continuation statement used for an isolated canonical orbit is planar. The next proposition specifies exactly the version obtained from Chemin's primary theorem.

Proposition 3.5. *Let $0 < \gamma < \gamma_0 < 1$. Let a planar compactly supported vorticity be a finite real linear combination of indicators of bounded domains whose positively oriented boundaries are pairwise disjoint C^{1,γ_0} Jordan curves. Domains may be nested and coefficients may have either sign. The Euler material flow transports these curves for all finite positive and negative times. On each compact time interval their material parametrizations have bounded C^{1,γ_0} norms and a positive chord-arc lower bound, and different interfaces have positive separation. In particular their trajectory is a compact subset of the finite product of $C^{1,\gamma}$ contour spaces.*

Proof. Let Γ be the union of the initial interfaces. On disjoint collars of these curves construct C^{1,γ_0} defining functions with nonvanishing gradient. A smooth cutoff of the defining functions, constant off a slightly larger collar, produces one global field $X_0 \in C^{\gamma_0}(\mathbb{R}^2; \mathbb{R}^2)$ of the form $\nabla^\perp \psi$. It

is tangent and nonzero on every interface, and remains nonzero on a closed sufficiently thin collar A_0 containing Γ . Its divergence is zero. These constructions can also be obtained by gluing local graph defining functions with a smooth partition of unity; the nonvanishing normal is preserved by shrinking each collar.

The distribution $X_0 \cdot \nabla w_0 = \operatorname{div}(X_0 w_0)$ is zero: each boundary measure has coefficient equal to the normal component of the tangent field. This argument is linear in the signed coefficients. Outside A_0 , w_0 is globally C^{γ_0} . More explicitly, take $A_0 = \{\operatorname{dist}(x, \Gamma) \leq d\}$ with $d > 0$ small enough for the field to be nonzero. If two points outside A_0 have different vorticity values, their joining segment crosses Γ , and their distance is at least $2d$. Thus the $C^{\gamma_0}(\mathbb{R}^2 \setminus A_0)$ seminorm is finite. Compact support also gives $w_0 \in L^1$. These are precisely the hypotheses of Chemin's [3] Definition 1 and Theorem 1, with exponent γ_0 and $p = 1$. That theorem supplies a global velocity with locally bounded Lipschitz norm and locally bounded C^{γ_0} norm of $X_0 \cdot \nabla \Phi$, where Φ is the material flow.

For any initial contour $Z_j(0, \theta)$, there is a nonvanishing C^{γ_0} scalar $\lambda_j(\theta)$ with $X_0(Z_j(0, \theta)) = \lambda_j(\theta) Z_j'(0, \theta)$. The following argument justifies the chain rule on the initial curve, rather than using an almost-everywhere spatial chain rule on a set of measure zero. Mollify the Lipschitz function $\Phi(t, \cdot)$ in space. If $g = X_0 \cdot \nabla \Phi$, then $X_0 \cdot \nabla \Phi_\varepsilon - g * \varrho_\varepsilon$ has uniform norm at most $C\varepsilon^{\gamma_0} \|\nabla \Phi\|_\infty \|X_0\|_{C^{\gamma_0}}$: move X_0 through the convolution and use its Hölder modulus. The right side tends to zero and $g * \varrho_\varepsilon \rightarrow g$ uniformly. Restricting the smooth chain rule to the initial curve and taking this uniform limit proves

$$\partial_\theta \Phi(t, Z_j(0, \theta)) = \lambda_j(\theta)^{-1} (X_0 \cdot \nabla \Phi)(t, Z_j(0, \theta)). \quad (\text{C.10})$$

This proves the uniform C^{1, γ_0} bounds at the times where Chemin's locally essential bounds hold. They extend to every time: approach any exceptional time by such times, use uniform material continuity and Arzelà–Ascoli for the bounded contour derivatives, and pass the γ_0 -Hölder bound to their limit. The Lipschitz velocity makes $\Phi(t)$ and its inverse Lipschitz, with constants bounded by $\exp \int \|\nabla u(s)\|_\infty ds$. Initial chord-arc and separation lower bounds therefore remain strictly positive. Material trajectories are continuous in the uniform norm because the velocity is bounded on these time intervals. Interpolation gives, for bounded C^{1, γ_0} functions,

$$\|f\|_{C^{1, \gamma}} \leq C \|f\|_{C^0}^{(\gamma_0 - \gamma)/(1 + \gamma_0)} \|f\|_{C^{1, \gamma_0}}^{(1 + \gamma)/(1 + \gamma_0)}. \quad (\text{C.11})$$

One obtains this estimate by first bounding a derivative by a first finite difference and its γ_0 -Hölder remainder, optimizing the difference scale, and then interpolating the derivative's Hölder seminorm between exponents zero and γ_0 . Apply it to contour differences at two times. Their uniformly bounded higher norms and uniform-norm continuity imply time continuity in $C^{1, \gamma}$. Smoothing a C^{1, γ_0} function and using the same interpolation shows membership in $c^{1, \gamma}$. The continuous image of a compact time interval is compact in this little space, proving the last assertion. This argument does not mistake a bounded Hölder ball for a compact set at its own exponent. $\square$

For the planar canonical signed annulus, conservation of its L^1 and L^∞ norms also gives

$$\|K * w(t)\|_\infty \leq C \|w(t)\|_1^{1/2} \|w(t)\|_\infty^{1/2}. \quad (\text{C.12})$$

Indeed, split the kernel integral at radius d : its near part is at most $Cd \|w\|_\infty$, and its far part at most $Cd^{-1} \|w\|_1$, and optimize. Hence a support initially in B_{R_0} remains in $B_{R_0 + C|t|}$. Every fourth spatial moment is bounded by $C(1 + |t|)^4$. If quarter-turn symmetry is present and the origin begins strictly inside both boundaries, flow uniqueness preserves the symmetry and fixes the origin. The positive finite-time inverse-Lipschitz bound then preserves a positive distance from the origin to each boundary on every compact time interval.

3.6 Lipschitz comparison on a compact contour tube

For a positively oriented planar contour define

$$b(Z) = \inf_{\theta \neq \phi} \frac{|Z(\theta) - Z(\phi)|}{|e^{i\theta} - e^{i\phi}|}.$$

Verdera's [4] contour-dynamics appendix establishes a locally Lipschitz vector field on the open set of $C^{1,\gamma}$ contours with $b(Z) > 0$. Its input is the self-boundary operator for a single planar patch. We check explicitly that his degree-zero boundary operator is the physical Euler operator used here. For fixed x , put $t = x - y$. Homogeneity $t \cdot \nabla K_a(t) = -K_a(t)$ gives, for each vector component a ,

$$\sum_{j=1}^2 \partial_{y_j} (t_j K_a(t)) = -K_a(t).$$

Integrating this identity on the domain with a disk of radius ε about x removed gives

$$\int_{\Omega} K(x - y) dy = - \int_{\partial\Omega} K(x - y)(x - y) \cdot n(y) d\sigma(y).$$

The removed-circle flux is $O(\varepsilon)$, because $tK(t)$ is bounded and its circle has length $O(\varepsilon)$. The same argument applies at a boundary point by intersecting the removed disk with the domain. For positive orientation $n d\sigma = -J dZ$, where $Jt = t^\perp$; hence the right side is $\int K(x - Z) \langle -J(x - Z), dZ \rangle$, precisely the operator in Verdera's equation (16). Green's logarithmic identity (C.6) expresses the same volume integral; its removed-circle error is $O(\varepsilon |\log \varepsilon|)$. This proves exact equality of the two boundary operators, including their self-interface traces. A finite signed collection inherits this local result: sum the self operators with their real coefficients; all remaining boundary logarithms have a positive separation and are locally smooth operators. Nesting does not change that separation argument. A smooth torus Green-function remainder has the same local property in coordinate charts, although a global torus continuation theorem is a separate statement.

Lemma 3.6. *Let $\mathcal{U}$ be an open subset of a Banach space and $F : \mathcal{U} \rightarrow E$ locally Lipschitz. If $\mathcal{K} \subset \mathcal{U}$ is compact, some tube $\mathcal{T}_\delta = \{z : \text{dist}(z, \mathcal{K}) < \delta\}$ is contained in $\mathcal{U}$, and F is bounded and globally Lipschitz on that tube. Convexity of the tube is unnecessary.*

Proof. Choose a finite collection of balls $B(x_a, 4r_a) \subset \mathcal{U}$ on which F has Lipschitz constants L_a , with their smaller balls $B(x_a, r_a)$ covering $\mathcal{K}$. Choose $0 < \delta < \min_a r_a$. Each point of the tube is within δ of a point of $\mathcal{K}$, hence lies in one of $B(x_a, 2r_a)$. This gives a uniform bound M for F there. If two tube points z, y have distance less than δ , use the ball chosen for z : both are in $B(x_a, 4r_a)$, and $\|F(z) - F(y)\| \leq (\max_a L_a) \|z - y\|$. If their distance is at least δ , the uniform bound gives the same inequality with constant $2M/\delta$. No segment through the union of the balls is required. $\square$

Consequently, if $Z_*(t)$ is the planar trajectory in Proposition 3.5 for $0 \leq t \leq S$, and Z solves $Z_t = V(Z) + f(t)$ while it remains in the above tube, then

$$\|Z(t) - Z_*(t)\| \leq e^{Lt} \left(\|Z(0) - Z_*(0)\| + \int_0^t \|f(r)\| dr \right). \quad (\text{C.13})$$

The integral form, the tube's Lipschitz constant, and Gronwall prove this estimate. When its right side is less than half the selected tube margin, the usual first-exit argument proves that the tube stop cannot occur before S . The same argument keeps any prescribed positive separation or origin-hole margin by choosing the tube small in the uniform norm. A label gauge can be included by its isometric group and the mild equation (C.9).

3.7 Identifying local torus contours inside a global bounded solution

The following localization avoids assuming that every foreign interface has a global Hölder parametrization. We state explicitly the bounded vorticity input needed for the uniqueness calculation: for the normalized torus Biot–Savart operator,

$$\|\nabla K_{\mathbb{T}} * \zeta\|_{L^p} \leq Cp \|\zeta\|_{\infty} \quad (p \geq 2), \quad \|K_{\mathbb{T}} * \zeta\|_{\infty} \leq C \|\zeta\|_{\infty}. \quad (\text{C.14})$$

The constant mode of ζ is immaterial. This is the periodic Calderón–Zygmund estimate used in the bounded-vorticity existence and uniqueness input; the following argument requires no stronger patch regularity theorem on the torus.

Lemma 3.7. *Let $\omega(t)$ be a bounded-vorticity torus Euler solution on $[0, S]$, with its unique incompressible material flow. Split it into two transported components $\omega = \omega^a + \omega^f$, each bounded, and set $f(t) = K_{\mathbb{T}} * \omega^f(t)$. Assume that $\omega^a(0)$ is a finite signed patch with pairwise disjoint positively oriented material parametrizations in $C^{1,\gamma}$, each with positive chord-arc lower bound. Let $\mathcal{K}$ be a compact set of such candidate active contours in one coordinate chart, with positive pairwise separation. Assume there is an open spatial neighborhood of all these contours on which f is continuous in time in $C^{2,\gamma}$, with a uniform bound.*

The local contour equation

$$Z_t = V_{\mathbb{T}}^a(Z) + f(t, Z) \quad (\text{C.15})$$

is well posed in the $C^{1,\gamma}$ contour space near $\mathcal{K}$. For as long as its solution stays in this neighborhood, it is exactly the active material contour of the given global solution. If the comparison estimate on a fixed compact candidate trajectory is strictly smaller than the chosen tube margin, its existence extends through that candidate time window by the tube first-exit argument. Only the indicated smooth bounds for f enter local well-posedness; no regularity of the foreign boundaries is required.

Proof. The self contour vector field is locally Lipschitz by the preceding construction and Verdera’s local estimate; separated active cross terms are smooth. Composition $Z \mapsto f(t, Z)$ is locally Lipschitz in $C^{1,\gamma}$, with uniform bounds from $\|f(t)\|_{C^{2,\gamma}}$; it maps the little space into itself. The integral-equation contraction proves local existence and uniqueness for (C.15). Let η be the signed active patch enclosed by its contours. Reynolds’ formula on each moving C^1 domain shows distributionally that

$$\partial_t \eta + \operatorname{div}((K_{\mathbb{T}} * \eta + f)\eta) = 0. \quad (\text{C.16})$$

The velocity is globally continuous and divergence free; its boundary trace is (C.15). The formula follows directly by differentiating $\int_{\Omega_j(t)} \varphi$ and replacing the boundary flux by the volume integral of the divergence. In particular each area is constant. The patch has a bounded global vorticity norm throughout the local contour interval.

The actual transported active component satisfies the same equation with η replaced by ω^a , because its actual velocity is $K_{\mathbb{T}} * \omega^a + f$. We prove uniqueness for these two solutions with the same prescribed f . Their means coincide, so put

$$d = \eta - \omega^a, \quad \psi = \Delta^{-1}d, \quad U = \nabla^{\perp} \psi, \quad E(t) = \int_{\mathbb{T}^2} |U|^2.$$

Writing $v = K_{\mathbb{T}} * \eta + f$, the difference equation is

$$\partial_t d + \operatorname{div}(vd) + \operatorname{div}(U\omega^a) = 0.$$

The energy calculation is justified by spatial smoothing followed by integration by parts: all vorticities are bounded, and (C.14) gives $W^{1,p}$ velocities for every finite p . Equivalently, $\psi \in W^{2,p}$ and $d_t \in W^{-1,p}$ allow pairing with ψ and then passing through these smooth approximations. The

term containing ω^a is zero because $U \cdot \nabla \psi = 0$ pointwise. In the remaining term substitute $d = \Delta \psi$; an integration by parts and $\operatorname{div} v = 0$ give

$$|E'(t)| \leq 2 \int_{\mathbb{T}^2} |\nabla v| |U|^2.$$

By (C.14), $\|\nabla v\|_p \leq CpM$, where M is a finite bound for both active vorticities and the foreign vorticity. Also $\|\tilde{U}\|_\infty \leq CM$. Hölder and interpolation yield

$$E'(t) \leq C_M p E(t)^{1-1/p} \quad (p \geq 2). \quad (\text{C.17})$$

For $0 < E < e^{-2}$, take $p = \log(1/E)$; for larger E , use $p = 2$ and enlarge the constant. The resulting upper modulus near zero is $C_M E \log(1/E)$, whose reciprocal has divergent integral at zero. The scalar integral comparison (or its regularization with $E + \varepsilon$ followed by $\varepsilon \downarrow 0$) proves $E(0) = 0 \Rightarrow E(t) = 0$. Thus $\eta = \omega^a$. Their total velocities agree, and uniqueness of the log-Lipschitz material flow identifies the individual domains and their labels.

Finally apply Lemma 3.6 to the active self field and the uniformly locally Lipschitz composition field. A compact-tube comparison such as (C.13), followed by a strict first-exit improvement, keeps (C.15) within its regular neighborhood through the prescribed time window. Identification then proves precisely the same local regularity for the actual active contours. $\square$

One common sufficient condition for the smooth-field hypothesis is that the foreign vorticity is constant on a fixed open neighborhood of the active tube for every time in the stopped interval. The constant may be zero, as when all foreign supports are separated. The identities $\operatorname{div} f = 0$ and $\operatorname{curl} f = \omega^f - \langle \omega^f \rangle$ then make both components of f harmonic there. Interior harmonic estimates bound every spatial derivative on a smaller neighborhood by the global velocity bound and its distance to the neighborhood's boundary. Transport gives strong L^1 time continuity of bounded foreign components, and splitting the integrable singular kernel shows uniform time continuity of their velocities. Applying the same interior harmonic estimates to velocity differences gives continuity in every spatial derivative on the smaller neighborhood. Thus irregular foreign interfaces outside that neighborhood cause no loss of active-contour regularity. In a cascade argument the required support or plateau separation must be proved on the stopped interval before this lemma is invoked.

Scope of the continuation input. The last global continuation proposition is for $\mathbb{R}^2$, exactly the geometry of the cited Chemin theorem. The local analytic and local Lipschitz statements above include the square torus. Neither Chemin's cited planar Theorem 1 nor Verdera's single-plane-patch appendix, by itself, is a citation for global continuation of finitely many signed torus patches. A manuscript needing that additional global statement must prove a periodic version or supply a primary theorem with those hypotheses; it does not follow merely by quoting the planar statements.

We next study the planar, isolated two-interface problem. Set

$$\rho^2 = \frac{3}{5}, \quad r_1 = \rho, \quad r_2 = 1, \quad c_1 = -1, \quad c_2 = \rho^2, \quad W = c_1 \mathbf{1}_{B_{r_1}} + c_2 \mathbf{1}_{B_{r_2}}.$$

Our convention is $\Delta \psi = W$, $u = \nabla^\perp \psi$, with $\nabla^\perp = (-\partial_y, \partial_x)$. Thus $u_r = -r^{-1} \partial_\theta \psi$ and $u_\theta = \partial_r \psi$. Let $0 < \gamma < 1$. The real Banach space is the closed subspace

$$E = \{h \in c^{1,\gamma}(S^1; \mathbb{R}^2)^2 : h_i(\theta + \pi/2) = i h_i(\theta)\}$$

of little Hölder perturbations of $X_i^0(\theta) = r_i e^{i\theta}$. Here and below real vectors in the plane are written as complex numbers, without complexifying the real phase space. The little space is the closure of smooth functions in the usual $C^{1,\gamma}$ norm.

The contour input, and the precise extent of this module. By Proposition 3.3, the physical contour velocity V is a bounded C^2 map on an open chord-arc neighborhood of X^0 in E , preserves smooth approximations and the enclosed domains, and gives the planar Euler velocity there. Its first two derivatives are bounded on a smaller fixed neighborhood. This is the differentiability assertion proved in the preceding section; no unstable-manifold theorem, spectral theorem for multipliers, or unproved shape differentiability statement is being used here. All remaining conclusions are proved below. An analytic contour input makes the moment chart analytic; C^2 is sufficient for the strong unstable graph.

4 The material gauge and the layer calculation

The angular velocity of the reference field is

$$\Omega(r) = \begin{cases} -1/5, & 0 < r < \rho, \\ 3/10 - \rho^2/(2r^2), & \rho < r < 1, \\ 0, & r > 1. \end{cases}$$

Consequently $\Omega_1 = -1/5$, $\Omega_2 = 0$. Independent label translations commute with the material equation. The stationary gauge is

$$X_{i,s} = V_i(X) - \Omega_i \partial_\theta X_i.$$

Writing $X = X^0 + h$, this is the semilinear equation

$$h_s = \mathcal{A}_0 h + \mathcal{B}(h), \quad \mathcal{A}_0 h_i = -\Omega_i \partial_\theta h_i, \quad \mathcal{B}(h) = V(X^0 + h) - V(X^0).$$

Label translations form a strongly continuous group of isometries on the little space: continuity holds on smooth functions and then on their norm closure. The mild equation and its first two differentiated integral equations therefore give a local C^2 flow Φ_s . For clarity, on any fixed interval $[-P, P]$, contraction on a short time interval follows from the local Lipschitz bound for $\mathcal{B}$; Gronwall extends the flow on a sufficiently small fixed ball. Its first derivative solves a bounded Volterra equation; the second has the same linear equation with source $D^2 \mathcal{B}(h)[D\Phi_s, D\Phi_s]$. These bounds are uniform for $|s| \leq P$. Negative times use the same group and integral equation. No differentiation in time into E is asserted for the unbounded label generator.

Untwist the perturbation by writing

$$h_i(\theta) = r_i e^{i\theta} \{z_i(\theta) + i\xi_i(\theta)\}, \quad z_i, \xi_i \text{ real and } \pi/2\text{-periodic.}$$

This is a bounded real isomorphism of the phase space with four scalar little Hölder spaces. Fourier coefficients use $f(\theta) = \sum_m f_m e^{im\theta}$, so $f_{-m} = \overline{f_m}$. The allowed nonzero geometric indices are $m \in 4\mathbb{Z}$.

A relative normal perturbation z_l of disk l produces the distributional first variation $c_l r_l z_l(\theta) \delta(r - r_l)$ of vorticity. For $m > 0$, its streamfunction coefficient is the unique continuous solution, regular at zero and decaying at infinity, of the radial Poisson equation. Matching the derivative jump at r_l gives exactly

$$\psi_{l,m}(r) = -\frac{c_l r_l^2}{2m} z_{l,m} \begin{cases} (r/r_l)^m, & r < r_l, \\ (r_l/r)^m, & r > r_l. \end{cases} \quad (1)$$

Indeed its outward-minus-inward radial derivative is $c_l r_l z_{l,m}$, as required. Its radial velocity, divided by r_i , at interface i is

$$\frac{ic_l r_l^2}{2r_i^2} \left(\frac{\min(r_i, r_l)}{\max(r_i, r_l)} \right)^m z_{l,m}.$$

Its tangential velocity, divided by r_i , is the same coefficient without i , with a minus sign for $r_i < r_l$ and a plus sign for $r_i > r_l$.

The reference-field evaluation derivative must be included. Away from an interface the ordinary polar calculation gives

$$z_{i,s} = -\Omega_i \partial_\theta z_i + \delta u_{r,i}/r_i, \quad (2)$$

$$\xi_{i,s} = -\Omega_i \partial_\theta \xi_i + r_i \Omega' (r_i) z_i + \delta u_{\theta,i}/r_i. \quad (3)$$

For example, the variation of the reference vector under tangential displacement is $-\Omega_i r_i \xi_i e_r$. This cancels the term $+\Omega_i \xi_i$ coming from differentiating $e^{i\theta}$ in the label generator; the normal evaluation derivative leaves $r_i \Omega' z_i$ in the second equation.

At a self interface neither Ω' nor the radial derivative in (1) has a single trace. There is no averaging assumption: their sums have identical one-sided limits. At $r_1 = \rho$ the two values of $r\Omega'$ are 0, 1, and the self-layer tangential values divided by r_1 are $+z_1/2, -z_1/2$. Their sum is $z_1/2$ from either side. At $r_2 = 1$ the corresponding values are $\rho^2, 0$ and $-\rho^2 z_2/2, +\rho^2 z_2/2$, giving $\rho^2 z_2/2$. Equivalently, one may evaluate the moving boundary from either side and differentiate the continuous physical velocity. This cancellation resolves the apparent trace ambiguity.

Combining these terms proves, for every positive allowed m ,

$$\begin{aligned} z_s &= L_m z, & \xi_s &= T_m \xi + C_m z, \\ L_m &= \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}, & T_m &= \begin{pmatrix} im/5 & 0 \\ 0 & 0 \end{pmatrix}, \\ C_m &= \frac{1}{2} \begin{pmatrix} 1 & -\rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}. \end{aligned} \quad (4)$$

Negative indices have the conjugate matrices.

For the mean layer, Gauss' law gives zero velocity inside disk l and tangential velocity $c_l r_l^2 z_{l,0}/r$ outside it. Using the same one-sided cancellation gives the entire mean block

$$z_{1,0,s} = z_{2,0,s} = 0, \quad \xi_{1,0,s} = 0, \quad \xi_{2,0,s} = \rho^2 (z_{2,0} - z_{1,0}). \quad (5)$$

It is a nilpotent block of order two. In particular, deleting mean normal modes before constructing the manifold would delete legitimate ambient area variations and is unnecessary.

5 Exact bounded shear and the complete spectral split

On positive indices set $H_m = \text{diag}(i, -i)$, on negative indices set $H_{-m} = \overline{H_m}$, and set $H_0 = 0$. Direct entrywise multiplication in (4) yields

$$H_m L_m - T_m H_m = C_m. \quad (6)$$

Thus H is the negative periodic Hilbert transform on the first component and the positive periodic Hilbert transform on the second, when the standard transform has symbol $-i \text{sgn } m$.

Here are the elementary operator bounds needed in this proof. The periodic Hilbert transform is bounded on C^γ , $0 < \gamma < 1$. Its principal-value kernel is a constant multiple of $\cot(t/2)$. Subtracting $f(\theta)$ uses its odd cancellation and bounds the value by $C[f]_\gamma \int_0^\pi t^{\gamma-1} dt$. For a difference at separation d , split the integrals into the two arcs within $2d$ of the evaluation points and their complement. The first part is bounded by $C[f]_\gamma d^\gamma$ by the same subtraction. In the second part, subtract the value at one evaluation point and use $|K(t+d) - K(t)| \leq Cd/|t|^2$; the bound is $C[f]_\gamma d \int_d^\pi t^{\gamma-2} dt \leq C_\gamma [f]_\gamma d^\gamma$. The constant term created by changing the subtracted value is bounded by $C[f]_\gamma d^\gamma$, using the integral cancellation of the kernel outside the removed arcs. This

proves the seminorm estimate. The transform commutes with the derivative on smooth functions, so it is bounded on $C^{1,\gamma}$. It preserves smooth functions and hence their closed little subspace. Secondly, an absolutely summable matrix Fourier symbol (A_m) defines convolution with a uniformly convergent kernel, and has operator norm at most $C \sum_m \|A_m\|$ on each of these Hölder spaces. Such a multiplier is compact, since its finite Fourier truncations converge in operator norm. Both facts remain valid on the symmetry subspace.

Consequently the change of variables $\zeta = \xi - Hz$ is a bounded invertible real linear map, and (6) gives

$$z_s = Lz, \quad \zeta_s = T\zeta$$

on every nonzero mode. The identities first hold on trigonometric polynomials; boundedness and density give the exact mild-flow identities on E . The derivative terms in L and T belong to the label-translation generator. The ungauged derivative DV is order zero: in untwisted variables its normal component is $-\Omega_i \xi_i + \delta u_{r,i}/r_i$ and its tangential component is $u'_\theta(r_i)z_i + \delta u_{\theta,i}/r_i$, with the self combination understood as above. Thus the shear introduces no unbounded operator and loses no derivative.

Set

$$\sigma = \frac{\rho^5}{2}, \quad \tau = \frac{3}{10}, \quad v_+ = (1, -i\rho)^t, \quad v_- = (1, i\rho)^t, \quad \ell_+ = (1/2, i/(2\rho)), \quad \ell_- = (1/2, -i/(2\rho)).$$

At $m = 4$, $L_4 v_\pm = (\pm\sigma + i\tau)v_\pm$ and $\ell_\alpha v_\beta = \delta_{\alpha\beta}$. On the full material space the unstable and stable eigenvectors are respectively $(v_+, H_4 v_+)$ and $(v_-, H_4 v_-)$, with conjugate negative Fourier coefficients to impose reality. In particular, $H_4 v_+ = (i, -\rho)^t$.

For $m \geq 8$, write

$$a_m = m/5 - 1/2, \quad d = 3/10, \quad g_m = a_m - d = (m-4)/5, \quad b_m = \rho^m/2, \quad c_m = -\rho^{m+2}/2.$$

The exact real eigenfrequencies of L_m are

$$\mu_{m,+} = a_m - \delta_m, \quad \mu_{m,-} = d + \delta_m, \quad (7)$$

$$\delta_m = \frac{g_m - \sqrt{g_m^2 - \rho^{2m+2}}}{2} = \frac{\rho^{2m+2}}{2(g_m + \sqrt{g_m^2 - \rho^{2m+2}})}. \quad (8)$$

The square root is real: $g_m \geq 4/5$ and $\rho^{2m+2} \leq (3/5)^9 < 1/64$. Thus $0 < \delta_m \leq C\rho^{2m}/m$. A diagonalizer is

$$D_m = \begin{pmatrix} 1 & -b_m/(g_m - \delta_m) \\ c_m/(g_m - \delta_m) & 1 \end{pmatrix}, \quad D_m^{-1} L_m D_m = i \operatorname{diag}(a_m - \delta_m, d + \delta_m).$$

Its determinant is $1 + b_m c_m / (g_m - \delta_m)^2$, uniformly positive. Both $D_m - I$ and $D_m^{-1} - I$ have absolutely summable symbols. Their conjugate negative-index symbols define bounded real diagonalizers on the little Hölder space. This justifies operator diagonalization, beyond the formal matrix eigenvalue calculation.

Take $P = 10\pi$. The nonzero tangential period operator is the identity. The normal high-mode period multipliers are exactly $-e^{-iP\delta_m}$, $-e^{iP\delta_m}$. They are uniformly separated from 1: for example $P\delta_m \leq (25\pi/4)(3/5)^9 < \pi/2$, and hence $|1 + e^{\pm iP\delta_m}| \geq \sqrt{2}$. If $M = e^{PL}$, the full nonzero-mode material period operator in the original (z, ξ) variables is exactly

$$(z, \xi) \mapsto (Mz, \xi + H(M - I)z). \quad (9)$$

The mean period block is $I + PN_0$, where N_0 is given by (5) and $N_0^2 = 0$.

For every integer $n \geq 1$,

$$\sum_{m=8,12,\dots} |e^{\pm inP\delta_m} - 1| \leq C \sum_m \min(2, n\rho^{2m}/m) \leq C(1 + \log(1 + n)). \quad (10)$$

For the last inequality split at $m = C_\rho \log(2 + n)$, chosen large enough that $n\rho^{2m} \leq 1$ past the splitting point. There are $O(1 + \log(1 + n))$ terms before it and the remaining geometric tail is bounded. The base multiplier is $(-1)^n I$. The summable-multiplier bound and bounded diagonalizers therefore bound all normal high-mode period powers by $C(1 + \log(1 + n))$. This proof does not assume that uniformly bounded scalar multipliers are automatically bounded on Hölder space.

Define $J : \mathbb{C} \rightarrow E$, as a real linear map, by assigning positive fourth coefficients $z_4 = v_+ \alpha$, $\xi_4 = H_4 v_+ \alpha$, their conjugate negative coefficients, and no other coefficients. Define $d_u(h) = \ell_+ z_4$. Then

$$P_u = Jd_u, \quad P_c = I - P_u, \quad d_u J = I_{\mathbb{C}}. \quad (11)$$

These are bounded real complementary projections commuting with the linearized flow. The complement contains the stable fourth modes, all high normal modes, every tangential relabeling, and the whole mean block. In particular $d_u P_c = 0$ despite the material shear.

Let $S(s) = D\Phi_s(0)$, $s \geq 0$, and let $S_c = S|_{P_c E}$. The stable fourth modes decay exponentially, the high normal modes obey (10), the tangential modes are translations, and the means grow at most linearly. Bounded evolution on $[0, P]$ then proves

$$\|S_c(s)\| \leq C(1 + s), \quad \|S(s)\| \leq C e^{\sigma s}, \quad \|S(-s)|_{P_u E}\| \leq C e^{-\sigma s}. \quad (12)$$

The second bound follows since $1 + s \leq C_\sigma e^{\sigma s}$. One can also directly bound continuous high-mode powers: their base inner factor is a translation composed with the bounded sign-phase multiplier $e^{-is \operatorname{sgn}(m)/2}$, and their outer factor is $e^{ids \operatorname{sgn}(m)}$; each sign-phase multiplier is a linear combination of the identity and Hilbert transform. The δ_m corrections satisfy the same summation bound with nP replaced by $|s|$.

For completeness the complexified period spectrum is precisely

$$\{-e^{\sigma P}, -e^{-\sigma P}, 1, -1\} \cup \{-e^{\pm i P \delta_m} : m = 8, 12, \dots\}.$$

The point 1 includes the tangential identity and the mean Jordan block. The high normal operator is a bounded diagonal conjugate of $-I$ plus a compact summable multiplier, and each Fourier subspace supplies its displayed eigenvalues. Conversely, outside the closure of these values, the inverses of the diagonal symbols are bounded constants plus absolutely summable corrections, so give a bounded inverse. This proves equality of the spectrum and the stated set. For the continuous generator the unstable and stable exponents are $\pm\sigma \pm i\tau$; the remaining spectrum consists of 0, the tangent frequencies $\pm im/5$, the normal frequencies $\pm i\mu_{m,\pm}$, and the accumulation points $\pm id$, where $m = 4, 8, \dots$ in the tangent list and $m \geq 8$ in the normal list. To check the converse assertion for this unbounded generator, the pure-translation resolvent follows by integrating the translation group over its period when the spectral parameter is not a translation eigenvalue. Positive and negative frequency projections are bounded combinations of identity, mean projection, and Hilbert transform. On each sign sector the inner normal resolvent differs from such a translated and shifted resolvent by an absolutely summable symbol, after removing any finite resonant slots of the unperturbed shifted translation. This qualification is needed when the spectral parameter equals an unperturbed inner frequency but not its perturbed value. There are only finitely many such slots for a fixed parameter, since their frequencies tend to infinity. Their Fourier projections are bounded finite-rank maps; on each slot invert the actual finite matrix, which is invertible because the parameter is outside the displayed actual spectrum. On the remaining tail the unperturbed denominators are bounded away from zero and grow linearly, and the resolvent difference has an absolutely summable symbol. The outer normal resolvent differs from its constant limiting resolvent by an absolutely summable symbol. Parameters off the displayed closed set therefore have bounded resolvents. All calculations initially on polynomials extend to the generator domain and then by the resolvent identity.

6 A direct C^2 strong unstable construction

Write the local period map as

$$\Phi_P(h) = Mh + R(h), \quad M = S(P), \quad R(0) = DR(0) = 0.$$

On a fixed ball, $\|R(h)\| \leq C\|h\|^2$, $\|DR(h)\| \leq C\|h\|$, and $\|D^2R(h)\| \leq C$. Set $M_u = M|_{P_u E}$, $M_c = M|_{P_c E}$. Choose any $0 < \kappa < \sigma$ and let

$$\mathcal{X}_\kappa = \{(h_n)_{n \leq 0} : \|h\|_{\mathcal{X}_\kappa} := \sup_{n \leq 0} e^{-\kappa n P} \|h_n\| < \infty\}.$$

Prescribe $u_0 \in P_u E$. The backward-sequence equations are

$$u_n = M_u^n u_0 - \sum_{l=n}^{-1} M_u^{n-1-l} P_u R(h_l), \quad (13)$$

$$c_n = \sum_{l=-\infty}^{n-1} M_c^{n-1-l} P_c R(h_l), \quad h_n = u_n + c_n. \quad (14)$$

The unstable exponents in the first sum are strictly negative. The complementary exponents in the second sum are nonnegative. Their weighted convolution bounds are respectively bounded by constants times

$$\sum_{j \geq 0} e^{-(\sigma-\kappa)jP}, \quad \sum_{j \geq 0} (1+j)e^{-\kappa jP}.$$

Thus all series converge in E and define a map on $\mathcal{X}_\kappa$. On a sequence ball of radius r , its nonlinear part has norm at most Cr^2 and Lipschitz constant at most Cr ; the free term has norm at most $C\|u_0\|$. Choosing r fixed and small, and then u_0 small, gives a unique fixed point in this ball. This requires no smooth cutoff or smooth norm on the ambient Banach space.

The fixed-point map is C^2 jointly in the sequence and u_0 . For example, in the second derivative the pointwise source obeys

$$\|D^2R(h_n)[k_n, l_n]\| \leq C e^{2\kappa n P} \|k\|_{\mathcal{X}_\kappa} \|l\|_{\mathcal{X}_\kappa},$$

which is stronger than the weight required by either convolution. Continuity of the second derivative is uniform on the shrinking sequence tail because $h_n \rightarrow 0$; finitely many remaining indices use ordinary continuity of D^2R . The derivative of the fixed-point equation with respect to the sequence has inverse given by a convergent Neumann series, since its nonlinear operator norm is less than one. Differentiating it once and twice proves C^2 dependence on u_0 . At zero the solution is $h_n = M_u^n u_0$, and hence $c_0 = G(u_0)$ satisfies

$$G(0) = DG(0) = 0, \quad \|G(u_0)\| \leq C\|u_0\|^2.$$

This proves a two-real-dimensional local graph $\mathcal{W}^u = \{u + G(u)\}$ tangent to the full material unstable eigenspace.

The fixed-point equations imply $h_{n+1} = \Phi_P(h_n)$. Conversely any sufficiently small backward period orbit in $\mathcal{X}_\kappa$ satisfies (13): solve its unstable recursion toward time zero, and solve the complementary recursion from time N to time n . The leftover complementary term is bounded by $C(1+n-N)e^{\kappa NP}\|h\|_{\mathcal{X}_\kappa}$, which tends to zero as $N \rightarrow -\infty$. This proves the uniqueness characterization needed below.

Subperiod flow preserves the graph locally. Indeed, for $|t| \leq P$, the sequence $k_n = \Phi_t(h_n)$ is a backward period orbit because the local autonomous flow commutes with its period map. Its norm is at most $C_P\|h\|_{\mathcal{X}_\kappa}$, since $\Phi_t(0) = 0$ and its first derivative is uniformly bounded. After shrinking the parameter ball this sequence lies in the uniqueness neighborhood of the same construction. It is

therefore the fixed-point sequence prescribed by $P_u \Phi_t(h_0)$. This proves continuous local invariance whenever the segment stays in the local neighborhood. Gluing the subperiod segments of the backward sequence constructs an actual ancient isolated trajectory with $\|h(s)\| \leq C e^{\kappa s}$, $s \leq 0$.

Both enclosed areas are conserved along the physical material flow: their derivative is $\int_{\partial D_i} u \cdot n \, dS = 0$. Relabeling does not change an enclosed area, so they are also conserved in the stationary gauge. Since every graph trajectory tends backward to X^0 , its two areas are precisely $\pi \rho^2$ and π . Thus the graph lies in the correct area level set automatically, although the construction was in the full ambient space. In particular its total circulation is zero. This argument does not differentiate an unbounded generator through a nonlinear area chart.

Compatibility of the graph across Hölder exponents. Fix $\gamma < \gamma_0 < 1$. Apply the same construction also in the little space E_{γ_0} , with the contour input and its fixed-time maps at exponent γ_0 . The natural inclusion $\iota : E_{\gamma_0} \rightarrow E_\gamma$ is continuous: smooth approximation in the stronger norm proves that its image lies in the lower little space. The physical velocities and the label-translation groups agree under this inclusion. Uniqueness of the lower-space mild equation therefore makes the two local fixed-time maps agree wherever the higher-space trajectory is defined in their common neighborhood. The finite-dimensional map J , coefficient functional d_u , and the projections P_u, P_c have the same formulas at both exponents.

Choose the same weight κ in the two backward constructions. For the prescribed datum $u_0 = J\alpha$, the higher-space fixed-point sequence has norm at most $C_{\gamma_0} |\alpha|$. Its included lower-space sequence has norm at most $\|\iota\| C_{\gamma_0} |\alpha|$ and solves the lower-space period recursion with exactly the same u_0 . After shrinking $|\alpha|$ this sequence belongs to the lower-space uniqueness ball. The characterization following (13) identifies it with the lower-space fixed-point sequence. In particular

$$\iota\{J\alpha + G_{\gamma_0}(J\alpha)\} = J\alpha + G_\gamma(J\alpha)$$

on a common fixed parameter disk. The two geometric moment coordinates also agree, since their polynomial boundary formulas agree. Thus the canonical graph, its sufficiently small cap, and every fixed compact segment of its ancient trajectory can be chosen in little C^{1,γ_0} , while all comparisons use the lower $C^{1,\gamma}$ norm. This supplies the higher boundary regularity needed for a later contour continuation argument. Constants may depend on the fixed pair (γ, γ_0) , but remain independent of amplitude, radius, and stack depth.

7 Exact moment coordinates, symmetry, and the cap family

For each enclosed domain define

$$Q_i(X) = \frac{1}{2\pi r_i^6} \int_{D_i(X)} \bar{z}^4 \, dA(z), \quad a(X) = \ell_+ Q(X).$$

The boundary expression is the real polynomial functional

$$Q_i(X) = \frac{1}{20\pi i r_i^6} \int_0^{2\pi} \overline{X_i(\theta)}^5 X_i'(\theta) \, d\theta. \quad (15)$$

Indeed $d(\bar{z}^5 dz) = 10i\bar{z}^4 dA$. This proves analyticity as a map on E , as well as invariance under orientation-preserving independent relabelings. The first shape derivative is

$$DQ_i(X^0)h = \frac{1}{2\pi} \int_0^{2\pi} z_i(\theta) e^{-4i\theta} \, d\theta = z_{i,4};$$

it has no tangential contribution. Consequently

$$Da(X^0)h = d_u(h), \quad Da(X^0)P_c = 0, \quad Da(X^0)J = I_{\mathbb{C}}. \quad (16)$$

In particular this is a full material left dual, a complex notation for exactly two real coordinates. It is not a complex one-dimensional restriction of the real phase space.

The map

$$h \longmapsto (a(X^0 + h), b = P_c h) \in \mathbb{C} \times P_c E$$

has derivative (d_u, P_c) , whose bounded inverse is $(\alpha, b) \mapsto J\alpha + b$. The Banach inverse function argument (equivalently solve the quadratic remainder by contraction) gives a real analytic local coordinate chart. In these coordinates the unstable graph is $b = g(a)$, with $g \in C^2$, $g(0) = Dg(0) = 0$, and $\|g(a)\| \leq C|a|^2$.

Only the geometric coordinate requires a continuous differential equation. Its label-generator term vanishes by relabeling invariance, and thus

$$a_s = \ell_+ DQ(X)V(X) =: F(a, b), \quad F(a, b) = (\sigma + i\tau)a + O((|a| + \|b\|)^2).$$

This is a bounded C^2 (analytic for analytic V) finite-dimensional functional. The linear term follows from $\ell_+ L_4 = (\sigma + i\tau)\ell_+$ and (16). The computation can first be done along smooth material solutions; boundedness and continuity extend its integral form to the little-space flow. There is no conjugation of the unbounded label generator by the nonlinear coordinate chart.

The action

$$(\mathcal{R}_\phi X)_i(\theta) = e^{i\phi} X_i(\theta - \phi)$$

fixes X^0 , preserves fourfold equivariance, and commutes with the planar gauged flow. It is a bounded action on E , with norms uniform in ϕ ; using the natural rotation-invariant Hölder norms it is isometric. It sends a to $e^{-4i\phi}a$ and commutes with P_u, P_c . The backward-sequence uniqueness therefore makes the graph equivariant.

On the graph, $a_s = (\sigma + i\tau)a + O(|a|^2)$. Hence

$$\frac{d}{ds}|a| = \sigma|a| + O(|a|^2) > 0$$

for $0 < |a| \leq \delta$, with a fixed sufficiently small δ . The cap $\{|a| = \delta, b = g(a)\}$ is transverse and all its points are a single rotation family. There are no additional material relabeling parameters on this graph; those neutral modes were in the complementary space and are uniquely slaved by the backward condition. Fourfold rotations are the expected isotropy. Any chosen cap point has one ancient backward orbit. Rotating that orbit gives every phase of the cap and its ancient history.

For a nonzero such orbit, $a(s) \neq 0$ at every finite backward time, by uniqueness of the reduced two-dimensional equation. Since $|a(s)| \leq Ce^{\kappa s}$, the function $(F(a(s), g(a(s))) - (\sigma + i\tau)a(s))/a(s)$ is integrable on $(-\infty, 0]$. Integrating the scalar complex logarithmic derivative yields a nonzero number a_* and

$$a(s) = a_* e^{(\sigma + i\tau)s} + O(e^{(\sigma + \kappa)s}), \quad s \rightarrow -\infty.$$

Nonvanishing follows directly from the exponential representation with a finite integral, rather than from merely taking a possibly zero limit.

The whole fourth moment of the physical signed annulus is exactly

$$B(X) = \int \bar{z}^4 \{c_1 \mathbf{1}_{D_1(X)} + c_2 \mathbf{1}_{D_2(X)}\} dA(z) \tag{17}$$

$$= 2\pi[-\rho^6 Q_1 + \rho^2 Q_2] = 2\pi(-\rho^6 - i\rho^3)a + O(|a|^2 + \|b - g(a)\|). \tag{18}$$

The coefficient $2\pi(-\rho^6 - i\rho^3)$ is nonzero. On the chosen ancient orbit this proves

$$B_*(s) = b_* e^{(\sigma + i\tau)s} + O(e^{(\sigma + \kappa)s}), \quad b_* = 2\pi(-\rho^6 - i\rho^3)a_* \neq 0.$$

The whole moment is essential: no positivity statement about a single interface, a radial strip, or a selected part of the pulse has been used.

8 Normalization and uniformity

For an amplitude $A > 0$ and radius $R > 0$, put $x = Rz$, $s = At$ and divide the material embedding by R . The planar velocity scales as $u(x) = ARV(z)$. Thus all Banach norms, fixed chart radii, multiplier bounds, contraction constants, cap radii, and subperiod bounds in this module depend only on ρ, γ and the fixed unit-scale contour neighborhood. They are independent of A, R . The physical growth rate is $A\sigma$, physical period is P/A , and the whole physical fourth moment is AR^6B . A known ambient solid rotation can be removed by a rotation of the spatial frame; rotational covariance and uniform rotation bounds preserve all these norm constants. The isolated module contains no truncation-depth parameter, and its constants can therefore be reused at every level of every finite stack. Uniform estimates for the genuinely nonconstant fields of other packets require the interaction and forced-comparison sections below; they do not follow from this isolated statement.

9 Material colours, exact moments, and interaction estimates

Identify the square torus with $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and use $u = \nabla^\perp \psi$, $\Delta \psi = \omega$, where $\nabla^\perp = (-\partial_2, \partial_1)$. Fix

$$\rho^2 = 3/5, \quad W = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}, \quad A_j = A_1 \beta^{j-1}, \quad R_j = q^j, \quad 4 \leq \beta \leq 5.$$

An overall common amplitude factor does not change any estimate below. Constants are independent of the number N of colours and of sufficiently small q . Unless specified otherwise, their other dependencies are the fixed exponent $0 < \gamma < 1$ and fixed contour neighbourhoods.

For a finite transported stack let

$$w_j(t) = A_j[-\mathbf{1}_{D_{j,1}(t)} + \rho^2 \mathbf{1}_{D_{j,2}(t)}]. \quad (19)$$

The two domains are transported by the same area-preserving flow as all other colours. Initially they have areas $|D_{j,1}| = \pi \rho^2 R_j^2$, $|D_{j,2}| = \pi R_j^2$, they are invariant under $z \mapsto iz$, and $D_{j,1} \subset D_{j,2}$. We also require the initial cross-colour inclusions $D_{j+1,2} \subset D_{j,1}$ whenever they are used. A common flow preserves these inclusions. The finite Euler solution is quarter-turn invariant by uniqueness, so its flow commutes with the quarter turn and fixes the origin. Thus every colour retains its symmetry, both areas, zero circulation, and

$$\int w_j = 0, \quad \|w_j\|_\infty = \frac{3}{5} A_j, \quad \|w_j\|_1 = \frac{12\pi}{25} A_j R_j^2. \quad (20)$$

The last equality uses $D_{j,1} \subset D_{j,2}$; its upper bound alone would suffice. We will use the material packet $\mathcal{P}_k = \bigcup_{l \geq k} \overline{D_{l,2}}$ and its radius, so that an accidental cancellation in the signed sum cannot invalidate a support or separation assertion.

For a real, integrable vorticity w supported in a coordinate disk put

$$M_n(w) = \int \zeta^n w(\zeta) dA(\zeta), \quad B_n(w) = \int \bar{\zeta}^n w(\zeta) dA(\zeta) = \overline{M_n(w)}.$$

Quarter-turn invariance gives $M_n = i^n M_n$, and therefore

$$M_n = B_n = 0 \quad (n \not\equiv 0 \pmod{4}). \quad (21)$$

Circulation gives $M_0 = 0$ separately. The moments with degree a positive multiple of four are generally *not* conserved by transport. For a positively oriented boundary X the exact Stokes identities are

$$\int_D \bar{z}^n dA = \frac{1}{2i(n+1)} \int_{S^1} \bar{X}^{n+1} X' d\theta, \quad \int_D z^n dA = -\frac{1}{2i(n+1)} \int_{S^1} X^{n+1} \bar{X}' d\theta. \quad (22)$$

They follow by differentiating $\bar{z}^{n+1}dz$ and $z^{n+1}d\bar{z}$. These formulas show directly that the moments are bounded polynomial functionals on the full material $E = c^{1,\gamma}(S^1; \mathbb{R}^2)^2$, independent of relabelling.

Write a local colour's boundaries as $R_l X_{l,a}$, where $X_{l,a} = r_a e^{i\theta} + h_{l,a}$, $r_1 = \rho$, $r_2 = 1$, and $x_l = \|h_l\|_E$. Choose once and for all a sufficiently small neighbourhood with uniformly bounded E norms, chord-arc constants, support in B_{bR_l} , and $B_{aR_l} \subset D_{l,1}$, for fixed $0 < a < \rho < b$. Put $\delta w_l = w_l - A_l W(\cdot/R_l)$. The boundary is within $CR_l x_l$ of its reference circle, so its symmetric difference with the reference disk has area at most $CR_l^2 x_l$. For $n \geq 1$ this gives

$$|M_n(w_l)| = |M_n(\delta w_l)| \leq CA_l R_l^{n+2} b^n x_l. \quad (23)$$

Here all positive holomorphic moments of each reference disk vanish. The same bound follows from (22). The domains and δw_l have the stated symmetry; δw_l vanishes on B_{aR_l} and outside B_{bR_l} .

9.1 The two planar multipole expansions

The complex velocity $U = u_1 + iu_2$ is exactly

$$U(z) = \frac{i}{2\pi} \int \frac{w(\zeta)}{\bar{z} - \bar{\zeta}} dA(\zeta). \quad (24)$$

It is $2i\partial_{\bar{z}}\psi$ for $\psi = (2\pi)^{-1} \int \log|z - \zeta| w(\zeta) dA$. If the source is supported in B_r and $|z| > r$, the convergent geometric series in (24) gives

$$U(z) = \frac{i}{2\pi} \sum_{n \geq 0} \frac{B_n(w)}{\bar{z}^{n+1}} = \frac{i}{2\pi} \sum_{m \geq 1} \frac{B_{4m}(w)}{\bar{z}^{4m+1}}. \quad (25)$$

The second equality requires both zero circulation and quarter-turn symmetry. This also checks all complex conjugations in the formulas. For a local inner source $l > i$, assume $bR_l \leq \eta aR_i$ with one fixed $\eta < 1$, and that both receiving contours stay in the chosen local neighbourhood. Normalizing the velocity by R_i , the leading moment is of size $A_l R_l^6 x_l / R_i^6$. Termwise differentiation of (25) through two spatial derivatives, followed by a uniform geometric-series estimate, gives

$$\sup_{aR_i \leq |z| \leq bR_i} |D_z^m U_l(z)| \leq C_m A_l R_l^6 x_l R_i^{-5-m}, \quad m = 0, 1, 2, 3.$$

Consequently

$$\left\| R_i^{-1} U_l(R_i X_i) \right\|_E \leq CA_l (R_l/R_i)^6 x_l = CA_l q^{6(l-i)} x_l. \quad (26)$$

More precisely, the first term in the normalized velocity is $iB_4(w_l)/(2\pi R_i^6 \bar{X}_i^5)$; the E norm of the remaining terms is at most $CA_l (R_l/R_i)^{10} x_l$.

For $|z| < aR_l$ the reference radial colour is exactly $U_l^0(z) = -iA_l z/5$. The perturbation has the nonsingular expansion

$$\begin{aligned} U_l(z) + iA_l z/5 &= -\frac{i}{2\pi} \sum_{n \geq 0} \bar{z}^n \int \frac{\delta w_l(\zeta)}{\bar{\zeta}^{n+1}} dA \\ &= -\frac{i}{2\pi} \sum_{m \geq 1} \bar{z}^{4m-1} I_{4m,l}, \quad I_{4m,l} = \int \frac{\delta w_l(\zeta)}{\bar{\zeta}^{4m}} dA. \end{aligned} \quad (27)$$

Symmetry kills every index except $n+1 \equiv 0 \pmod{4}$. The symmetric-difference estimate gives $|I_{4m,l}| \leq CA_l R_l^{2-4m} a^{-4m} x_l$. For a target core $i > l$, require $bR_i \leq \eta aR_l$. Evaluating (27) and differentiating its series yields

$$\sup_{|z| \leq bR_i} |D_z^m [U_l(z) + iA_l z/5]| \leq C_m A_l x_l R_i^{3-m} / R_l^2, \quad m = 0, 1, 2, 3,$$

and hence

$$\left\| R_i^{-1} [U_l(R_i X_i) + i A_l R_i X_i / 5] \right\|_E \leq C A_l (R_i / R_l)^2 x_l = C A_l q^{2(i-l)} x_l. \quad (28)$$

The subtracted field is the *actual exact* solid rotation in the constant-vorticity plateau, not a linearized rotation. Quarter-turn symmetry removes harmonic constant, linear, and quadratic velocities; the first remaining harmonic velocity is cubic.

For completeness, these are full material estimates. If a smooth spatial field F on a fixed neighbourhood of the two normalized target curves has $\sup_{m \leq 2} \|D^m F\| \leq M$, the chain rule gives

$$\|F \circ X\|_{C^{1,\gamma}} \leq CM \quad (29)$$

uniformly for their bounded E norms. Indeed $(F \circ X)' = DF(X)X'$, and its Hölder seminorm is bounded by $\|DF\|_\infty [X']_\gamma + \|D^2 F\|_\infty \|X'\|_\infty^{1+\gamma} (2\pi)^{1-\gamma}$. The estimates include both vector components and all tangential material velocities. The resulting composition is in the little space: approximate X in that space by smooth curves and use the same bound. The displayed third spatial derivative also bounds the full normalized foreign field in $C^{2,\gamma}$ on a fixed collar of the target curves. All these series can be differentiated to any prescribed fixed order, with a corresponding constant depending on that order; separation is maintained on the slightly larger collar used for the derivative bound. Separated cross-contour integral formulas (or the uniformly convergent moment series above) also show analytic dependence on local source and target embeddings. No derivative of a radial-graph parametrization is used. The numerical gains are asserted on the actual area constraints and quarter-turn symmetry, not for arbitrary ambient area variations. A perturbation of the target alone cannot create a missing source term: a circular complete colour has exactly zero exterior field and exactly the stated interior solid rotation.

At a receiving circle $z = r e^{i\theta}$, the first exterior term gives

$$\frac{u_r}{r} = \Re \left(\frac{i B_4(w)}{2\pi r^6} e^{4i\theta} \right), \quad (u_r/r)_4 = \frac{i B_4(w)}{4\pi r^6}.$$

The last expression is the positive complex Fourier coefficient, not the cosine amplitude. With receiver radii $(\rho R_i, R_i)$ and the specified left dual $\ell = (1/2, i/(2\rho))$, its projection is

$$\ell f_4 = D_4 \frac{B_4(w)}{R_i^6}, \quad D_4 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0. \quad (30)$$

In receiver time $\theta = A_i t$ divide the right side by A_i . If $a(X) = \ell Q(X)$ is the geometric moment coordinate with $Q_a = (2\pi r_a^6)^{-1} \int_{D_a(X)} \bar{z}^4 dA$, then (22) gives $Da(X^0)$ equal to that normal left dual and $\|Da(X) - Da(X^0)\| \leq C x_i$. Thus an actual perturbed recipient has leading scalar forcing (30) plus a planar remainder bounded by

$$C A_l (R_l / R_i)^6 x_l x_i + C A_l (R_l / R_i)^{10} x_l \quad (31)$$

for an inner local source $l > i$. The first term includes both evaluation on the perturbed material curve and the changed moment derivative. For a packet of mass μ and radius r , its analog is $C \mu r^4 R_i^{-6} x_i + C \mu r^8 R_i^{-10}$. Image terms are bounded separately by (36) or (37). The leading coefficient is exact on the reference circles; it is not asserted to be exact on a finite perturbed contour.

9.2 An arbitrary spent packet

Let w_P be any real zero-circulation quarter-turn invariant source, with $\|w_P\|_1 \leq \mu$ and support in B_r . No boundary regularity is required. Then $|B_{4m}(w_P)| \leq \mu r^{4m}$. If $r \leq \eta a R_i$, the same differentiated series proves

$$\left\| R_i^{-1} U_P(R_i X_i) \right\|_E \leq C \frac{\mu r^4}{R_i^6}, \quad |B_4(w_P)| \leq \mu r^4. \quad (32)$$

These bounds replace (26) once the inner packet has left every shape neighbourhood. The separation condition is essential; a conserved mass bound without a support/separation bound does not control the field at a receiving interface.

9.3 Square-torus regular kernel and the exact image cancellation

Let G be the mean-zero Green function satisfying $\Delta G = \delta_0 - 1$ on the unit square torus. Its logarithmic singularity has coefficient $(2\pi)^{-1}$. On a fixed disk B_{r_0} containing no nonzero lattice point write $G(z) = (2\pi)^{-1} \log |z| + H(z)$. Then $\Delta H = -1$, so $H + |z|^2/4$ is harmonic and real analytic. Uniqueness of G gives $G(iz) = G(z)$ and hence

$$H(z) = c - \frac{1}{4}|z|^2 + \Re \sum_{n \geq 1} c_{4n} z^{4n}. \quad (33)$$

To verify the allowed powers, represent the harmonic part as the real part of a holomorphic power series. Quarter-turn invariance kills every nonconstant power not divisible by four. Convergence and all derivative bounds hold on every smaller fixed disk. This derivation requires no formula for the numerical coefficients c_{4n} .

Convolve (33) with a *complete* constrained colour, or a sum of complete colours, w . All differences $z - \zeta$ must remain in that fixed disk. The quadratic polynomial becomes

$$\int |z - \zeta|^2 w(\zeta) dA = |z|^2 M_0 - 2\Re(z \overline{M_1}) + \int |\zeta|^2 w(\zeta) dA.$$

It is constant in z because $M_0 = M_1 = 0$. For the harmonic part,

$$\int (z - \zeta)^{4n} w(\zeta) dA = \sum_{m=0}^n \binom{4n}{4m} z^{4n-4m} M_{4m}. \quad (34)$$

The signs are positive since $4m$ is even. The $m = 0$ term vanishes by circulation and the $m = n$ term is constant in z . Thus the degree-four term is only M_4 , whereas the degree-eight term is $70z^4 M_4 + M_8$. For a potential $\Re P(z)$ its complex velocity is $i\overline{P'(z)}$. Consequently the first *possible* nonzero image velocity is

$$U^{\text{im}}(z) = 280i\overline{c_8 M_4(w)} \bar{z}^3 + \text{terms of higher total degree}. \quad (35)$$

In particular the degree-four potential contribution produces no image velocity. The conjugation in (35) agrees with the exterior moment $B_4 = \overline{M_4}$ in (25).

Choose source and target radii small enough that their sum lies in a strictly smaller Green disk. Differentiating the normally convergent series (34), and using (23), gives for local source l and local target i , including $i = l$,

$$\sup_{|z| \leq bR_i} |D_z^m U_l^{\text{im}}(z)| \leq C_m A_l R_l^6 x_l R_i^{3-m}, \quad m = 0, 1, 2, 3,$$

and therefore

$$\left\| R_i^{-1} U_l^{\text{im}}(R_i X_i) \right\|_E \leq C A_l R_l^6 R_i^2 x_l. \quad (36)$$

Every nonconstant surviving term has source degree at least four and potential target degree at least four. The higher powers are summed uniformly on the smaller disk; (29) then applies. For an arbitrary packet of mass μ and radius r the corresponding bound is

$$\left\| R_i^{-1} U_P^{\text{im}}(R_i X_i) \right\|_E \leq C \mu r^4 R_i^2. \quad (37)$$

Thus the ratios to the planar estimates are respectively

$$\begin{array}{l|l} \text{inner local source } l > i & R_i^8 \\ \text{outer local source } l < i & R_l^8 \\ \text{local self interaction } l = i & R_i^8 \\ \text{arbitrary separated inner packet} & R_i^8. \end{array}$$

Since $R_j \leq q$, each gain is at most q^8 . These statements apply to the two disks *only after their signed strengths are combined*. A single disk has nonzero circulation and does not satisfy these cancellations. For a complete circular radial colour all positive holomorphic moments vanish as well, so its entire image velocity in this coordinate neighbourhood is exactly zero.

10 Uniform confinement of an aggregate material packet

The torus kernel satisfies $|K_{\mathbb{T}}(z)| \leq C/|z| + C$ in coordinate distance. Splitting its convolution at distance h gives

$$\|K_{\mathbb{T}}w\|_{\infty} \leq C\|w\|_{\infty}h + C\|w\|_1/h + C\|w\|_1.$$

Choose h proportional to $\min\{1, \sqrt{\|w\|_1/\|w\|_{\infty}}\}$, with the zero case understood by continuity. Hence

$$\|K_{\mathbb{T}}w\|_{\infty} \leq C\sqrt{\|w\|_1\|w\|_{\infty}} + C\|w\|_1, \quad \|K_{\mathbb{T}}w_l\|_{\infty} \leq CA_lR_l. \quad (38)$$

For $\beta q < 1/2$ and $\beta q^2 < 1/2$ this yields

$$\sum_{l \geq k} \|K_{\mathbb{T}}w_l\|_{\infty} \leq \frac{CA_kR_k}{1 - \beta q}, \quad \left\| \sum_{l \geq k} w_l \right\|_1 \leq \frac{CA_kR_k^2}{1 - \beta q^2}. \quad (39)$$

Neither constant depends on N or the geometry of the spent colours.

Suppose all colours $l < k$ stay in the fixed local neighbourhood, and $\mathcal{P}_k$ lies in $B_{\eta a R_l}$ for every $l < k$. This is an explicit central-plateau assumption, to be improved by the radius bootstrap. For all such l , (27) and (36) give on that ball

$$U_l(z) = -iA_l z/5 + O(A_l x_l |z|^3/R_l^2).$$

The image correction is included; its cubic coefficient is smaller by R_l^8 . The first term is perpendicular to the radius exactly. Let $s = A_k(t - t_0)$ and $d(s) = R_k^{-1} \sup_{z \in \mathcal{P}_k(t)} |z|$. Taking the upper right Dini derivative of the supremum over material particles gives

$$\begin{aligned} D^+d(s) &\leq \frac{C}{1 - \beta q} + Cd(s)^3 \sum_{l < k} \frac{A_l}{A_k} \left(\frac{R_k}{R_l} \right)^2 x_l \\ &\leq C + C\beta^{-1}q^2d(s)^3 \leq C + Cq^2d(s)^3. \end{aligned} \quad (40)$$

In the second line x_l is bounded by the fixed neighbourhood radius and the geometric row is $\sum_{m \geq 1} (q^2/\beta)^m$. For finite N , the material particle paths are continuously differentiable in time; the radius is locally Lipschitz, and the usual maximum-of-paths argument proves this Dini inequality. It is valid also when the maximizing particle changes.

Fix $H \leq C_{\beta}L$, $L = \log(1/q) \geq 1$, and $d(0) \leq d_0$ with d_0 fixed. Choose $D \geq 4(d_0 + C)$ first. Stop if $d(s) = D(1 + s)$ or a central separation fails. On the stopped interval (40) implies

$$d(s) \leq d_0 + Cs + \frac{Cq^2D^3}{4}[(1 + s)^4 - 1].$$

For small q the ratio of this bound to $1 + s$ is at most $D/2$, uniformly up to H , since $q^2(1 + H)^3 \rightarrow 0$. Moreover require $qD(1 + H) < \eta a/2$; then $D(1 + s)R_k < \eta a R_{k-1}/2$, and hence the same strict

inclusion holds in every larger local plateau. This simultaneously improves both stopping margins. Consequently

$$d(s) \leq C(1+s), \quad \left| B_4 \left(\sum_{l \geq k} w_l \right) \right| \leq C A_k R_k^6 (1+s)^4 \quad (0 \leq s \leq H). \quad (41)$$

If a different part of the induction has its own earlier exit, these are conclusions on that stopped interval; this radius argument excludes radius and central-separation exits independently of it. All material supports must also stay in the common coordinate disk; $qD(1+H) \ll 1$ provides the required margin.

For any fixed $c > 0$, (41) gives the explicit tail estimate

$$\int_S^H e^{-cs} \frac{|B_4(\sum_{l \geq k} w_l)|}{A_k R_k^6} ds \leq C_c e^{-cS} (1+S)^4. \quad (42)$$

At a following cap the packet's radius relative to the next larger scale is $O(q(1+L))$. The assertion that this is also the deeper packet's radius at an already completed source cap is induction data: it follows by applying the preceding estimate at the completed stage, not by assuming the next recipient cap exists.

11 A causal deeper-packet bound on a compact source window

Here is the support lemma needed when the current source ceases to be close to a circle. Fix a compact regular planar material trajectory $X_*(s)$, $0 \leq s \leq S$, with the reference two areas and quarter-turn symmetry. Require uniformly separated $c^{1,\gamma}$ interfaces, bounded contour norms and chord-arc constants, support in $B_{b_S R_k}$ after scaling, and an origin plateau $B_{h_S R_k} \subset D_{k,1}^*(s)$ with $h_S > 0$. The last requirement follows, for a regular flow that fixes the origin, from an initially interior origin and compactness in time; it does not follow merely from a conserved area. Take a fixed small material E tube of this trajectory such that every actual source in the tube has the same uniform geometric bounds and contains $B_{h_S R_k/2}$.

In the source's origin plateau its vorticity is exactly $-2A_k/5$. Even without a small shape norm, its planar velocity has the form

$$U_k(z) = -iA_k z/5 + \mathcal{H}_k(z), \quad |\mathcal{H}_k(z)| \leq C_S A_k |z|^3 / R_k^2 \quad (|z| \leq h_S R_k/8). \quad (43)$$

One elementary proof is to subtract the circular patch $(-2A_k/5)\mathbf{1}_{B_{h_S R_k/4}}$. The difference vanishes near zero and has mass at most $C_S A_k R_k^2$. Its inverse-moment series as in (27) is convergent on $B_{h_S R_k/8}$; quarter-turn symmetry makes the first term cubic and bounds its coefficient by $C_S A_k / R_k^2$. The subtracted patch produces the displayed solid rotation. Complete-colour torus images are cubic as well by (34), and their coefficient obeys the same bound after increasing C_S .

Let the actual completed cap at $s = 0$ have full material distance $\eta_0(q) \rightarrow 0$ from a rotated canonical cap and let the deeper material packet initially obey

$$e(0) := \text{rad}(\mathcal{P}_{k+1}(0))/R_k \leq C_d q(1+L).$$

The larger colours are assumed to remain in their fixed local neighbourhoods on the receiving stopped interval. Remove their exact solid rotations, which has no effect on any radius or any absolute moment bound. The joint stopping conditions are: leaving the source tube, reaching the source's origin-plateau margin, reaching the deeper-packet barrier below, or any independently specified larger contour exit. On this joint stopped interval, while $e \leq h_S/8$, (38) and (43) imply in source time

$$D^+ e(s) \leq \frac{C\beta q}{1-\beta q} + C_S e(s)^3 \leq C\beta q + C_S e(s)^3. \quad (44)$$

The larger local colours contribute only another bounded cubic coefficient, with row

$$\sum_{m \geq 1} \beta^{-m} q^{2m} x_{k-m}.$$

The source's linear rotation has zero radial component. The entire deeper packet's own velocity is bounded by $CA_{k+1}R_{k+1}/(1 - \beta q)$, independently of its shape or N .

Choose $M \geq 4(C_d + C)$ and impose the barrier $e(s) \leq Mq(1 + L + \beta s)$. On $0 \leq s \leq S$, integration yields

$$e(s) \leq C_d q(1 + L) + C\beta q s + C_S M^3 q^3 (1 + L + \beta S)^3 S \leq \frac{1}{2} M q (1 + L + \beta s)$$

for sufficiently small q . This improves the barrier strictly, and the improved bound lies strictly inside $h_S/8$. The tube's origin plateau therefore contains the deeper packet with a fixed separation margin; this fact was proved on the joint stop rather than assumed from future source shadowing.

The deeper mass is at most $CA_k R_k^2 \beta q^2 / (1 - \beta q^2)$. Applying (32) on the separated actual source interfaces therefore bounds its full normalized material forcing, in source time, by

$$\frac{C_S \beta q^2}{1 - \beta q^2} e(s)^4 \leq C_S q^6 (1 + L)^4. \quad (45)$$

The compact source's own image forcing is $C_S R_k^8 \leq C_S q^8$ in that time scale, using its conserved mass, bounded radius, and complete-colour cancellation. For the local larger colours the nonradial forcing is bounded by

$$J_k(s) = C_S \sum_{l < k} \frac{A_l}{A_k} (R_k/R_l)^2 x_l(s); \quad (46)$$

their image row is smaller by at least q^8 . In particular, if the adjacent receiver has $x_{k-1} = O(q^6(1 + L)^6)$ and the remaining prefix gives a no-larger geometric row, then $J_k = O(q^8(1 + L)^6)$.

The local contour-flow comparison theorem on this fixed regular tube gives, by bounded-time Grönwall,

$$\sup_{0 \leq s \leq S} \|X_k(s) - X_*^{\text{rot}}(s)\|_E \leq C_S \left[\eta_0(q) + \int_0^S J_k(s) ds + q^6(1 + L)^4 + q^8 \right]. \quad (47)$$

This is the only contour regularity input here: the autonomous material contour vector field is locally Lipschitz on the specified compact chord-arc tube, including both signed self-interface terms, and forced solutions satisfy the corresponding mild comparison. That theorem must be established in the contour section; smooth kernel estimates alone would not establish its self-interface part. If the bracket tends to zero, the estimate strictly improves the tube and its origin-hole margins. Together with the independent radius improvement this rules out every source-tube, deeper-radius, or source-hole exit before S . Any independent larger-contour exit is excluded by its own induction estimate. No future receiving cap or long-time shadowing of the source is invoked. By (22), the source moment is Lipschitz on this compact tube and satisfies the same error bound. The negative half of a two-sided compact window, when required, comes from a previously completed source stage and bounded-time backward comparison from the already known cap; it is distinct from this forward support argument.

Hypotheses that cannot be dropped. The q^6 outward gain needs complete-colour zero circulation and quarter-turn symmetry, plus separated supports. The q^2 inward gain needs an actual constant-vorticity origin plateau, with its solid rotation removed. The q^8 image gain needs the square torus and complete constrained colours; neither an unconstrained individual disk nor a mere signed mass bound supplies it. The compact source-window argument needs an independently regular source trajectory with an origin plateau and an initial $O(q(1 + L))$ deeper-packet radius. Within those hypotheses the interaction and support estimates above are uniform in N .

12 The finite-depth nonlinear cascade

Let $E = c^{1,\gamma}(S^1; \mathbb{R}^2)^2$ be the real little Hölder space of two material contours with fourfold symmetry, including their tangential components. Put

$$r_1 = \rho, \quad r_2 = 1, \quad \rho^2 = 3/5, \quad \sigma = \rho^5/2 > 0, \quad \tau = 3/10, \quad X_i^0(\theta) = r_i e^{i\theta}.$$

Every domain has its reference area, and a colour has signed vorticity $-\mathbf{1}_{D_1} + \rho^2 \mathbf{1}_{D_2}$ in normalized units. Its circulation is zero. The constraints and symmetry are preserved by the common transport flow.

The following hypotheses summarize the facts established above. Their application to the actual finite torus stacks is verified in Proposition 21.1; their constants are independent of the finite number of colours and of q .

(H1) In a fixed reference neighbourhood the stationary material gauge has the mild equation

$$h_t = Ah + N(h) + f(t), \quad G(t) = e^{tA}, \quad A = -\text{diag}(-1/5, 0)\partial_\theta.$$

G is a strongly continuous isometric group on E ; $N(0) = 0$ and N is C^2 (analytic is more than sufficient), with bounded derivatives on a slightly larger neighbourhood. The bounded perturbation $A + DN(0)$ generates a group $S(t)$. A commuting projection P_c has complementary group $S_c(t)$ satisfying

$$\|S_c(t)\| \leq C(1+t), \quad \|S(t)\| \leq M_\eta e^{(\sigma+\eta)t}, \quad t \geq 0.$$

The unstable real plane is identified with $\mathbb{C}$ and has exponent $\lambda_0 = \sigma + i\tau$; its growing relative-normal fourth-mode vector is $(1, -i\rho)$. The complementary projection is the spectral complement of that plane. All these bounds concern the full E space.

(H2) The relabeling-invariant moment coordinate is

$$Q_i(X) = \frac{1}{2\pi r_i^6} \int_{D_i(X)} \bar{z}^4 dA(z), \quad a(X) = \ell Q(X), \quad \ell = (1/2, i/(2\rho)).$$

It is C^2 ; its derivative at X^0 is the unstable left dual, and $(a, b) = (a(X^0 + h), P_c h)$ is a C^2 local chart with inverse $H(a, b)$. There is a C^2 invariant unstable graph $b = g(a)$, $g(0) = Dg(0) = 0$, with $\|g(a)\| \leq C_g |a|^2$. The graph is invariant for every isolated bounded-time flow, not only for its period map. The reference-fixing rotation

$$(\mathcal{R}_\phi X)_i(\theta) = e^{i\phi} X_i(\theta - \phi)$$

is an isometry, commutes with the isolated equation, and obeys $a(\mathcal{R}_\phi X) = e^{-4i\phi} a(X)$, $P_c \mathcal{R}_\phi = \mathcal{R}_\phi P_c$, and $g(e^{-4i\phi} a) = \mathcal{R}_\phi g(a)$. The isolated differentiated moment has the exact local form

$$\dot{a} = \lambda_0 a + \mathcal{N}(a, b), \quad |\mathcal{N}(a, b)| \leq C_a(|a|^2 + |a| \|b\| + \|b\|^2).$$

Here and below $\|b\|^2$ means $(\|b\|)^2$.

(H3) A nonconstant ancient orbit on that graph converges to X^0 as $t \rightarrow -\infty$. Its physical colour continues for every finite positive time, with regular separated chord-arc interfaces. On every fixed compact segment of this orbit the material contour operator is locally Lipschitz in E on a fixed tube. The colour preserves its mass and L^∞ bound, so that its normalized support radius is at most $C(1+t)$ for $t \geq 0$. This last support bound may equivalently be supplied by the transport velocity estimate in (H5).

- (H4) For local source and receiver contours, after removal of the exact constant solid rotations, the physical external fields satisfy the full contour norm estimates

$$l > i : CA_l q^{6(l-i)} x_l, \quad l < i : CA_l q^{2(i-l)} x_l, \quad x_l = \|h_l\|_E.$$

The constrained torus image fields gain a factor at most q^8 ; the local self-image term is at most $CA_i q^8 x_i$. For a fourfold zero-circulation source interior to circular receiving interfaces, with physical fourth moment $B = \int \bar{x}^4 w(x) dx$, the unstable projection of its degree-four velocity is exactly

$$D_4 B / R_i^6, \quad D_4 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0.$$

The same coefficient evaluated on a perturbed receiver differs by at most C times its full leading field norm times x_i . The exterior remainder starts at degree eight: its normalized field is bounded by the source mass times its support radius to the eighth power, divided by the receiving radius to the tenth power. Thus a source at scale qR_i and with radius $Cq(1+L)R_i$ produces a remainder at most $CA_{i+1} q^{10}(1+L)^8$. These last two assertions, including their E norm, are part of the exact multipole input, rather than a consequence of the q^6 upper bound alone.

- (H5) Every transported colour has $\|w_l\|_1 \leq CA_l R_l^2$, $\|w_l\|_\infty \leq CA_l$, and velocity at most $CA_l R_l$. For an aggregate packet $l \geq k$, use the material envelope

$$\mathcal{P}_k(t) = \bigcup_{l \geq k} \overline{D_{l,2}(t)}, \quad \mathcal{P}_{>k}(t) = \bigcup_{l > k} \overline{D_{l,2}(t)}.$$

These are unions of physical outer domains, not the support of a signed sum, which could be smaller by cancellation. Put $d(s) = \text{rad}(\mathcal{P}_k(t))/R_k$, $s = A_k(t - T_k)$. While the larger colours are local and the packet is in their central constant-vorticity regions,

$$D^+ d(s) \leq C_v + C_r q^2 d(s)^3.$$

Its initial radius is at most a fixed d_0 . If $d(s) \leq D_0(1+s)$, its fourth moment is at most $CA_k R_k^6 (1+s)^4$ and its full field on receiver $j = k-1$, divided by A_j , is at most $C_p q^6 (1+s)^4$. The same weighted bound on the inactive system is at most $C_p q^6 (1+s)^4 / D$. Here C_p depends on the fixed D_0 , not on the depth. The moment, exterior-remainder, and image bounds also apply to arbitrary spent packets. No boundary norm of a spent packet is asserted.

- (H6) For each finite stack a common bounded-vorticity transport solution exists. The initially nested domains retain their inclusions and fourfold symmetry. Each currently followed contour is an E mild solution whenever its self interfaces remain in the local reference chart or the compact canonical tube and the other colours are separated from that contour. Local continuation holds under a strict tube and separation margin. Larger local colours contain a fixed central disk $B_{r_{pl} R_i}$. In that disk, after the constant rotation is subtracted, their harmonic velocities are cubic with bounded coefficients. The same cubic assertion with a constant C_S applies inside the central hole of a compact regular canonical source tube.

The strength C^2 in (H2) cannot be replaced here by continuity and $g(a) = O(a^2)$. Those two facts alone do not prove the flattened map estimate used below. The contour comparison assumption on compact trajectories in (H3) likewise cannot be inferred from analyticity only near circles.

Set

$$A_i = A_1 \beta^{i-1}, \quad R_i = q^i, \quad 4 \leq \beta \leq 5, \quad L = \log(1/q) \geq 1, \quad \varepsilon = q^6.$$

The deepest colour starts on one fixed ancient orbit at normalized time $-S_0$; every other colour starts circular. Physical time starts at $t = 0$. Its initialization is not an $O(\varepsilon)$ perturbation.

Theorem 12.1. *Under inputs (H1)–(H6), one can fix a good $\beta \in [4, 5]$, constants D, D_0, K , a cap $\delta > 0$, a preselected window S , then $S_0 \geq S + 1$, and finally a threshold $q_* > 0$, all independently of the finite depth N , so that for every $0 < q < q_*$ and every finite depth N the deepest source reaches its first cap, and thereafter every receiver $j = N - 1, \dots, 1$ reaches its first cap $|a_j| = \delta$. If T_k is the completed incoming cap and $\theta = A_j(t - T_k)$, the recipient's first q^3 crossing lies in $[3L/\sigma - C, 3L/\sigma + C]$, and its cap occurs by $7L/\sigma + C < 9L/\sigma$. Throughout that stage its weighted inactive prefix is at most $K\delta/2$ and its packet radius is at most $D_0(1 + \beta\theta)/2$. At its cap the flattened error satisfies*

$$\|b_j - g(a_j)\| \leq Cq^{6-8\eta/\sigma}(1+L)^6 = o(1), \quad 0 < \eta < \sigma/32.$$

For each colour used as an incoming source, $k \geq 2$, the preselected source window $[-S, S]$ is available in actual physical time, with its positive half in the next recipient's first phase. On this window the normalized aggregate moment converges, as $q \downarrow 0$, to that of a rotated canonical source, uniformly in k, N . The window S is chosen before the base initialization time S_0 ; no values before physical time zero are asserted. The actual signed recipient seed has size at least $c_s\epsilon$ after the fixed incoming window and before the first q^3 crossing. All constants are uniform in j, N .

The proof is organized so that the signed estimate is used only after all non-amplitude exits in the recipient's first phase are excluded. The source's preceding cap is already completed by descending induction.

13 A causal bounded-time forced map lemma

Choose nested local neighbourhoods $U_0 \Subset U_1$ so every isolated orbit from U_0 exists in U_1 for $|r| \leq P$, where $P = 10\pi$. This is possible by the mild local existence theorem and bounded-time Grönwall around the stationary state. All chart derivatives and Lipschitz constants below are bounded on U_1 .

Lemma 13.1. *Let h, u solve respectively the forced and unforced mild equations on $[t, t+r]$, with $0 \leq r \leq P$, in U_1 . For strongly measurable $f \in L^1([t, t+r]; E)$,*

$$\|h(t+r) - u(t+r)\| \leq e^{K_0 P} \left(\|h(t) - u(t)\| + \int_t^{t+r} \|f(v)\| dv \right).$$

The same assertion holds backwards on an existing interval. If Φ_r is the isolated endpoint map and χ is either the raw chart (a, b) or the flattened chart $(a, y) = (a, b - g(a))$, then

$$\chi(h(t+r)) = (\chi\Phi_r\chi^{-1})(\chi(h(t))) + \xi_{t,r}, \quad \|\xi_{t,r}\| \leq C \int_t^{t+r} \|f\| dv.$$

These estimates involve only forcing on the interval actually completed.

Proof. Subtract the two mild equations. Since G is an isometry and N has Lipschitz constant K_0 on U_1 ,

$$\|h(t+v) - u(t+v)\| \leq \|h(t) - u(t)\| + \int_0^v \{K_0 \|h(t+w) - u(t+w)\| + \|f(t+w)\|\} dw.$$

Integral Grönwall gives the displayed bound. Reversing time uses the same isometric group $G(-v)$. Applying the Lipschitz chart only to the endpoints proves the second assertion. It includes the change of $g(a)$ caused by the force; no derivative of the unbounded label generator through that graph is used. $\square$

Differentiating the mild fixed point with respect to its initial state once and twice gives uniformly bounded derivatives of Φ_r for $|r| \leq P$: the variational equations have bounded coefficients $DN(h)$ and $D^2N(h)$ and the isometric group, so Grönwall bounds both variations. Thus Φ_r is uniformly C^2 . At the origin its derivative is $S(r)$. Since P_c commutes with this group, the complementary derivative has no unstable-to-complementary term.

Lemma 13.2. *For every $0 \leq r \leq P$, uniformly on the chosen local chart,*

$$\begin{aligned} b(t+r) &= S_c(r)b(t) + R_{b,r}(a(t), b(t)) + \xi_{b,t,r}, \\ \|R_{b,r}(a, b)\| &\leq C(|a|^2 + |a| \|b\| + \|b\|^2), \\ y(t+r) &= S_c(r)y(t) + R_{y,r}(a(t), y(t)) + \xi_{y,t,r}, \\ \|R_{y,r}(a, y)\| &\leq C(|a| + \|y\|) \|y\|, \quad \|\xi_{b,t,r}\| + \|\xi_{y,t,r}\| \leq C \int_t^{t+r} \|f\| \, dv. \end{aligned}$$

Moreover $\|y(t+r)\| \leq C \|y(t)\| + C \int_t^{t+r} \|f\|$.

Proof. The raw remainder is the second-order Taylor remainder of the C^2 endpoint map in the raw chart. The flattened endpoint fixes $y = 0$ because the isolated graph is continuously invariant. If its complementary component is $\Psi_r(a, y)$, then $\Psi_r(a, 0) = 0$ and $D_y \Psi_r(0, 0) = S_c(r)$. Integrating $D_y \Psi_r$ along the segment from 0 to y and using the bounded second derivatives proves the flattened remainder. Apply Lemma 13.1 for the forcing error. For the last assertion, compare with the isolated graph trajectory starting at $H(a(t), g(a(t)))$. Its initial distance from the actual state is at most $C \|y(t)\|$; it remains on the graph, and the flattened chart is Lipschitz. $\square$

At complete periods $M_c = S_c(P)$ satisfies $\|M_c^n\| \leq C(1+n)$. Iterating the raw or flattened recurrence is therefore an ordinary discrete Duhamel estimate with a genuine $L_t^1 E$ forcing error. Lemma 13.2 handles the last incomplete period and supplies causal interpolation without a spurious linear $|a|$ contribution to b .

The differentiated moment is legitimate for mild material solutions. Undo the label translations. In the physical material equation, the embedding is absolutely continuous in E on compact intervals and its velocity is $V(X) + f$. Differentiate the C^1 moment functional there and restore the labels. Relabeling invariance gives

$$\dot{a} = Da(X)[V(X) + f] \quad \text{almost everywhere.}$$

This does not assert that AX is an E -valued vector. Spatial rotating-frame terms are computed by $a \mapsto e^{-4i\phi} a$. The same proof of Lemma 13.1, on finitely many local Lipschitz neighbourhoods, compares a forced trajectory to any fixed compact regular canonical trajectory, with constants depending on that trajectory and window.

14 Frames and the canonical bilateral seed

The larger-colour reference rotation for receiver j is $\Omega_j^{\text{ext}} = -(1/5) \sum_{i < j} A_i$. For $k = j+1$, $\Omega_k^{\text{ext}} = \Omega_j^{\text{ext}} - A_j/5$. Fix the additive angle of both frames at the actual completed cap T_k . For physical position x , their normalized coordinates are

$$y_j^{\text{nat}} = e^{-i\Omega_j^{\text{ext}}(t-T_k)} x / R_j, \quad y_j^{\text{in}} = e^{-i\Omega_k^{\text{ext}}(t-T_k)} x / R_j.$$

Thus $y_j^{\text{in}} = e^{i\theta/5} y_j^{\text{nat}}$. Reference-fixing rotation includes the common compensating label shift, so these spatial rotations act on local displacement norms through the isometric action $\mathcal{R}$. The moment

is independent of that compensating relabeling. For physical time $t \geq 0$, define the dimensional aggregate moment in these native axes by

$$B_{\geq k}^{\text{nat}}(t) = e^{4i\Omega_k^{\text{ext}}(t-T_k)} \int \bar{x}^4 \sum_{l \geq k} w_l(t, x) dA(x). \quad (48)$$

On the whole actual interval $s = A_k(t - T_k) \geq -A_k T_k$, its normalized source-time form is

$$\mathcal{B}_{\geq k}(s) = \frac{B_{\geq k}^{\text{nat}}(T_k + s/A_k)}{A_k R_k^6}. \quad (49)$$

Thus $B_{\geq k}^{\text{nat}}$ has physical time as argument, whereas $\mathcal{B}_{\geq k}$ has source time as argument and already includes one division by $A_k R_k^6$. All source-time integrals below use $\mathcal{B}_{\geq k}$ directly. This one continuously rotating native frame is used also on the negative history. Different phases of individual deeper colours are retained in this exact sum, rather than assigned a canonical phase. Passing from receiver native axes to source native axes adds $\mathcal{R}_{\theta/5}$ in receiver time θ and changes its moment exponent to

$$\lambda = \sigma + i\tau - 4i/5 = \sigma - i/2.$$

The complementary norm is unchanged. Because $P/5 = 2\pi$, this additional action is the identity over a receiver period; the same period map estimates apply. For a partial period it is an isometry. More precisely the incoming-frame subperiod derivative is $S_c^{\text{in}}(r) = \mathcal{R}_{r/5} S_c(r)$, and its raw and flattened maps are the maps of Lemma 13.2 followed by $\mathcal{R}_{r/5}$. Equivariance of g makes their remainders and force-error norms unchanged. Thus one need not incorrectly identify the two subperiod maps themselves. In source time $s = \beta\theta$ the exponent is $z_\beta = \lambda/\beta$. The force is divided by β under this time change, so its integrated norm is preserved. The torus image is included in the forcing, rather than treated as rotation equivariant.

Fix a small canonical cap $|a| = \delta$, with cap time zero. On the graph, $\dot{a} = \lambda_0 a + O(|a|^2)$. For sufficiently small reference caps,

$$\sigma_0 |a| \leq \frac{d}{dt} |a| \leq (\sigma + C\delta) |a|, \quad \sigma_0 = 3\sigma/4,$$

throughout its local ancient part. In particular $|a(t)| \leq \delta e^{\sigma_0 t}$ for $t \leq 0$. It never vanishes. The scalar coefficient $\dot{a}/a - \lambda_0 = O(|a|)$ is integrable backwards; integrating it shows

$$a(t) = a_- e^{\lambda_0 t} (1 + O(e^{\sigma_0 t})), \quad a_- \neq 0.$$

The nonzero constant follows explicitly by exponentiating the convergent integral from any fixed nonzero $a(t_1)$.

The physical fourth moment is

$$B_*(t) = 2\pi[-\rho^6 Q_1(t) + \rho^2 Q_2(t)] = b_- e^{\lambda_0 t} + O(e^{(\sigma+\sigma_0)t}),$$

where $b_- = 2\pi(-\rho^6 - i\rho^3)a_- \neq 0$ because the growing normal vector is $(1, -i\rho)$. This vector and its left dual are part of the full spectral input. On positive times, $|B_*(t)| \leq C(1+t)^4$ by mass and support.

Lemma 14.1. *The bilateral transform*

$$F(z) = \int_{\mathbb{R}} e^{-zt} B_*(t) dt, \quad 0 < \text{Re } z < \sigma,$$

is holomorphic and not identically zero. There is $\beta \in [4, 5]$ for which $F(z_\beta) \neq 0$. Shrinking the cap by ancient time translation preserves this nonvanishing, and $|F_\delta(z_\beta)| \asymp \delta^{1/\beta}$.

Proof. On every compact substrip the negative exponential and positive polynomial bounds provide an integrable majorant also after any fixed number of z derivatives. They prove holomorphy by differentiation under the integral. For $0 < c < \sigma$, $e^{-ct}B_*(t)$ is a nonzero $L^1(\mathbb{R})$ function. If F were identically zero, its Fourier transform on the line $c + i\mathbb{R}$ would vanish. Fourier uniqueness can be seen without an inversion hypothesis: convolve the function with Gaussians, whose Fourier inversion is valid and whose transforms never vanish; each convolution is zero, and the Gaussian approximate identity converges in L^1 . This contradicts the nonzero ancient asymptotic. The points z_β form a nonconstant line segment inside the strip. If all were zeros, the identity theorem would make F identically zero.

Let $t_\delta \rightarrow -\infty$ be the old orbit time at the smaller cap. The new orbit is $B_\delta(t) = B_*(t + t_\delta)$, so a change of variable gives $F_\delta(z) = e^{zt_\delta}F(z)$. The ancient asymptotic gives $\delta \asymp e^{\sigma t_\delta}$, proving the magnitude assertion. $\square$

Fix such a β and, after the final cap has been chosen, define

$$c_0 = |D_4 F(z_\beta)| > 0.$$

The whole bilateral integral is used. $B_*(t)$ need not have any pointwise sign or phase condition for positive time.

15 The depth-independent inactive estimate

In a stage with receiver j and source $k = j + 1$, put

$$\theta = A_j(t - T_k), \quad s = \beta\theta, \quad H = 9L/\sigma, \quad w_i = (D\varepsilon)^{j-i}, \quad I_j(\theta) = \max_{i < j} x_i(\theta)/w_i.$$

For $j = 1$, $I_j = 0$. Fix $D > \beta$ and eventually $D\varepsilon < 1$. At the completed incoming cap the induction data are

$$I_j(0) \leq DK\delta\varepsilon, \quad x_j(0) \leq DK\delta\varepsilon, \quad d(0) \leq d_0.$$

All preceding stage clocks and pointwise prefixes are also part of the induction data; norms at time zero alone do not determine the signed seed.

Lemma 15.1. *Choose $0 < \eta < \sigma/32$. One can choose D , a fixed $\alpha_* < \sigma/2$, and then sufficiently small q , so that on a stopped interval with $I_j \leq 2K\delta$, all inactive colours local, and the packet radius and separation barriers,*

$$I_j(\theta) \leq C_w e^{\alpha\theta} I_j(0) + C_w \int_0^\theta e^{\alpha(\theta-v)} [x_j(v) + \varepsilon(1+v)^4] dv, \quad 0 < \nu \leq \alpha \leq \alpha_*.$$

Here $\nu = (\sigma + \eta)/\beta$. The current normalized feedback and its discounted integral satisfy

$$\begin{aligned} \|F_{\text{out}}(\theta)\| &\leq CDq^8 I_j(\theta), \\ q^8 \int_0^\theta e^{-\sigma v} I_j(v) dv &\leq Cq^8 [I_j(0) + \varepsilon] + Cq^8 \int_0^\theta e^{-\sigma v} x_j(v) dv. \end{aligned}$$

All constants are independent of the number of inactive rows.

Proof. For an inactive receiver $i < j$ and inner inactive source $l = i + d < j$, its coefficient in receiver time θ , after weights, is

$$\frac{A_l}{A_j} q^{6d} \frac{w_l}{w_i} = \frac{A_i}{A_j} (\beta/D)^d.$$

For an outer source $l = i - d$ it is

$$\frac{A_l}{A_j} q^{2d} \frac{w_l}{w_i} = \frac{A_i}{A_j} (Dq^8/\beta)^d.$$

The weighted current-source coefficient is $D^{-(j-i)} \leq 1/D$. Since $A_i/A_j \leq 1/\beta$ and $w_i \leq D\varepsilon$, the isolated quadratic remainder divided by w_i is at most $2CDK\delta\varepsilon I_j/\beta$. Add image rows and the packet input. The maximum of the finite mild inequalities therefore obeys

$$\begin{aligned} I_j(\theta) &\leq M_\eta e^{\nu\theta} I_j(0) \\ &\quad + M_\eta \int_0^\theta e^{\nu(\theta-v)} \{r(D, q) I_j(v) + C[x_j(v) + \varepsilon(1+v)^4]/D\} dv, \\ r(D, q) &\leq \frac{C}{\beta} \left[\frac{\beta}{D-\beta} + \frac{Dq^8}{\beta-Dq^8} + q^8 + 2DK\delta\varepsilon \right]. \end{aligned}$$

This takes a maximum of inequalities; it does not sum their number. Multiply by $e^{-\nu\theta}$ and apply scalar integral Grönwall. The result has $\alpha = \nu + M_\eta r(D, q)$. Choose D sufficiently large to make $\nu + M_\eta r(D, 0) < \alpha_* < \sigma/2$, then choose q with $Dq^8 < \beta/2$ and $\alpha \leq \alpha_*$.

Feedback from $i = j - d$ has coefficient $(A_i/A_j)q^{2d}w_i = (Dq^8/\beta)^d$. Summing this geometric row proves its bound. Insert the Volterra estimate into its discounted integral. Tonelli applied to nonnegative majorants gives, for $v \leq \theta$,

$$\int_v^\theta e^{-\sigma u} e^{\alpha(u-v)} du \leq e^{-\sigma v} / (\sigma - \alpha_*).$$

The discounted polynomial is integrable, proving the final estimate. $\square$

Lemma 15.2. *Suppose at a stopped current time $\theta \leq H$, for every $v \leq \theta$,*

$$x_j(v) \leq C_x |a(\theta)| e^{-\sigma_0(\theta-v)} + C_n \varepsilon (1+L)^{12}, \quad \sigma_0 = 3\sigma/4.$$

Then

$$I_j(\theta) \leq C_{\text{pr}} |a(\theta)| + C_{\text{rem}} \varepsilon (1+L)^{12} e^{\alpha\theta}.$$

C_{pr} depends only on the fixed $C_x, C_w, \sigma_0, \alpha_*$, not on K, C_n or later window constants. If $K \geq 4C_{\text{pr}}$ and $|a| \leq \delta$, sufficiently small q improves the prefix to $I_j \leq K\delta/2$.

Proof. Insert the displayed bound into Lemma 15.1. The above-noise integral is bounded by $C_w C_x |a(\theta)| / (\sigma_0 - \alpha_*)$. The initial term and the noise term have the claimed form. The packet term has this form because $\alpha \geq \nu > 0$ and $\int_0^\infty e^{-\nu v} (1+v)^4 dv < \infty$. Finally

$$\varepsilon (1+L)^{12} e^{\alpha H} \leq q^{6-9\alpha_*/\sigma} (1+L)^{12} \rightarrow 0,$$

since $\alpha_* < \sigma/2$. Choose q so its fixed multiple is at most $K\delta/4$. The amplitude term is at most the other $K\delta/4$. $\square$

16 Before q^3 : raw centre, upper amplitude, and actual errors

Choose $D_0 \geq 4(d_0 + C_v)$ before choosing K . On a packet stop $d(s) \leq D_0(1+s)$ and $s \leq \beta H$, the Dini inequality integrates to

$$d(s) \leq d_0 + C_v s + \frac{C_r q^2 D_0^3}{4} [(1+s)^4 - 1].$$

For small q , division by $1+s$ improves its bound to $D_0/2$, because $q^2(1+\beta H)^3 \rightarrow 0$. Choosing also $qD_0(1+\beta H) < r_{\text{pl}}/2$ improves separation. This argument is independent of a signed seed.

While $I_j \leq 2K\delta$, x_j is local, and these packet barriers hold, the full force in receiver time is

$$\|F_j(\theta)\| \leq C_p \varepsilon (1+\theta)^4 + CDq^8 I_j(\theta) + Cq^8 x_j(\theta) \leq F_* \varepsilon (1+L)^4. \quad (50)$$

F_* is fixed after the displayed constants, because $q^8/\varepsilon = q^2$.

Stop the first phase at the first $|a| = q^3$, $\|b\| = 2B\varepsilon(1+L)^6$, $I_j = 2K\delta$, packet or separation departure, local chart departure, or $\theta = H$. Before this first event $|a|^2 \leq \varepsilon$. Iterate Lemma 13.2 over at most CL complete periods and use $\sum_{m \leq CL}(1+m) \leq C(1+L)^2$. The last incomplete period is handled by the same lemma. It gives

$$\|b\| \leq C_b\varepsilon(1+L)^6 + C_bq^3(1+L)^2(2B\varepsilon(1+L)^6) + C_b(1+L)^2(2B\varepsilon(1+L)^6)^2.$$

The first term includes the inherited $O(\varepsilon)$ centre, the a^2 remainder, and the integrated force. Choose $B \geq 4C_b$; then small q makes the other two terms at most $B\varepsilon(1+L)^6/2$. Consequently

$$\|b(\theta)\| \leq B\varepsilon(1+L)^6. \quad (51)$$

This proves the centre bound on the current stopped interval without a source sign or a future recipient cap.

In incoming axes the exact moment equation is

$$a_\theta = \lambda a + \mathcal{N}(a, b) + F_a(\theta), \quad \lambda = \sigma - i/2.$$

The current circular leading packet force is bounded by a discounted polynomial. Moving the receiver contours introduces at most $C\varepsilon(1+\theta)^4(|a| + \|b\|)$, plus the smaller multipoles. Use the weighted feedback estimate with $x_j \leq C_{\text{ch}}(|a| + \|b\|)$ and put $U(\theta) = e^{-\sigma\theta}|a(\theta)|$. The centre square is an additive term at most $C\varepsilon^2(1+L)^{12}$, and

$$q^8 \int_0^\theta e^{-\sigma v} I_j(v) dv \leq Cq^8\varepsilon(1+L)^6 + Cq^8 \int_0^\theta U(v) dv.$$

Scalar Duhamel therefore gives

$$U(\theta) \leq C_u\varepsilon + C[q^3 + B\varepsilon(1+L)^6 + q^8 + \varepsilon(1+L)^4] \int_0^\theta U(v) dv.$$

The coefficient times H tends to zero. Grönwall proves

$$|a(\theta)| \leq C_u\varepsilon e^{\sigma\theta}. \quad (52)$$

In particular the first q^3 crossing cannot occur before $3L/\sigma - C$. Fixed outgoing source windows will thus be available once non-amplitude recipient exits are excluded.

Choose $a_* = C_*\varepsilon(1+L)^{12}$ with fixed $C_* \geq 1$. When $a_* \leq |a| \leq q^3$, equations (50) and (51) give

$$\frac{|\mathcal{N}|}{|a|} \leq C[q^3 + B\varepsilon(1+L)^6 + B^2\varepsilon/C_*], \quad \frac{|F_a|}{|a|} \leq (F_*/C_*)(1+L)^{-8}.$$

They tend to zero. Thus, for small q ,

$$\frac{d}{d\theta}|a| \geq \sigma_0|a|, \quad a_* \leq |a| \leq q^3. \quad (53)$$

The crossing of a_* is one-way: a first downward return would contradict this strict lower derivative near that level. If the current amplitude is below a_* , no earlier amplitude exceeded it. If it is above, every preceding above-noise amplitude is at most its current value times $e^{-\sigma_0(\theta-v)}$. Below noise the contour norm is $C_n\varepsilon(1+L)^{12}$; above noise $\|b\|/|a| \leq (B/C_*)(1+L)^{-6}$, so $x_j \leq C_x|a|$ for a fixed C_x chosen before B . This proves the hypothesis of Lemma 15.2 at every current time.

Choose $K \geq 4C_{\text{pr}}$. That lemma strictly improves the prefix; the packet bound already improves; and $x_j \leq C_{\text{ch}}(q^3 + B\varepsilon(1+L)^6) < r_*/2$ for small q , while $x_i \leq K\delta(D\varepsilon)^{j-i}/2 < r_*/2$ for inactive colours. All non-amplitude first-phase exits are excluded. This is the point at which a fixed outgoing source window may be invoked.

For later use, keep the complex interaction-picture coefficient. Before the first q^3 crossing, current recipient remainders satisfy the following explicit discounted bounds:

$$\begin{aligned}
\int_0^\theta e^{-\sigma v} |a|^2 dv &\leq C\varepsilon q^3(1+L), \\
\int_0^\theta e^{-\sigma v} (|a| \|b\| + \|b\|^2) dv &\leq C\varepsilon^2(1+L)^{13}, \\
\int_0^\theta e^{-\sigma v} \|F_{\text{out}}\| dv &\leq C\varepsilon q^8(1+L)^7, \\
\int_0^\theta e^{-\sigma v} [\varepsilon(1+v)^4 x_j(v)] dv &\leq C[\varepsilon q^3 + \varepsilon^2](1+L)^{12}, \\
\int_0^\theta e^{-\sigma v} \|F_{\text{image}}\| dv &\leq C\varepsilon q^8(1+L)^{12}.
\end{aligned} \tag{54}$$

For the first line use $|a| \leq q^3$ and (52), rather than integrating the exponentially growing upper bound squared up to an unspecified time. For the second use (51) and (52); the centre square has a bounded discounted integral. For feedback use Lemma 15.1. The remaining lines follow from the moving-contour and image inputs. Each is $o(\varepsilon)$; their sum is at most $C(q^9 + q^{12} + q^{14})(1+L)^{20}$.

17 The entire actual source pulse and its negative history

We now prove the signed seed; we do not impose it as induction data. Fix incoming $k = j + 1$ with its already completed cap T_k . Every earlier recipient stage $m \geq k$ has a completed clock at most $9L/(\sigma A_m)$, and the base initialization lasts $(S_0 + O(q^8))/A_N$. Thus

$$A_k T_k \leq \frac{9\beta}{\sigma(\beta-1)} L + C_{S_0}, \quad A_j T_k \leq \frac{9}{\sigma(\beta-1)} L + C_{S_0}.$$

Define

$$g_\beta = 9/(\beta-1) \leq 3.$$

On the actual past, the recipient discount is at most Cq^{-g_β} , and the past has length at most $C(1+L)$ in recipient units. This bound is uniform in depth. It is essential that the gain is $q^{-9/(\beta-1)}$, not an unbounded gain depending on N .

Throughout that past the recipient j was inactive. The completed prefix estimates give $x_j \leq CDK\delta\varepsilon$. Every colour $i < j$ has the stronger bound $x_i \leq CK\delta(D\varepsilon)^{k-i}$. Consequently the recipient's own nonlinear past contributes at most $Cq^{12-g_\beta}(1+L)$ to its discounted coefficient. Its outer feedback and self-image contribute at most $Cq^{14-g_\beta}(1+L)^{12}$: the nearest outer contour has size $O(\varepsilon^2)$ and its outer interaction has an extra q^2 . The past moving-receiver correction is at most $Cq^{12-g_\beta}(1+L)^{12}$ by $x_j = O(\varepsilon)$ and the leading packet force $O(\varepsilon(1+L)^4)$. These quantities are $o(\varepsilon)$. The past inner packet's torus image field is $O(\varepsilon q^8(1+L)^4)$ before discount, and hence contributes at most $Cq^{14-g_\beta}(1+L)^{12}$ after the actual-history gain. It is included in the same $o(\varepsilon)$ image remainder.

There is a depth-uniform bound on the support of the deeper packet $l > k$ before T_k : in source units it is at most $Cq(1+L)$. Indeed at a preceding stage with current index $m \geq k+1$, its local current source has radius at most CR_m and its aggregate packet has radius at most $C(1+L)R_{m+1}$. Since $R_m \leq R_{k+1}$, their union has radius at most $C(1+L)R_{k+1}$. The same statement holds at base initialization. After the cap of $k+1$ and during receipt by k , it is the confined packet. This covers the whole past interval, without adding one support constant per stage. Its normalized fourth moment is therefore at most $Cq^6(1+L)^4$, by its mass $CA_{k+1}R_{k+1}^2$.

For a non-base source $k < N$, the first-phase noise estimate and the post- q^3 estimate proved in the next section concatenate to give, relative to its completed cap at source time $s = 0$,

$$x_k(s) \leq C\delta e^{\sigma_0 s} + C\varepsilon(1+L)^{12}, \quad s \leq 0. \tag{55}$$

To avoid a logical cycle, that post- q^3 estimate is applied only to the *preceding completed source stage*, not to the current recipient. Here is the partition of its whole actual past. On the completed receiving stage for k , below-noise times have $x_k \leq C\varepsilon(1+L)^{12}$. Above-noise first-phase times and all second-phase times satisfy $x_k \leq C_x|a_k|$, and their concatenated radial rate is at least σ_0 . Integrating that rate backwards from the *completed* amplitude δ gives $|a_k(t)| \leq \delta e^{\sigma_0 A_k(t-T_k)}$ at every such time. Before that receiving stage, the actual history is the base interval and the finitely many completed stages with current index $m \geq k+1$. Their pointwise prefixes give $x_k(t) \leq K\delta(D\varepsilon)^{m-k} \leq DK\delta\varepsilon$; the base prefix gives the identical assertion with $m = N$. This covers every physical time $0 \leq t \leq T_k$ and proves (55) on the entire actual negative interval, including the times before k first began to receive a pulse.

The source's below-noise term and the deeper packet thus contribute at most

$$Cq^{6-g_\beta}(1+L)^{13} = o(1) \quad (56)$$

to the source transform before $s = 0$. Its above-noise negative tail before $-S$ contributes at most

$$C\delta \int_{-\infty}^{-S} e^{(\sigma_0 - \sigma/\beta)s} ds \leq C\delta e^{-(\sigma_0 - \sigma/\beta)S}.$$

This does not assume a cap yet to be reached: T_k is completed.

For the actual positive pulse, aggregate support gives, for every available stopped recipient time with $s \geq S$,

$$\int_S^{\beta\theta} e^{-\sigma s/\beta} |\mathcal{B}_{\geq k}(s)| ds \leq C \int_S^\infty e^{-\sigma s/\beta} (1+s)^4 ds \leq Ce^{-\sigma S/\beta} (1+S)^4. \quad (57)$$

The last constant is finite and uniform for the fixed β . It follows, for example, by substituting $s = S + u$ and expanding $(1+S+u)^4$. This is a bound for the actual full aggregate pulse; it does not compare it to the canonical trajectory at large times.

Lemma 17.1. *Fix the preselected window S used in the seed comparison, and then choose $S_0 \geq S+1$, before choosing q . For all sufficiently small q , uniformly in the depth and source index, the actual times $T_k + s/A_k$, $|s| \leq S$, are nonnegative; the positive half-window lies in the next recipient's first phase. For each completed non-base incoming source cap,*

$$\sup_{|s| \leq S} |\mathcal{B}_{\geq k}(s) - e^{i\phi_k} B_*(s)| \leq r(q), \quad r(q) \leq C_S q^{6-8\eta/\sigma} (1+L)^{20} \rightarrow 0.$$

For the base source $k = N$, the same assertion on this preselected window has error $C_{S,S_0,\delta} q^8$. The native axes and source-time normalization are exactly those of (48)–(49). In particular, the base assertion requires the window to be selected before S_0 ; it does not quantify over arbitrary backward windows once S_0 is fixed.

Proof for a non-base source. Its completed cap has flattened error at most $Cq^{6-8\eta/\sigma}(1+L)^6$. The inverse chart puts it that close in the full material norm to the graph point with the same a . $\text{SO}(2)$ equivariance identifies that graph point with a rotated canonical cap, because the group acts transitively on $|a| = \delta$. The upper radial bound in the completed source stage puts its fixed negative window after its q^3 crossing for small q . Its forcing on that window is $O(\varepsilon(1+L)^4)$. Backwards compact comparison from the completed cap proves the negative-window assertion.

For the positive half, (52) and (51) give $x_j \leq C\varepsilon(1+L)^6$ on $0 \leq s \leq S$, so the current receiver's nonradial field on source k is $O(q^8(1+L)^6)$; inactive rows are smaller. The regular torus image on a compact source tube is $O(q^8)$, using constrained mass and moments. A separate support stop is needed for the deeper packet since source k may leave its near-circle chart. Write

$$e(s) = \text{rad}(\mathcal{P}_{>k}(T_k + s/A_k))/R_k, \quad e(0) \leq C_d q(1+L).$$

The regular canonical trajectory on $[0, S]$ has a positive central hole and a compact chord-arc tube with the same property. To justify this geometric fact, its common fourfold transport homeomorphism fixes the origin, which started strictly inside both domains; it therefore keeps the origin strictly inside them. Every regular interface on this compact interval has positive origin distance. Joint continuity of the embeddings in time and label, and compactness of the interval and label circle, make the minimum distance positive. A sufficiently small E tube retains half that distance and positive chord-arc and cross-interface separation margins. Inside this hole the constant rotation is removed; the remaining field is cubic with coefficient C_S . Larger local colours satisfy the same cubic bound. Summed self velocities of the deeper colours give $C\beta q$. Hence, on that tube and before any coupled exit,

$$D^+e(s) \leq C\beta q + C_S e(s)^3.$$

Stop with $e(s) \leq 4C_d q(1 + L + \beta s)$, enlarging fixed C_d once to include the linear velocity. Its cubic increment is at most $C_S q^3(1 + L + \beta S)^3 S = o(q(1 + L))$. It improves the bound to $2C_d q(1 + L + \beta s)$, which is inside one quarter of the canonical hole for small q . Deep mass $O(A_k R_k^2 \beta q^2)$ now gives moment and full contour field $O(q^6(1 + L)^4)$ on the source.

Start a simultaneous stop at compact source-tube departure, central hole loss, the displayed deep-radius barrier, or $s = S$. The forcing just bounded, together with the completed cap error, tends to zero. Compact bounded-time comparison improves the source tube and hole margins strictly. The deep support improvement was independent of that comparison. Thus none of these stops can fail first. The moment is Lipschitz on the compact tube and the deep packet moment is $O(q^6(1 + L)^4)$, yielding the asserted error. The current recipient cannot reach q^3 before $3L/\sigma - C > S/\beta$, by (52); all its other first-phase exits were already excluded. $\square$

Choose S once the final cap and c_0 are fixed, so the actual and canonical positive tails and the above-noise negative tails are less than $c_0/(100(1 + |D_4|))$. For the base source choose in addition $S_0 \geq S + 1$ so its omitted canonical past before $-S_0$ has that same bound. The window lemma and (56) then give, for every available $\beta\theta \geq S$ before the recipient reaches q^3 ,

$$\left| D_4 \int_{-A_k T_k}^{\beta\theta} e^{-z\beta s} \mathcal{B}_{\geq k}(s) ds - e^{i\phi_k} D_4 F(z\beta) \right| \leq c_0/10 + C_S r(q) + C q^{6-g\beta} (1 + L)^{13}. \quad (58)$$

Here $\mathcal{B}_{\geq k}$ is the actual aggregate moment in the source native rotating axes, divided by $A_k R_k^6$. The proof partitions the actual integral into its actual past tail, its fixed window, and its actual future tail. It neither extends an actual trajectory to an unknown future cap nor replaces its whole trajectory by the canonical orbit.

The recipient's exact initial moment is zero at physical time zero, because it was initialized circular. Integrating its moment equation from that actual initial time, in incoming axes, gives

$$e^{-\lambda\theta} a(\theta) = \varepsilon D_4 \int_{-A_k T_k}^{\beta\theta} e^{-z\beta s} \mathcal{B}_{\geq k}(s) ds + \mathcal{E}_j(\theta). \quad (59)$$

The factor β in the receiver-time leading force $\beta \varepsilon D_4 \mathcal{B}_{\geq k}(\beta\theta)$ cancels in the change of integration variable. The inherited $a(0)$ at the source cap is included in this exact past integral. Merely assuming $a(0) = O(\varepsilon)$ with no history would not justify a lower bound; that free complex number could cancel the seed.

The higher exterior harmonics contribute at most $C q^{10-g\beta} (1 + L)^{13}$ on the actual negative history and $C q^{10} (1 + L)^8$ on the positive side, by the exact degree-eight remainder input. Their fourth-moment projection on imperfect contours is bounded by the full field, so this estimate is safely crude. Combine these with the recipient's past remainder estimates and (54). Uniformly before the

first q^3 crossing,

$$\begin{aligned} |\mathcal{E}_j(\theta)| &\leq C\{q^{12-g_\beta} + q^{14-g_\beta}\}(1+L)^{13} \\ &\quad + Cq^{10-g_\beta}(1+L)^{13} + C(q^9 + q^{12} + q^{14})(1+L)^{20} \\ &\leq C(q^7 + q^9 + q^{12} + q^{14})(1+L)^{20} = o(\varepsilon). \end{aligned}$$

In the last line use $g_\beta \leq 3$. The additional source-past noise error in (58), after multiplication by ε , is $O(q^{12-g_\beta}(1+L)^{13})$ and belongs to this same bound.

Equations (58)–(59) therefore give, for small q and all such current times,

$$|e^{-\lambda\theta}a(\theta)| \geq (c_0/2)\varepsilon =: c_s\varepsilon, \quad \theta \geq \Theta_w = S/\beta. \quad (60)$$

The phase $e^{i\phi_k}$ is arbitrary and harmless, because it is a single phase multiplying the whole nonzero bilateral transform.

18 The post- q^3 reset and the ordered first-exit argument

The first-phase strict improvements and (60) already imply existence of a first q^3 crossing $\theta = \tau$: if it had not occurred by $[3L + \log(2/c_s)]/\sigma < H$, the signed inequality would give $|a| \geq 2q^3$, a contradiction. Thus

$$3L/\sigma - C \leq \tau \leq 3L/\sigma + C.$$

At that crossing $\|y(\tau)\| \leq (B + C_g)\varepsilon(1+L)^6$. This establishes the first-phase endpoint before the second phase is introduced.

Put $u = \theta - \tau$ and $\underline{\sigma} = 2\sigma/3$. In phase two retain the prefix, packet, separation and local barriers. Add the stops

$$|a| = \delta, \quad \|y\| = 2Y\varepsilon(1+L)^6 e^{2\eta u}, \quad |a| < q^3 e^{\underline{\sigma}u}, \quad \theta = H.$$

The growth stop means a first *downward* violation for $u > 0$; equality at the restart does not stop the phase. We prove a strict radial improvement at that restart and thereafter.

Choose q with $2Y\varepsilon(1+L)^6 e^{2\eta H} < \delta$. On the stopped phase, $|a|, \|y\| \leq \delta$. At periods counted from τ , the flattened recurrence has remainder at most $2C_E\delta\|y\|$, and forcing error at most $PC_E F_*\varepsilon(1+L)^4$. Use $\|M_c^n\| \leq C_\eta e^{\eta nP}$, multiply the Duhamel sum by $e^{-\eta nP}$, and sum the forcing geometric series. This yields

$$e^{-\eta nP} \|y_n\| \leq C_\eta [\|y_0\| + \varepsilon(1+L)^4] + \kappa_\eta \delta \sum_{r < n} e^{-\eta rP} \|y_r\|.$$

Choose the cap earlier with $\kappa_\eta \delta \leq \eta P/2$. The elementary discrete Grönwall estimate follows by induction from $Z_n \leq A + k \sum_{r < n} Z_r$, which implies $Z_n \leq A(1+k)^n \leq Ae^{kn}$. Together with the uniform last-subperiod comparison it gives

$$\|y(u)\| \leq C_Y \varepsilon (1+L)^6 e^{3\eta u/2}.$$

Choose $Y \geq 2C_Y$, including the initial value and the interpolation constant. Then

$$\|y(u)\| \leq Y\varepsilon(1+L)^6 e^{2\eta u}, \quad (61)$$

a strict half-size improvement of the stated stop.

On the weaker growth barrier,

$$\frac{\|y\|}{|a|} \leq 2Yq^3(1+L)^6 e^{-(\underline{\sigma}-2\eta)u}, \quad \frac{|F_a|}{|a|} \leq F_*q^3(1+L)^4.$$

Since $b = g(a) + y$, the exact moment remainder obeys $|\mathcal{N}|/|a| \leq C\delta + C_Y q^3(1+L)^6$. Choose the cap with $C\delta < \sigma/8$, then small q so that the total remainder and force ratio is below $\sigma/4$. At every current time in this stopped phase,

$$\frac{d}{du}|a| \geq \sigma_0|a| > \underline{\sigma}|a|, \quad \frac{d}{du} \log |a| \leq \sigma + C_{\text{rad}}\delta + \zeta(q), \quad \zeta(q) \leq Cq^3(1+L)^6 \rightarrow 0. \quad (62)$$

These are almost-everywhere inequalities for an absolutely continuous amplitude. Integrating the first proves a nonempty initial growth interval and excludes every later downward growth exit.

Below noise in phase one the norm is $C_n \varepsilon(1+L)^{12}$; above noise it is $C_x|a|$. In phase two, graph quadraticity and the small y/a ratio give the same $C_x|a|$, after the cap was chosen small. The radial rates concatenate. Therefore at each stopped current time in phase two, every earlier above-noise amplitude is bounded by its current amplitude times $e^{-\sigma_0(\theta-v)}$, and the below-noise contribution has the stated uniform noise bound. Lemma 15.2 strictly improves the prefix again. The independent packet improvement persists. The local norms are strictly smaller than their local margin because

$$C_{\text{ch}}(\delta + C_g \delta^2) + C_{\text{ch}} 2Y \varepsilon(1+L)^6 e^{2\eta H} < r_*/2,$$

and inactive contours have the additional weight $D\varepsilon$. All non-amplitude second-phase exits are now excluded.

Integrating the lower radial inequality shows that the cap must occur within

$$u_{\text{cap}} \leq \sigma_0^{-1} \log(\delta/q^3) \leq 4L/\sigma + C$$

of τ . Hence it occurs by $7L/\sigma + C < H$, excluding the deadline. Only *after* existence of that cap has been established do we integrate the upper radial bound to obtain

$$u_{\text{cap}} \geq \frac{3L + \log \delta}{\sigma + C_{\text{rad}}\delta + \zeta(q)} \geq cL.$$

Thus every fixed backward cap window is contained in this completed second phase for small q . At the actual cap, (61) gives

$$\|y_{\text{cap}}\| \leq Cq^6(1+L)^6 e^{2\eta(4L/\sigma+C)} \leq Cq^{6-8\eta/\sigma}(1+L)^6 = o(1).$$

This is the weak reset needed for the next fixed-window comparison; no long-time canonical shadowing has been asserted.

19 The deepest fixed ancient source: a separate base

The deepest source starts at a fixed ancient time $-S_0$ on the canonical orbit. As $q \downarrow 0$ its nonzero initial amplitude is fixed, and can exceed q^3 . Neither the raw-centre bootstrap nor the above-noise first phase is applied to this source.

Choose a fixed small forward time $r_b > 0$, independent of the final cap and windows, on which the graph orbit remains local and its radial derivative is at least $\sigma_0|a|$. The graph and inverse chart yield a cap-independent constant C_{anc} such that

$$x_*(s) \leq C_{\text{anc}} \delta e^{\sigma_0 s}, \quad s \leq r_b,$$

with the fixed forward factor included in C_{anc} . This constant is distinct from the later raw-centre bootstrap constant. Use deepest time $s = A_N t - S_0$, and weights $w_i = (D\varepsilon)^{N-i}$ for $i < N$. All outer colours start circular, so $I_N(-S_0) = 0$. The same weighted row computation as before gives

$$I_N(s) \leq \frac{C}{D} \int_{-S_0}^s e^{\alpha_b(s-v)} x_N(v) dv, \quad 0 < \alpha_b < \sigma/2, \quad (63)$$

on its stopped local prefix, with constants independent of S_0, N . Feedback into the deepest source, including its self-image, is

$$\|f_N(s)\| \leq Cq^8[x_N(s) + DI_N(s)]. \quad (64)$$

Choose K before δ, S, S_0 , large enough for the recipient prefix bound and for four times $C(C_{\text{anc}} + 1)e^{\sigma_0 r_b}/[D(\sigma_0 - \sigma/2)]$. Stop the base at first source-local-tube departure, $I_N = 2K\delta$, first cap, or $s = r_b$. Its force is $O(q^8\delta)$. Compare from the identical initial state at $-S_0$ over the fixed interval using Lemma 13.1. It gives

$$\sup \|X_N(s) - X_*(s)\| \leq C_{S_0, \delta} \delta q^8.$$

The possibly large constant constrains q , not K . Choose q so this error is less than the fixed tube margin and than $\delta e^{-\sigma_0 S_0}$. Then the causal envelope $x_N(s) \leq (C_{\text{anc}} + 1)\delta e^{\sigma_0 s}$ holds on the stop. Insert it in (63); since $\sigma_0 - \alpha_b \geq \sigma_0 - \sigma/2 > 0$, it improves the prefix to $K\delta/2$. The source local tube also improves by comparison.

The canonical cap is transverse: its radial derivative at zero is at least $\sigma_0\delta$. The exact bounded moment functional and the comparison imply convergence of the actual moment and its moment derivative near zero, by (64). Choose a fixed short interval around zero where the canonical radial derivative is $\geq c\delta$. The actual derivative is $\geq c\delta/2$ for small q . On the earlier compact interval $[-S_0, -r]$, ancient monotonicity gives a positive fixed gap below the cap. Thus the intermediate value theorem and the derivative lower bound establish its unique first cap at

$$T_N = (S_0 + \zeta_N)/A_N, \quad |\zeta_N| \leq C_{S_0, \delta} q^8, \quad \|y_N(T_N)\| \leq C_{S_0, \delta} \delta q^8.$$

The source's first cap precedes $s = r_b$; no stopped exit failed first. The time estimate follows by dividing its $O(\delta q^8)$ amplitude error by the fixed $c\delta$ radial derivative.

We next prove its outgoing fixed source window directly. Recenter at the completed cap and choose the compact canonical tube on $[0, S + 1]$. Its source support is $C_S R_N$. Far-field mass and support bounds give a weighted source term C_S/D in the outer equations, even if the source is no longer close to a circle. The causal estimate is

$$I_N(T_N + s/A_N) \leq C e^{\alpha_b s} K\delta + \frac{C_S}{D} \int_0^s e^{\alpha_b(s-v)} dv.$$

Let M_S be twice a fixed upper bound for this expression on $[0, S + 1]$, allowing $\alpha_b \leq \sigma/2$. Stop at $I_N = 2M_S$, an outer local departure, source compact-tube departure, or $s = S + 1$. Choose q after M_S so that $D\epsilon M_S$ is in the outer chart and $C_S q < r_{\text{pl}}/2$. Weighted nonlinear rows are then small. The preceding bound improves the prefix to M_S , and source feedback is $\|f_N\| \leq C_S q^8(1 + DM_S)$. Compact comparison from the completed cap strictly improves its source tube. Thus the outgoing interval exists, independently of any later recipient cap, and its full source error is $O_{S, S_0, \delta}(q^8)$.

The actual recentered base history begins at $s_{\min} = -A_N T_N = -S_0 - \zeta_N$. Since $S_0 \geq S + 1$, choosing q so that $|\zeta_N| < 1/2$ gives $s_{\min} < -S$. Hence the entire negative half-window $[-S, 0]$ consists of actual physical times. Backwards compact comparison from the completed cap is used only on this available interval and gives the same error there. Moment Lipschitz continuity proves the base case of Lemma 17.1. For the whole initialized base history, the preceding forward comparison from $-S_0$, the outgoing comparison, and the $O(q^8)$ cap-time shift give a uniform $O_{S, S_0, \delta}(q^8)$ moment error. The endpoint shift also changes the canonical integral by $O_{S, S_0, \delta}(q^8)$, since its integrand is bounded on a fixed neighbourhood of $-S_0$. Consequently,

$$\int_{-S_0 - \zeta_N}^S e^{-z\beta s} \mathcal{B}_N(s) ds = e^{i\phi_N} \int_{-S_0}^S e^{-z\beta s} B_*(s) ds + O_{S, S_0, \delta}(q^8).$$

The omitted canonical past is controlled by the fixed choice of S_0 , and the seed comparison error is $O(q^6 q^8) = O(q^{14})$.

M_S is deliberately different from $K\delta$. No statement says the spent deepest source retains the universal cap prefix throughout its outgoing window. At its cap the stronger universal prefix holds;

through the outgoing window the reindexed next recipient still has norm $O(D\varepsilon M_S) = O(\varepsilon)$. One must not enlarge K afterwards to dominate M_S/δ : doing so would destroy the stated choice order and can create a cap-window circularity.

20 Reindexing, finite choices, and closure of the induction

At a completed ordinary cap T_j , the new weights for receiver $j - 1$ differ by exactly $D\varepsilon$. Hence

$$I_{j-1}(T_j) \leq D\varepsilon I_j(T_j) \leq DK\delta\varepsilon/2, \quad x_{j-1}(T_j) \leq DK\delta\varepsilon/2.$$

The already spent packet has radius $O(q(1+L)R_j)$. Adding newly capped colour j produces a packet of radius at most a fixed $d_0 R_j$, for sufficiently small q . The new recipient therefore has inherited norm $O(\varepsilon)$ with no growing power of L . The reset supplies a fresh fixed-window error; it does not multiply the previous source's window error by a stage-dependent constant. Its proof used only current mass, support, weighted prefix, and forcing bounds.

The descending induction now has an ordered logical structure:

1. The base comparison proves the deepest cap and its outgoing window, with universal cap prefix and separate outgoing constant.
2. The inherited local data permit the next recipient's stopped first phase. Raw centre, upper amplitude, current-time growth above noise, prefix improvement, packet and separation improvements exclude all non-amplitude exits independently of the signed seed.
3. The source cap is completed. Its past envelope, reset, and outgoing compact-tube comparison provide the actual whole pulse. The exact initialization at physical time zero gives the seed identity, and the entire discounted remainder is $o(\varepsilon)$. This proves the signed seed and forces the first q^3 crossing.
4. The flattened second-phase recurrence, with the weaker radial barrier, gives the stronger radial derivative, improved prefix and separation, and then the recipient cap. Only after that cap exists is its backward fixed window located and its new reset used.
5. Reindexing reproduces the next local data and packet radius.

At no point is the recipient's future cap an input to an earlier prefix or source comparison. This completes Theorem 12.1.

The first-exit statements use continuous quantities: the finite maximum I_j , the contour norm and moment, and the support radius under the continuous transport homeomorphism. The support radius is locally Lipschitz in time whenever the physical velocity is bounded; the supplied upper Dini inequality therefore integrates in the usual scalar comparison sense. At a putative endpoint of a strict tube and separation improvement, local contour continuation from (H6) extends the currently followed contours, while the globally existing bounded-vorticity transport continues the other colours. Thus an endpoint cannot be a loss of existence hidden behind the listed barriers. For each fixed N only finitely many descending stages occur. The uniform estimates concern their constants, rather than an unproved infinite-stage contour continuation theorem.

For clarity, the finite parameter order is as follows.

1. Fix cap-independent local analytic and chart constants, a small reference unstable orbit, and its bilateral transform. Select a good $\beta \in [4, 5]$ by Lemma 14.1.
2. Choose $0 < \eta < \sigma/32$, then D and $\alpha_* < \sigma/2$. Fix the packet constant D_0 from d_0, C_v . These precede K because packet forcing constants can depend on D_0 .

3. Fix C_x from the inverse chart and graph, uniformly for all sufficiently small caps. Choose K from the prefix lemma and the cap-independent ancient envelope needed for the base.
4. Shrink δ by translating the same ancient orbit so that its local chart, radial, flattened-map, and cap-tube inequalities hold. The transform remains nonzero. Fix $c_0 > 0$ for this final cap.
5. Choose S , then $S_0 \geq S + 1$, to bound actual and canonical tails relative to c_0 . Fix compact-tube and deep-radius constants. Choose M_S for the base outgoing window separately from $K\delta$. Fix B, C_*, Y and their finite forcing and remainder constants.
6. Choose q last to satisfy all strict inequalities.

There are finitely many fixed inequalities. Their vanishing factors are positive powers of q times fixed powers of $1 + L$:

$$q^2(1+L)^3, \quad q^3(1+L)^{12}, \quad q^{6-9\alpha_*/\sigma}(1+L)^{12}, \quad q^{6-18\eta/\sigma}(1+L)^6, \\ q^{6-8\eta/\sigma}(1+L)^{20}, \quad q^{6-g_\beta}(1+L)^{13}, \quad q^{10-g_\beta-6}(1+L)^{20}.$$

All exponents are positive because $\alpha_* < \sigma/2$, $\eta < \sigma/32$, and $g_\beta \leq 3$. Any additional fixed requirement $\beta q < 1$ or $\beta q^{2/p} < 1$ can be imposed at this final step. The bounded-time constants may grow with the fixed windows; they only constrain this one final choice of q . There is no selection of q depending on j or N .

21 Verifying the cascade inputs for the actual finite stacks

We record explicitly how the preceding analytic assertions apply to exact torus Euler solutions. Fix exponents $0 < \gamma < \gamma_0 < 1$. Construct the isolated graph first in c^{1,γ_0} and view it in $E = c^{1,\gamma}$, using the compatibility proved in the spectral section. Thus the canonical trajectory has the higher initial regularity needed in Proposition 3.5.

Proposition 21.1. *The inputs of Theorem 12.1 hold for the complete signed colours in the exact bounded torus Euler stacks, with constants independent of their finite depth.*

Proof. For the local flow input, Proposition 3.3 gives a bounded analytic material velocity on the full little contour space, and Proposition 3.4 gives its C^2 mild endpoint maps. Restricting to quarter-turn symmetry is legitimate because it is a closed invariant linear subspace. The spectral calculation and its bounded multipliers give the stated full-space projections and semigroup bounds, including all tangential and mean modes. The exact polynomial moment formula gives the chart and its full left dual; the backward-sequence construction gives the C^2 graph and continuous invariance. The differentiated moment calculation and the reference-fixing rotation action give the chart input.

For the orbit input, the higher exponent graph supplies a finite signed C^{1,γ_0} planar patch at its cap. Proposition 3.5 applies to this patch with $p = 1$ in that proposition's planar statement; signs and nesting were explicitly checked there. Its material parametrizations are a compact set in $c^{1,\gamma}$ on each finite time window. The local chord-arc contour estimate and Lemma 3.6 give a Lipschitz tube there. The velocity split (C.12) gives the polynomial support and moment bounds. Equivariance and the fixed origin preserve the positive central hole on each such compact window.

For the interaction and packet inputs, the expansions in the interaction section are real volume identities for complete area-constrained colours. They give the q^6 , q^2 , and q^8 full material norm gains, the exact coefficient D_4 , the higher exterior remainder, the arbitrary bounded-packet moment bounds, and the two independent support comparisons. Constrained colours have zero circulation because their two areas retain their reference values. A common flow preserves quarter-turn symmetry, so the first three complex moments vanish exactly. The regular torus field uses the convergent square-symmetric Green expansion. The degree-four potential contribution is constant

and hence produces no velocity; the resulting relative image gain is q^8 , with the first possible nonzero velocity arising from the degree-eight potential. The finite solutions use the exact torus kernel.

For the transport input, at physical time zero take the deepest colour to be the canonical signed domains at time $-S_0$, scaled by R_N and A_N . All other colours are circles. The local canonical perturbation can be chosen so small that each outer disk lies strictly inside the next larger inner disk. Small q makes these inclusions strict with a fixed normalized margin. The finite sum is bounded and has zero mean. Proposition 2.2 provides a global bounded Euler solution with a common measure-preserving flow. Transport its individual signed indicators by that flow. Thus their areas, coefficients, nesting, symmetry, L^1 , and L^∞ bounds hold without a contour continuation assumption for the spent colours.

Each stopped active contour can now be identified locally rather than by an unverified global torus patch theorem. The active pair's foreign vorticity is constant on a fixed collar of its current local or compact canonical contour tube: the smaller packet is strictly inside it, and each larger local colour has its exact constant plateau there. The packet and central-hole stops in the induction prove precisely these margins before comparison is invoked. Multipole differentiation through order three provides continuous $C^{2,\gamma}$ foreign fields on the collars, with the asserted scaled bounds. Lemma 3.7 therefore identifies the local contour mild solution with the actual active colour of the globally existing bounded solution. The same assertion holds simultaneously for the finite inactive prefix on any stopped interval: each of its collars has positive separation, and the finite system of locally Lipschitz contour equations can be solved together, with the actual spent packet field prescribed. Apply the prescribed-foreign-field difference-energy argument of Lemma 3.7 to the aggregate followed prefix. It identifies the new aggregate active vorticity with its actual transported counterpart, and then uniqueness of their equal total velocity's material flow identifies each followed domain. A strict local and separation improvement extends these equations past every putative non-amplitude exit.

Inside the plateau of a quarter-turn colour, its velocity is its exact constant solid rotation plus a harmonic cubic remainder. The interior series proves this statement with the appropriate full scaled coefficient. On a compact canonical source window the same assertion follows from its positive central hole and interior harmonic estimates. These are precisely the cubic assumptions used by the two radius comparisons. No norm of a spent boundary enters. All constants above are normalized constants, finite geometric row sums, or constants on fixed canonical time windows. None counts the number of colours. This verifies every input. $\square$

22 The stationary initial datum

After the finite parameter choice in the cascade, further decrease q , if necessary, to ensure $\beta q < 1$ and $\beta q^{2/p} < 1$. Set

$$\bar{\omega} = \sum_{j=1}^{\infty} A_j W(x/R_j), \quad \bar{\omega}_N = \sum_{j=1}^N A_j W(x/R_j). \quad (\text{L.1})$$

These are functions on the torus, using the central coordinate disk. The series converges absolutely in L^p , since

$$\sum_j \|A_j W(\cdot/R_j)\|_p = \|W\|_{L^p(\mathbb{R}^2)} \sum_j A_j R_j^{2/p} < \infty.$$

Every summand has zero integral and is quarter-turn invariant.

Here is an exact verification of stationarity on the torus. The planar radial streamfunction ψ_j of colour j has radial derivative

$$\psi'_j(r) = r^{-1} \int_0^r s A_j W(s/R_j) ds.$$

Zero circulation makes this derivative zero for $r \geq R_j$. Choose its additive constant so that $\psi_j = 0$ there. Then it is a compactly supported distributional solution of $\Delta\psi_j = A_j W(\cdot/R_j)$ on the plane and, by periodic extension, on the torus. Uniqueness of the zero-mean Poisson inverse up to an additive constant makes its velocity exactly $\nabla^\perp \psi_j$, including all torus effects. It is tangential, radial in its scalar angular speed, supported in B_{R_j} , and bounded by $CA_j R_j$. The latter bounds are summable because $\beta q < 1$. Therefore $\bar{u} = \mathcal{K}\bar{\omega}$ is the uniform sum of these tangential velocities.

Both $\bar{\omega}$ and $\beta(\bar{\omega})$ are radial on B_{R_1} , for every Lipschitz β . Outside that disk the velocity is zero. Polar integration against a smooth test function gives

$$\int_{\mathbb{T}^2} \beta(\bar{\omega}) \bar{u} \cdot \nabla \varphi = 0 :$$

at almost every radius the angular factor is a full-period derivative. The integral is justified by $\beta(\bar{\omega}) \in L^p$ and $\bar{u} \in L^\infty$; its constant value outside the disk causes no flux. The same calculation for $\beta(s) = s$ proves Euler. Thus $\omega^{\text{st}}(t) = \bar{\omega}$ is a renormalized stationary solution with the stated initial datum.

23 A separated limit of the finite exact solutions

Let ω_N be the exact bounded Euler solution whose initial colours are those of Proposition 21.1. The deepest normalized domains are a fixed canonical pair at $-S_0$; all other pairs are disks. Hence

$$\|\omega_N(0) - \bar{\omega}_N\|_p \leq C_{S_0, \delta} A_N R_N^{2/p} \rightarrow 0. \quad (\text{L.2})$$

Together with the convergent tail in (L.1), this proves $\omega_N(0) \rightarrow \bar{\omega}$ strongly in L^p . Minkowski's inequality and common transport give

$$\sup_{N, t} \|\omega_N(t)\|_p \leq C \sum_j A_j R_j^{2/p} < \infty. \quad (\text{L.3})$$

Apply the now verified Theorem 12.1. Let t_N be the outermost first cap time. The base time and the geometric sum of the stage deadlines give the depth-independent bound

$$0 \leq t_N \leq \frac{S_0 + C}{A_N} + \sum_{j=1}^{N-1} \frac{9L}{\sigma A_j} \leq S_0 + C + \frac{9\beta L}{\sigma(\beta - 1)} =: T_*. \quad (\text{L.4})$$

Continue every bounded solution to $T = T_* + 1$ by its global transport construction.

At t_N the outer colour has $|a_1| = \delta$ and flattened error $\|y_1\| \leq Cq^{6-8\eta/\sigma}(1+L)^6$. The whole-moment formula in the spectral section gives

$$|B_4(\omega_{N,1}(t_N))| \geq A_1 R_1^6 \left(c_B \delta - C\delta^2 - Cq^{6-8\eta/\sigma}(1+L)^6 \right), \quad c_B = 2\pi|\rho^6 + i\rho^3| > 0.$$

First the cap is chosen with $C\delta^2 \leq c_B \delta/4$, then the final q with the remaining error at most $c_B \delta/4$. The spent inner packet has mass at most $CA_2 R_2^2$ and support radius at most $Cq(1+L)R_1$. Consequently

$$|B_4(\omega_{N, \geq 2}(t_N))| \leq CA_1 R_1^6 \beta q^6 (1+L)^4.$$

Require that fixed quantity also to be at most $c_B A_1 R_1^6 \delta/4$. Then

$$|B_4(\omega_N(t_N))| \geq (c_B/4) A_1 R_1^6 \delta. \quad (\text{L.5})$$

All supports at this time lie inside B_{2R_1} , by the local outer chart and the packet bound. This is a genuine spatial moment on the chosen coordinate disk, not a nonperiodic torus test function.

Choose a smooth radial cutoff χ equal to one on B_2 and zero outside B_3 . Small $q = R_1$ ensures B_{3R_1} is a coordinate disk. The smooth periodic complex test function

$$\varphi(x) = \chi(x/R_1)(\overline{x_1 + ix_2}/R_1)^4$$

has norm $\|\varphi\|_{p'} \leq C_p R_1^{2/p'}$. The stationary datum has zero pairing with it by radial symmetry. Equation (L.5) and Hölder give the depth-independent separation

$$\|\omega_N(t_N) - \bar{\omega}\|_p \geq c_p A_1 R_1^{2/p} \delta =: d_* > 0. \quad (\text{L.6})$$

Here q , δ , and all other parameters are already fixed; d_* is strictly positive and does not tend to zero with N .

By Proposition 2.4, (L.2)–(L.3) yield a subsequence converging in $C([0, T]; L^p)$ to a renormalized Euler solution ω^{ns} with initial datum $\bar{\omega}$. After a further subsequence, $t_N \rightarrow t_* \in [0, T_*]$. Strong uniform convergence and strong time continuity pass (L.6) to

$$\|\omega^{\text{ns}}(t_*) - \bar{\omega}\|_p \geq d_*.$$

In particular $t_* > 0$ and this solution differs from the stationary one. Quarter-turn symmetry passes to the strong limit. Both velocities have zero mean and satisfy the unforced equation by their Biot–Savart definition and the verified transport equation. This proves Theorem 1.1.

A Checked external inputs

This appendix records the exact scope of the external analytic results. Their applications are verified in Section 3; the nonlinear and limiting arguments are internal to this manuscript.

1. Dacorogna–Moser [1], Theorem 2 and Remark (ii), printed pages 5–6: a bounded $C^{k+3, \alpha}$ domain, $k \geq 0$, $0 < \alpha < 1$, and a mean-zero $C^{k, \alpha}$ scalar admit a bounded linear zero-boundary-trace divergence inverse into $C^{k+1, \alpha}$. We use a disk, $k = 0$, and $\alpha = \gamma$. The determinant residual is proved to be mean-zero before this inverse is applied. The constant-Jacobian boundary extension is then constructed by the analytic implicit-function argument; it is not an extra assertion attributed to the cited theorem.
2. Hernandez [2], Definition 1.1 and Proposition 1.3, printed pages 3–4, with the proof in Section 4: the domain is bounded, convex, and C^2 , the reference density is its indicator, and the complex map is arbitrary in a small $C^{1, \gamma}$ ball about the identity. His kernel uses the bilinear complex denominator, and his uniform volume-map bound imposes no Jacobian constraint. Our disks satisfy every hypothesis. Norm-valued holomorphy, the boundary trace, the required physical Jacobian factor, all cross terms, and the restriction to little spaces are proved separately here.
3. Chemin [3], Definition 1 and Theorem 1, printed pages 521–522: the geometry is $\mathbb{R}^2$; the vorticity is bounded and in L^r , $1 \leq r < 2$; it is ordinary Hölder off a closed set on which a divergence-free Hölder field is nonvanishing; and its tangential derivative has the specified Hölder distribution regularity. Our isolated planar signed patch has compact support, so $r = 1$. A field tangent to every separated boundary makes its tangential derivative zero. The closed collar and the ordinary off-collar regularity are explicitly constructed. The graph is constructed at a higher exponent before this theorem is applied. This result is used only for the isolated planar canonical trajectory. It is not invoked as a periodic global patch theorem.
4. Verdera [4], contour-dynamics appendix, formula (16) and the local mapping/Lipschitz discussion on printed pages 12–13: the self-boundary operator of a single planar patch is locally Lipschitz on the open set of $C^{1, \gamma}$ chord-arc embeddings. The exact homogeneity identity identifies that boundary operator with Euler’s logarithmic contour velocity. Each signed interface

uses its own self operator; separated cross terms and the torus Green remainder are smooth local maps. A finite-cover proof gives the tube estimate on the compact canonical path. Torus active contours are identified with actual globally transported colours by the separately proved difference-energy argument.

The periodic multiplier bounds, bounded transported Euler construction, renormalization including all-time traces, and strong compactness are proved in Section 2. No form of a transport or patch theorem with unchecked torus hypotheses is required.

References

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- [3] J.-Y. Chemin, *Persistence de structures géométriques dans les fluides incompressibles bidimensionnels*, Ann. Sci. École Norm. Sup. (4) 26 (1993), 517–542. [Primary journal PDF](#).
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