

1 Finite actual forcing: density closure and exact-cap reset

A finite expansion with the actual foreign-field coefficients gives an actual density envelope and an exact-cap state reset. The recursion uses those coefficients as measurable functions of time. A nonzero growing coefficient and pointwise bounds on the protected late window supply the first-cap conclusion.

1.1 Spaces, operators and the derivative reserve

Let $\mathcal{C}$ be the union of the two closed derivative collars of the fixed smooth radial profile W . Choose a closed radial interval $I \Subset (0, \infty)$ containing these collars in its interior. For an integer $J \geq 1$, let E_j^r denote the real finite angular-sector space of functions supported in $\mathcal{C}$ whose sectors are $m \in 4\mathbb{Z}$, $|m| \leq 4Jj$, with radial coefficients in $H^r(I)$. The norm is the sum of their radial Sobolev norms. On these supported finite-sector functions it is equivalent to the full-plane H^r norm, with constants depending on I, J, j, r . Initially and in every recursion below the functions are smooth, flat at the collar endpoints, and have zero integral. The coefficient equations therefore use a common closed supported subspace on each finite Sobolev level; there is no assertion of an unrestricted all-angular-mode spectral estimate.

Write

$$\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W, \quad Q(f, g) = -(K * f) \cdot \nabla g.$$

The inherited finite-sector spectral lemma gives

$$e^{t\mathcal{L}} = e^{\lambda t} P_+ + e^{\bar{\lambda}t} P_- + \mathcal{R}(t), \quad \|\mathcal{R}(t)\|_{H^r \rightarrow H^r} \leq C_r e^{2\eta t}, \quad 2\eta < \sigma, \quad \lambda = \sigma + i\tau. \quad (1)$$

This is used only in one of the finitely many spaces E_j^r . The growing eigenfunction F_+ lies in sector $+4$ and is normalized by $\mathcal{A}(F_+) = 1$; its real conjugate lies in sector -4 . The global test defining $\mathcal{A}$ selects sector $+4$. For the pattern rotation set

$$(\mathcal{T}_\theta f)(r, \varphi) = f(r, \varphi + \theta/4), \quad \mathcal{L}_{\text{rot}} = \mathcal{L} - (\tau/4)\partial_\varphi.$$

For vector fields its definition is the pushforward $\mathcal{T}_\theta V(x) = \mathbf{R}_{-\theta/4} V(\mathbf{R}_{\theta/4} x)$, where $\mathbf{R}$ is the planar rotation matrix. Equivalently its polar components are shifted by $\theta/4$. This gives $K * (\mathcal{T}_\theta f) = \mathcal{T}_\theta(K * f)$ and $\mathcal{T}_\theta[V \cdot \nabla f] = (\mathcal{T}_\theta V) \cdot \nabla(\mathcal{T}_\theta f)$. This rotation is applied to the entire function, separately in every angular sector. It is not replaced by the scalar factor $e^{-ij\tau s}$ on an order- j coefficient. The operators commute, and rotations are isometries in radial Sobolev norms. Thus $e^{t\mathcal{L}_{\text{rot}}} \leq C e^{\sigma t}$ and the growing real space has eigenvalue σ .

Fix truncation orders $M, N \geq 1$ and a desired final Sobolev order k . Set

$$k_* := \max\{4, k\}, \quad r_{j\ell} = k_* + 2(M - j) + 2(N - \ell), \quad R = k_* + 2M + 2N + 3. \quad (2)$$

The reserve is explicit. A diffusion source uses $g_{j,\ell-1}$, whose order exceeds $r_{j\ell}$ by two. A quadratic or foreign-advection source uses a coefficient with strictly smaller seed order, whose order exceeds the requested order by at least two. A heat-interaction source uses $g_{j,\ell-r}$, again with at least two additional derivatives. All profile derivatives used below are fixed derivatives of W ; W is smooth in every finite order.

For zero-integral functions supported in one fixed ball,

$$\|K * f\|_{H^{r+1}(\mathbb{R}^2)} \leq C_r \|f\|_{H^r}, \quad \|Q(f, g)\|_{H^r} \leq C_r \|f\|_{H^r} \|g\|_{H^{r+1}}, \quad r \geq 2. \quad (3)$$

To check the first low-frequency estimate, compact support and zero mass give $|\widehat{f}(\xi)| \leq C|\xi| \|f\|_1$ for $|\xi| \leq 1$. The Biot–Savart multiplier has order -1 , so it is bounded there by this cancellation and is of order -1 at high frequency. Plancherel proves the first estimate. For the second, $H^r(\mathbb{R}^2)$ is an algebra for $r > 1$, and $\nabla g \in H^r$; multiplication and the first estimate prove the assertion. On the supported spaces H^{k*} also controls all the L^1, L^p product norms and the two extra derivatives in residual Laplacians. These elementary bounds justify, rather than hide, every transport derivative in the finite induction.

1.2 The finite recursion and its exact residual

Put $X = L^1 \cap L^p$, $p > 2$, with its sum norm. In a scale family write $L = \log(1/q)$, $\delta = L^{-4}$, with $0 < q < 1$; all error exponents $\chi > 0$ are fixed before q is decreased. Splitting Biot–Savart at radius one gives $\|K * f\|_\infty \leq C_p \|f\|_X$. Let $0 \leq H < \infty$, $t_0 \geq 0$, $\alpha \geq 0$, $\varepsilon > 0$, $0 < \kappa \leq 1$, and

$$s = t - t_0, \quad a(t) = \varepsilon e^{\sigma(t-t_0)}, \quad W_\kappa(t) = e^{\kappa t \Delta} W, \quad W_r(t) = \frac{t^r}{r!} \Delta^r W.$$

Let $B(t, x)$ be real and divergence free, spatially C^∞ for almost every time, and locally integrable in time with values in every fixed finite C^m norm. Only its C^R norm is used in the quantitative bounds. This qualitative regularity is automatic for the finite smooth stream basis with actual moment coefficients. Its polar velocity components have finitely many angular sectors $m \in 4\mathbb{Z}$, $|m| \leq 4J$. The retained field is $b_J = \varepsilon B$. Assume

$$\frac{\alpha}{\varepsilon} e^{\sigma t_0} \|\Phi\|_{H^{r_{10}}} \leq A_0, \quad D(t) := e^{-\sigma(t-t_0)} \|B(t)\|_{C^R}, \quad \int_0^H D(t) dt \leq B_0, \quad (4)$$

where $\Phi \in E_1^{r_{10}}$ is a real smooth zero-integral seed. The bounded constants A_0, B_0 are the input ball; no pointwise upper bound on D is imposed.

Set $g_{10}(0) = (\alpha/\varepsilon)\Phi$, with all other initial coefficients zero. For $1 \leq j \leq M$, $0 \leq \ell \leq N$, solve

$$\begin{aligned} (\partial_t - \mathcal{L})g_{j\ell} &= \mathbf{1}_{\ell \geq 1} \Delta g_{j, \ell-1} - \sum_{r=1}^{\ell} [(K * W_r) \cdot \nabla g_{j, \ell-r} + (K * g_{j, \ell-r}) \cdot \nabla W_r] \\ &+ \sum_{\substack{j_1+j_2=j, \, j_1, j_2 \geq 1 \\ \ell_1+\ell_2=\ell}} Q(g_{j_1 \ell_1}, g_{j_2 \ell_2}) - \mathbf{1}_{j=1} B \cdot \nabla W_\ell - \mathbf{1}_{j \geq 2} B \cdot \nabla g_{j-1, \ell}. \end{aligned} \quad (5)$$

Every coefficient on the right precedes (j, ℓ) in the ordering by $j + \ell$. The angular sectors are at most $4Jj$. All outputs are supported in $\mathcal{C}$: Laplacians preserve the support of smooth flat functions; $\Delta^r W$ with $r \geq 1$ has this support; and every advective term differentiates a supported function. Each forcing has zero integral by divergence freedom or the Laplacian identity. The bounded finite-sector generator and Duhamel therefore construct a unique smooth coefficient, absolutely continuous in time in every required lower Sobolev order. Measurable time coefficients are sufficient.

Define

$$F(t) = \sum_{j=1}^M \sum_{\ell=0}^N \varepsilon^j \kappa^\ell g_{j\ell}(t), \quad w_{\text{app}} = W_\kappa + F.$$

The recursion is an algebraic identity at the given run's B, κ . Dependence of B on the actual solution and on κ does not change that identity.

Lemma 1.1 (Coefficients and finite residual). *Under (4), with the reserve (2),*

$$\|g_{j0}(t)\|_{H^{r_{j0}}} \leq C e^{j\sigma s}, \quad \|g_{j\ell}(t)\|_{H^{r_{j\ell}}} \leq C(1+t)^{d_{j\ell}} e^{j\sigma s}, \quad d_{j\ell} = 2\ell(j+\ell), \quad \ell \geq 1. \quad (6)$$

The constants depend on the fixed finite orders and input ball, not on $t_0, H, \varepsilon, \kappa$ or the particular waveform. One may take $d_R = 4N(M+N) + N + 3$. If $a \leq 1$ and $\kappa(1+H)^{d_R} \leq c$, the residual in the retained field satisfies

$$\|\partial_t w_{\text{app}} + (K * w_{\text{app}} + \varepsilon B) \cdot \nabla w_{\text{app}} - \kappa \Delta w_{\text{app}}\|_X \leq C[a^{M+1} + \kappa^{N+1}(1+t)^{d_R}a](1+D(t)). \quad (7)$$

Proof. For g_{10} , (1) and the weighted initial bound show that its homogeneous contribution is at most $Ce^{\sigma s}$. Its forcing convolution is at most

$$C \int_0^t e^{\sigma(t-r)} \|B(r)\|_{C^R} dr = Ce^{\sigma s} \int_0^t D(r) dr.$$

The reserve supplies its requested derivatives. Suppose preceding coefficients have the claimed bounds. A quadratic pure-seed source has size $Ce^{j\sigma(r-t_0)}$ in the requested Sobolev order, by (3). For $j \geq 2$ its convolution has the gap $(j-1)\sigma$, so its bound is $Ce^{j\sigma s}$ with no polynomial. A pure external-advection source has size $Ce^{j\sigma(r-t_0)}D(r)$; dividing its convolution by $e^{j\sigma s}$ leaves $C \int_0^t e^{-(j-1)\sigma(t-r)} D(r) dr \leq CB_0$. This proves the pure-seed estimate.

For positive viscosity order, the heat terms carry $(1+r)^r$ from W_r , the diffusion term has the preceding polynomial, and the quadratic terms add their two polynomial degrees. A convolution at $j = 1$ can add at most one degree; for $j \geq 2$ the strict gap adds none. The choice $d_{j\ell} = 2\ell(j+\ell)$ dominates all these degrees:

$$d_{j\ell} \geq d_{j,\ell-1} + 1, \quad d_{j\ell} \geq r + d_{j,\ell-r} + 1, \quad d_{j\ell} \geq d_{j-1,\ell}, \quad d_{j\ell} \geq d_{j_1\ell_1} + d_{j_2\ell_2}.$$

The second inequality is used for $r \geq 1$ and $\ell \geq r$; the third for $j \geq 2$; the last for the displayed quadratic indices. For the direct $B\nabla W_\ell$ term its degree ℓ is smaller. The weighted D convolution adds no degree beyond the preceding factor. This proves (6) by the finite induction, using precisely the reserve already specified.

For completeness we give the residual identity. Put $G_N = W_\kappa - \sum_{r=0}^N \kappa^r W_r$ and $Q_r h = -[(K * W_r)\nabla h + (K * h)\nabla W_r]$. Write $Q_G h$ for the same expression with G_N replacing W_r . After cancellation of the retained monomials the residual is exactly

$$\begin{aligned} R^J = & - \sum_{j=1}^M \varepsilon^j \kappa^{N+1} \Delta g_{jN} - \sum_{\substack{1 \leq j \leq M, \\ \ell+r > N}} \varepsilon^j \kappa^{\ell+r} Q_r g_{j\ell} \\ & - \sum_{\substack{1 \leq j_1, j_2 \leq M, \\ j_1+j_2 > M \text{ or } \ell_1+\ell_2 > N}} \varepsilon^{j_1+j_2} \kappa^{\ell_1+\ell_2} Q(g_{j_1\ell_1}, g_{j_2\ell_2}) \\ & + \varepsilon^{M+1} \sum_{\ell=0}^N \kappa^\ell B \cdot \nabla g_{M\ell} + \varepsilon B \cdot \nabla G_N - Q_G F. \end{aligned} \quad (8)$$

Here juxtaposition with a gradient means a dot product. The exact radial heat base has zero residual on its own. The integral heat Taylor formula gives

$$\|G_N\|_X + \|\nabla G_N\|_X \leq C\kappa^{N+1}t^{N+1}.$$

Use the Biot–Savart X bound and (6) to estimate Q_GF . Every remaining factor in (8), other than the heat remainder, is supported on the fixed collars. Pure terms with seed order exceeding M are Ca^{M+1} ; their pure coefficient bounds have no polynomial. A positive viscosity power multiplied by any of the finitely many polynomial factors is absorbed by increasing d_R and requiring $\kappa(1+H)^{d_R} \leq c$. Terms with viscosity order exceeding N are $C\kappa^{N+1}(1+t)^{d_R}a$, since $a^j \leq a$. The external top-seed term uses $\|B\| = e^{\sigma s}D$ and is $Ca^{M+1}D$. The external heat remainder is $C\kappa^{N+1}(1+t)^{d_R}aD$. This proves (7), including all viscosity terms and without assuming a pointwise bound on D . $\square$

1.3 Comparison with the actual foreign velocity

Suppose the actual colour solves

$$\partial_t w + (K * w + b) \cdot \nabla w = \kappa \Delta w, \quad w(0) = W + \alpha \Phi.$$

The actual b is divergence free; it may depend on the entire interacting solution. Assume the usual positive-time smoothness and $C_t X$ initial trace, sufficient for the scalar Kato inequalities. A removed rigid rotation is allowed. Let $\mathcal{U}$ be a fixed annular neighbourhood of $\mathcal{C}$, separated from its complement by a positive radial heat-gradient gap. Assume

$$\sup_{x \in \mathcal{U}} |b - b_J| \leq m_q, \quad |b(t, x) \cdot e_r| \leq C_b(1 + |x|), \quad (9)$$

with fixed C_b . The second is used only against the exact radial heat gradient outside $\mathcal{U}$. No derivative of b is assumed. The stream extensions giving B agree with the retained harmonics on $\mathcal{U}$. These are exactly the collar omission and actual global radial-growth inputs supplied by the packet/core estimates.

Lemma 1.2 (Actual density comparison). *Fix $C_1 = C_p(\|\nabla W\|_X + 1)$ before choosing M, N, J . Choose $(M+1)\sigma > C_1$. There is a later threshold $a_* > 0$ such that if $a(t) \leq a_*$ and the finite viscosity smallness conditions hold, then $\|\nabla w_{\text{app}}\|_X$ gives the fixed exponent C_1 and*

$$\begin{aligned} \|w - w_{\text{app}}\|_X &\leq Ca^{M+1} + C\kappa^{N+1}(1+t)^{d_R+1}ae^{(C_1-\sigma)_+t} \\ &\quad + Cm_q(1+t)e^{C_1 t} + C(1+H)e^{C_1 H}(1+\varepsilon e^{\sigma H})e^{-c/(\kappa H)}. \end{aligned} \quad (10)$$

The last Gaussian term is understood as zero at $H = 0$. Consequently in a scale family

$$\begin{aligned} \varepsilon = q^r, \quad \sigma t_0 \leq \theta L, \quad H \leq C_H L, \quad \kappa \leq q^\gamma, \quad m_q \leq q^P L^{d_m}, \\ r, \gamma, \chi > 0, \quad \theta \geq 0, \quad 0 < C_H < \infty. \end{aligned}$$

if

$$\gamma(N+1) > C_H(C_1 - \sigma)_+ + \chi, \quad P > r + \theta + C_H(C_1 - \sigma)_+ + \chi, \quad (11)$$

then

$$\|w - w_{\text{app}}\|_X \leq Ca^{M+1} + Cq^\chi L^{d_*} a, \quad \|w - W_\kappa\|_X \leq Ca. \quad (12)$$

These estimates require no nonzero transmission coefficient.

Proof. The finite coefficient bounds make the pure perturbation gradient at most $1/2$ after choosing a_* sufficiently small. Choose the later viscosity threshold so the remaining finite gradients are

at most $1/2$. Heat contracts $\|\nabla W\|_X$. This yields the already fixed C_1 ; the coefficient constants change thresholds, not that exponent. For $z = w - w_{\text{app}}$ the exact equation is

$$(\partial_t + (K * w + b) \cdot \nabla - \kappa \Delta)z = -(K * z) \cdot \nabla w_{\text{app}} - R^J - (b - b_J) \cdot \nabla w_{\text{app}}.$$

The actual foreign field is part of divergence-free transport and cancels from the L^1, L^p norm inequalities. The Biot–Savart X bound gives

$$(\|z\|_X)' \leq C_1 \|z\|_X + \|R^J\|_X + \|(b - b_J) \cdot \nabla w_{\text{app}}\|_X.$$

The initial error is zero. Smooth norm regularizations and compact cutoffs justify the integrated inequality from time zero.

On $\mathcal{U}$ the mismatch is at most $Cm_q(1+a)$. Outside it, $\nabla F = 0$ and the weighted radial heat-gradient norm is $Ce^{-c/(\kappa t)}$, after decreasing c to absorb fixed polynomial heat factors. The actual radial-growth bound and $\|b_J\|_\infty \leq aD$ therefore bound the omitted force by $C(1+aD)e^{-c/(\kappa t)}$. Convolve with $e^{C_1(t-v)}$. For the top pure residual, division by $a(t)^{M+1}$ leaves

$$\int_0^t e^{-[(M+1)\sigma - C_1](t-v)}(1 + D(v)) \, dv \leq [(M+1)\sigma - C_1]^{-1} + B_0.$$

For the viscous residual, division by $a(t)$ leaves at most $C(1+t)^{d_R+1}e^{(C_1-\sigma)+t}(1+B_0)$. The constant collar mismatch gives $Cm_q(1+t)e^{C_1 t}$. The Gaussian force uses $\int D \leq B_0$ and monotonicity of its exponential, proving (10) after enlarging its harmless prefactor. In the scale family, $a(0) \geq q^{r+\theta}$ and (11) makes both non-Gaussian errors relative to $a(t)$ positive powers of q times fixed logarithms. The Gaussian dominates all these fixed power losses uniformly for smaller viscosities. Finally the finite coefficient sum has norm Ca and the error has the same bound, proving (12). $\square$

1.4 A normalized finite polynomial loses the forcing waveform

Extend B by zero after H , and put $S = H - t_0$. Suppose the decomposition holds on $-t_0 \leq s \leq S$, with the following future bound for $s \geq 0$:

$$\begin{aligned} B(t_0 + s) &= B^{\text{main}}(s) + E(s), \\ \|B^{\text{main}}(s)\|_{C^R} &\leq C_F(1+s)^{d_F}e^{hs}, \quad 0 \leq h \leq h_* < \sigma, \\ \int_{-t_0}^S e^{-\sigma s} \|E(s)\|_{C^R} \, ds &\leq \eta_q. \end{aligned} \tag{13}$$

The main and remainder are finite-sector fields of the same class as B and are zero-extended after S . Before $s = 0$ only the total weighted bound (4) is used; no past convergence hypothesis is needed for the coefficient reset. Define the *full* first-order growing amplitude

$$C_q = \frac{\alpha}{\varepsilon} e^{\lambda t_0} \mathcal{A}(\Phi) + \int_0^H e^{-\lambda(t-t_0)} \mathcal{A}(-B(t) \cdot \nabla W) \, dt. \tag{14}$$

It includes every retained input and the initial seed. It equals the infinite-future growing amplitude of the zero-extended first coefficient. It is not replaced by a canonical target coefficient.

Let $F_1(C) = CF_+ + \overline{CF_+}$, and, for $j \geq 2$, define

$$F_j(C) = -(j\sigma - \mathcal{L}_{\text{rot}})^{-1} \sum_{a+b=j} (K * F_a(C)) \cdot \nabla F_b(C). \tag{15}$$

The inverse is the convergent integral of the finite-sector semigroup $e^{t(\mathcal{L}_{\text{rot}} - j\sigma)}$. It is not a formal inverse. All coefficients are smooth, supported in $\mathcal{C}$, of zero mass, and belong to sectors $4n$ with $|n| \leq j$ and $n \equiv j$ modulo two. Induction proves that they are real polynomials of total degree j in $C, \bar{C}$, and that $F_j(C) = |C|^j \mathcal{T}_{\text{arg } C} F_j(1)$ for $C \neq 0$.

Lemma 1.3 (Finite coefficient memory loss). *Under (4) and (13), for any fixed $0 < d < \min(\sigma - 2\eta, \sigma - h_*)$, the pure coefficients obey*

$$\left\| e^{-j\sigma s} \mathcal{T}_{-\tau s} g_{j0}(t_0 + s) - F_j(C_q) \right\|_{H^{\tau_{j0}}} \leq C_j(1+s)^{d_j} e^{-ds} + C_j \eta_q, \quad 0 \leq s \leq S. \quad (16)$$

The constants depend only on the fixed finite orders, input ball, future envelope, and the strict rate gap. They do not depend on the waveform, preparation length, or an incoming canonical-profile error. No lower bound on $|C_q|$ is required for this coefficient statement.

Proof. The coefficient bounds imply that every normalized coefficient at $s = 0$ is uniformly bounded in its reserved order. Write $h_j(s) = e^{-j\sigma s} \mathcal{T}_{-\tau s} g_{j0}(t_0 + s)$. For $j = 1$ the growing component is $F_1(C(s))$, where the partial integral in (14) defines $C(s)$. Its missing main future tail has norm at most $C(1+s)^{d_F} e^{-(\sigma-h)s}$; its missing remainder has norm at most $C\eta_q$. Its complementary component has propagator decay $e^{-(\sigma-2\eta)s}$ after normalization. Its main forcing has bound $C(1+s)^{d_F} e^{-(\sigma-h)s}$ in this frame. Duhamel convolution of these two decays is bounded by $C(1+s)^{d_1} e^{-ds}$ for the stated strictly smaller d . The remainder convolution is bounded by $C \int_0^s e^{-\sigma v} \|E(v)\|_{C^R} dv \leq C\eta_q$. This proves the first case.

For $j \geq 2$ the normalized equation is exactly

$$h'_j = (\mathcal{L}_{\text{rot}} - j\sigma)h_j - \sum_{a+b=j} (K * h_a) \cdot \nabla h_b - e^{-\sigma s} (\mathcal{T}_{-\tau s} B(t_0 + s)) \cdot \nabla h_{j-1}.$$

Its homogeneous propagator decays as $Ce^{-(j-1)\sigma s}$ by (1). Subtract (15). Each quadratic difference has at least one lower difference, and its other factor is uniformly bounded in the needed reserved order. The reserve supplies one extra derivative in that lower factor, so (3) applies without loss. By induction these terms are bounded by $C(1+s)^d e^{-ds} + C\eta_q$ in the target order. The direct main external term is $C(1+s)^{d_F} e^{-(\sigma-h)s}$; its remainder has weighted time norm at most $C\eta_q$. Duhamel and $(j-1)\sigma > d$ yield the same bound for the normalized difference, including the decaying bounded initial value at zero. Increasing a finite polynomial degree handles a possible convolution polynomial. This proves the finite induction and (16). $\square$

1.5 Exact protected-coordinate matching

Define the real Banach-valued polynomial

$$\mathcal{P}_M(z) = W + \sum_{j=1}^M F_j(z), \quad z \in \mathbb{C}.$$

Its protected coordinate is $\Psi_M(z) = \mathcal{A}(\mathcal{P}_M(z))$. It is a real polynomial in $z, \bar{z}$, is rotation equivariant, $\Psi_M(e^{i\theta}z) = e^{i\theta}\Psi_M(z)$, and $D\Psi_M(0)$ is the identity on $\mathbb{R}^2$. In particular it has the form $z[1 + O(|z|)]$; the parity actually eliminates its quadratic term, but that stronger fact is not needed.

Lemma 1.4 (Exact polynomial and ancient cap gauge). *There is a fixed $\rho > 0$ such that Ψ_M has a unique inverse on $|\zeta| < \rho$, with inverse derivative uniformly bounded. Suppose $\|v - \mathcal{P}_M(z)\|_X \leq e$, $|z| < \rho$, and $\mathcal{A}(v) = \delta e^{i\theta}$ with $0 < \delta \ll \rho$. Then for $z_* = \Psi_M^{-1}(\delta e^{i\theta})$,*

$$|z - z_*| \leq Ce, \quad \|v - \mathcal{P}_M(z_*)\|_X \leq Ce. \quad (17)$$

For the inherited canonical ancient orbit, Corollary ?? supplies every fixed finite jet

$$\left\| \Omega_{\text{ref}}(u) - \mathcal{P}_M(c_{\text{ref}} e^{\lambda u}) \right\|_X \leq C_M |c_{\text{ref}} e^{\lambda u}|^{M+1}, \quad u \ll 0, \quad c_{\text{ref}} \neq 0.$$

If u_δ is its small exact cap, the rotation ϕ matching its protected coordinate exactly to $\delta e^{i\theta}$ satisfies

$$\|v - \mathcal{T}_\phi \Omega_{\text{ref}}(u_\delta)\|_X \leq C[e + \delta^{M+1}]. \quad (18)$$

Proof. Choose 2ρ so small that $\sup_{|z| \leq 2\rho} \|D\Psi_M(z) - I\| \leq 1/4$. The integral mean-value formula then gives the quantitative inverse estimate $|\Psi_M(z) - \Psi_M(z')| \geq (3/4)|z - z'|$ in that ball. The inverse function theorem, or its contraction proof applied to $z = \zeta - [\Psi_M(z) - z]$, supplies the inverse on a smaller ball. Since $\mathcal{A}$ is bounded on X , the state error bounds the coordinate error by Ce , and the inverse estimate proves the first bound in (17). The derivative of $\mathcal{P}_M$ is bounded on this fixed ball, so the second follows.

At the canonical exact cap, its jet remainder is $O(\delta^{M+1})$ because $|c_{\text{ref}} e^{\lambda u_\delta}| \asymp \delta$. The same inverse estimate changes its linear asymptotic parameter to the unique parameter matching its *full* protected coordinate, with error $O(\delta^{M+1})$. Rotation equivariance makes that parameter, after the exact coordinate-matching rotation, precisely z_* . The polynomial and ancient-state errors therefore add to the bound (18). Matching only the linear parameter would instead leave a lower-order error; it is not used here. $\square$

1.6 State reset and the separate instantaneous cap condition

Combine the lemmas in a scale family with $\eta_q \leq q^{\chi_E} L^{d_E}$, $\chi_E > 0$, and $0 < c_* \leq |C_q| \leq C^*$. Define

$$S_q = \sigma^{-1} \log \frac{\delta}{\varepsilon |C_q|}, \quad \delta = L^{-4},$$

and assume the prospective interval contains $t_0 + [S_q - L_0, S_q + L_0]$, where L_0 is fixed. At a time in that window put $z(t) = \varepsilon e^{\lambda(t-t_0)} C_q$. The coefficient memory estimate, the positive-viscosity correctors, the comparison bound, and $\|W_\kappa - W\|_X \leq C\kappa H$ give

$$\|w(t) - \mathcal{P}_M(z(t))\|_X \leq C\delta^{M+1} + q^{\chi_*} L^{d_*}, \quad (19)$$

for some fixed $\chi_* > 0$. Indeed $S_q = (r/\sigma)L + O(\log L)$, so the intrinsic memory transient is a positive power of q times fixed logarithms. The other errors have the strict powers already stated. The exponent χ_* is independent of an incoming canonical state error: that error may change the waveform and C_q , but is not used in the memory estimate. Exact coordinate matching by Lemma 1.4 then gives the same order of ancient-state reset at any exact cap in this window.

A weighted time norm alone does not supply a crossing slope. For a protected first-cap conclusion, impose the following *separate pointwise actual-field bounds* on the late window whenever $3\delta/4 \leq |\mathcal{A}(w)| \leq 5\delta/4$, writing $f = w - W_\kappa$:

$$\left| \int Tb \cdot \nabla W \right| + \int |f| |b| |\nabla T| + \left| \int Tb \cdot \nabla (W_\kappa - W) \right| \leq \rho_q \delta, \quad \rho_q \rightarrow 0. \quad (20)$$

These are instantaneous hypotheses, not consequences of (4). In the actual receiver application they follow, respectively, from the absolute packet moments and older collar bounds, the global weighted adjoint lemma, and the weighted smooth heat-profile estimate with the global radial-growth bound. These pointwise criteria are checked separately in the receiver application.

Corollary 1.5 (Protected actual first cap). *Assume the preceding scale-family bounds, the nonzero coefficient bound, (20), and a sufficiently wide prospective interval. Then for small q there is a unique first crossing $|\mathcal{A}(w(t_q))| = \delta$. It lies in the fixed late window and satisfies*

$$t_q = t_0 + S_q + o(1), \quad \frac{d}{dt}|\mathcal{A}(w)| \geq \sigma\delta/2 \quad \text{a.e.} \quad \text{for } 3\delta/4 \leq |\mathcal{A}(w)| \leq 5\delta/4 \text{ in that window.}$$

The upper slope is $C\delta$ almost everywhere, and its exact-cap state satisfies

$$\|w(t_q) - \mathcal{T}_{\phi_q}\Omega_{\text{ref}}(u_\delta)\|_X \leq C\delta^{M+1} + q^{\chi_*}L^{d_*},$$

where the canonical coordinate and actual coordinate are matched exactly. Any continuous family of actual solutions satisfying these uniform bounds has continuous cap times and phases.

Proof. The all-time density bound and boundedness of $\mathcal{A}$ give $|\mathcal{A}(w)| \leq Ca(t)$. Choose L_0 so large that $Ce^{-\sigma L_0}/c_* < 1/4$. Then no cap occurs for $t - t_0 \leq S_q - L_0$. In the fixed late window (19) and the polynomial coordinate yield $\mathcal{A}(w) = z(t) + O(\delta^2) + o(\delta)$ uniformly. The moduli at the window endpoints straddle δ , so continuity gives a crossing. The adjoint identity proved in Section ??, with (12) and (20), gives

$$\mathcal{A}' = \lambda\mathcal{A} + E_A, \quad |E_A| \leq C\delta^2 + \rho_q\delta + C\kappa(1 + H)\delta.$$

The last term tends to zero relative to δ . The coordinate is absolutely continuous. Its modulus derivative exists almost everywhere and equals $\sigma|\mathcal{A}| + \text{Re}(\overline{\mathcal{A}}E_A)/|\mathcal{A}|$. On the stated level interval it is at least $3\sigma\delta/4 - C\delta^2 - o(\delta) \geq \sigma\delta/2$. The upper bound follows from the same identity. Let C_{speed} be its fixed upper-speed constant. For all sufficiently small q , the crossing lies in the middle half of the late window. Choose

$$d_{\text{cap}} = \min\{L_0/4, (8C_{\text{speed}})^{-1}\}.$$

On $[t_q - d_{\text{cap}}, t_q + d_{\text{cap}}]$ the radius stays in the level band: before a first exit its change is at most $C_{\text{speed}}\delta d_{\text{cap}} \leq \delta/8$, whereas exit requires a change of $\delta/4$. The almost-everywhere lower slope integrates on this entire interval. At earlier times the envelope margin applies or the radius is below $3\delta/4$; a previous return from a larger radius would require a downward band crossing. Thus for each fixed $d \in (0, d_{\text{cap}}]$ there is the explicit uniform precap margin

$$\sup_{0 \leq t \leq t_q - d} |\mathcal{A}(w(t))| \leq \delta - \min\{\delta/4, \sigma\delta d/2\}.$$

A second crossing within the protected window would require a downward passage through this level interval, contradicted by the strictly positive slope. The endpoint estimate also gives $t_q - (t_0 + S_q) = o(1)$. Apply the exact-gauge lemma to (19) at this crossing. Finally a uniform early margin, common crossing slope, and uniform convergence of the continuous coordinate under solution parameters prove continuity of the first hit; its nonzero coordinate makes its circle-valued phase continuous. $\square$

The finite density and reset estimates use upper field-history bounds. The protected first-cap corollary additionally uses a nonzero C_q and (20).

2 An actual receiver and the loss of source memory

The receiving step starts from the child preparation envelope and its phase-matched exact-cap state. The finite expansion is applied to the actual harmonic coefficients of the coloured Navier–Stokes evolution.

Write $L = \log(1/q)$, $\delta = L^{-4}$, $\alpha = q^K$, where $K > 6$, and fix the smooth reference W , its protected coordinate $\mathcal{A}$, its growing eigenvalue $\lambda = \sigma + i\tau$, and its canonical ancient orbit $\Omega_{\text{ref}} := \Omega_{\delta_{\text{ref}}}$. The reference amplitude $\delta_{\text{ref}} > 0$ is fixed, sufficiently small, independently of q . After one fixed time translation and rotation, normalize the canonical orbit so $\mathcal{A}(\Omega_{\delta_{\text{ref}}}(0)) = \delta_{\text{ref}} > 0$. The bilateral transform is multiplied by a nonzero factor by this normalization; its nonvanishing selection is therefore unchanged. Choose a fixed β in a sufficiently large interval so that

$$C_* := \gamma_\epsilon \int_{\mathbb{R}} e^{-\mu_\beta u} B_4(\Omega_{\delta_{\text{ref}}}(u)) du \neq 0, \quad \mu_\beta = [\sigma + i(\tau - 4/5)]/\beta.$$

All choices below are made before q .

2.1 Discounted histories of an arbitrary prepared actual child

The following conditional history estimate applies to a child prepared by an incoming packet. It uses its preparation envelope and exact-cap reset independently of the receiving-stage conclusion.

Lemma 2.1 (Prepared-child reference history). *Fix the smooth profile, its all-order canonical reference orbit, $\beta > 1$, logarithmic clock constants and finite spatial derivative orders. Write $L = \log(1/q)$, $\delta = L^{-4}$ and $\mu_\beta = [\sigma + i(\tau - 4/5)]/\beta$. Assume the actual colours belong to the geometric stack of Lemma ??, with the clock and parameter margins of Corollary ?? in child units. Use child units, with all older constant rotations removed. Let the actual child w_c have normalized viscosity $\kappa_c \leq q^{3/2}$, an exact cap time τ_c satisfying $c_0 L \leq \tau_c \leq C_0 L$, and the common geometric initial colour bounds. Suppose, for fixed finite trial constants $C_{\text{past}}, C_{\text{reset}} > 0$,*

$$\|w_c(t) - e^{\kappa_c t \Delta} W\|_X \leq C_{\text{past}} \delta e^{\sigma(t-\tau_c)}, \quad 0 \leq t \leq \tau_c.$$

Let $s_q < 0$ be the canonical exact δ -cap time, and match the constant phase ϕ exactly in the protected coordinate. Suppose

$$\|\mathcal{T}_{-\phi} w_c(\tau_c) - \Omega_{\text{ref}}(s_q)\|_X \leq E_q, \quad E_q = C_{\text{reset}} \delta^{M+1} + q^\chi L^D, \quad \chi > 0,$$

and

$$\delta^{-C_W/\sigma-1/\beta} (1 + |s_q|) [E_q + q(1 + |s_q|)] \longrightarrow 0. \quad (21)$$

Here C_W is the fixed one-sided comparison constant on the small canonical ancient segment. The sufficient fixed choice $M + 1 > C_W/\sigma + 2$ has strict slack in this condition.

Put $t_{\text{ref}} = \tau_c - s_q$. On the interval ending at this reference time, assume the actual inactive-prefix estimates and the geometric preparation localization hypotheses of Lemma ??, with strict clock slack. On the following prospective interval assume the actual parent/larger constant-core and harmonic conclusions on the provisional parent density stop $\|w_i - W_{\kappa_i}\|_X \leq C_s \delta$, where $C_s > 0$ is fixed and finite. In particular the child's older radial drift in a fixed fraction of the parent core has coefficient at most $C(1 + C_s)\delta/\beta$. Use the Gaussian parent-radius handoff of Lemma ?? at t_{ref} . Let the following child horizon satisfy

$$0 \leq U_q \leq C_U \beta L.$$

The complete age and parent-unit horizon are assumed to lie in the fixed logarithmic windows of the cited inactive/core estimates.

For every such shorter stopped interval the absolute differentiated main-input history is bounded by one fixed ball independent of the trial constants, after a later q threshold depending on those constants. If the complete prospective interval has been reached and $U_q \rightarrow \infty$, define

$$B_q(u) = \begin{cases} B_4(\mathcal{T}_{-\phi} w_c(t_{\text{ref}} + u)), & -t_{\text{ref}} \leq u \leq U_q, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_{\mathbb{R}} e^{-\sigma u/\beta} |B_q(u) - B_4(\Omega_{\text{ref}}(u))| du \longrightarrow 0. \quad (22)$$

Consequently its complex discounted transform converges to the full canonical transform at μ_β .

For clarity, B_q is the actual physical history truncated at the controlled prospective endpoint and zero-extended thereafter. The canonical history in (22) is defined on the entire real line. No estimate for the genuine actual infinite-time future after parent density/core control has ended is asserted or needed.

The quantifiers are as follows. The fixed profile, reference, β , clocks and finite spatial orders determine the input ball first. For each fixed finite triple $(C_{\text{past}}, C_{\text{reset}}, C_s)$ satisfying the stated preparation/reset assumptions, there is a later q threshold. Within that class every estimate and convergence above is uniform in depth, source phase and allowed nondistinguished seed homotopies. The threshold need not be uniform over unbounded trial constants. Thus the constants may be selected once from the finite-forcing output bounds and then one common q chosen, without multiplying an inherited input constant at every descending stage.

Proof. Write $L_q = -s_q$. The canonical exact-cap asymptotic gives

$$L_q = O(\log L), \quad e^{-\sigma s_q/\beta} \leq C\delta^{-1/\beta}.$$

All phases below have already been removed by $\mathcal{T}_{-\phi}$. Rigid rotations preserve every radial positive-envelope estimate.

Localization before the reference. On the preparation interval the child's own nonradial radial velocity has time integral at most $CC_{\text{past}}\delta$. The descendant velocity contributes at most $C\beta q C_0 L$. The actual older nonrotational harmonic field and core leaks on a fixed child ball are smaller, by Corollary ?? . For any fixed trial C_{past} these displacements tend to zero. The initial support and a fixed radial margin therefore give, at every preparation time, positive-envelope Gaussian tails outside one fixed ball. Conditional global origin-growth estimates control every fixed post-exit moment; their fixed q -power amplification is absorbed by the Gaussian. The exact radial heat base has the corresponding Gaussian moment tails. These conclusions use the actual positive scalar envelope and the common drift, not a gradient of the actual child.

On the cap-to-reference segment, Lemma ?? gives the one-sided X comparison with the phase-matched canonical ancient solution

$$\sup_{0 \leq v \leq L_q} \|\mathcal{T}_{-\phi} w_c(\tau_c + v) - \Omega_{\text{ref}}(s_q + v)\|_X \leq C\delta^{-C_w/\sigma} [E_q + q(1 + L_q)]. \quad (23)$$

Indeed the canonical segment has fixed derivative bounds and fixed compact support. On that support the descendant field is $O(\beta q)$, the actual larger nonrotational field tends to zero with a strict positive q power, and $\kappa_c \leq q^{3/2}$. The comparison forcing is therefore $O(q)$ with fixed

parameters. Gronwall over L_q costs at most $C\delta^{-C_W/\sigma}$. No derivative of an actual moment coefficient or actual colour occurs.

The canonical nonradial radial displacement on this segment is $O(\delta_{\text{ref}})$; the comparison error contributes $o(1)$ by (21). Choose the fixed reference amplitude inside the fixed radial margin. Together with the descendant and older-field contributions, this gives the same fixed-radius positive-envelope moment reset at the reference. In particular no factor $\log \log(1/q)$ is introduced into its moment constants. The separate parent-radius handoff gives the Gaussian exterior-tail input required by Proposition ??; fixed-radius tightness alone is not used as a substitute for that input.

The negative history. Let $u = t - t_{\text{ref}}$. For $u \leq s_q$, angular symmetry gives $B_4(e^{\kappa_c t} \Delta W) = 0$. Inside the fixed preparation ball the whole-moment pairing is bounded by a fixed multiple of the perturbation L^1 norm. Outside it use the preceding Gaussian positive-envelope and heat tails. Thus

$$|B_q(u)| \leq CC_{\text{past}} \delta e^{\sigma(u-s_q)} + \mathcal{G}_q(u), \quad -t_{\text{ref}} \leq u \leq s_q,$$

where the Gaussian remainder is superpolynomial on the logarithmic preparation horizon. Its weighted integral is still superpolynomial: the earliest negative endpoint has size $O(L)$, so the discount costs only a fixed power of q^{-1} , whereas $\kappa_c \leq q^{3/2}$. The main term has the explicit bound

$$\int_{-t_{\text{ref}}}^{s_q} e^{-\sigma u/\beta} CC_{\text{past}} \delta e^{\sigma(u-s_q)} du \leq \frac{CC_{\text{past}} \delta e^{-\sigma s_q/\beta}}{\sigma(1-1/\beta)} \leq CC_{\text{past}} \delta^{1-1/\beta} \rightarrow 0.$$

The canonical moment on $(-\infty, s_q]$ has the same vanishing discounted bound, since it is $O(e^{\sigma u})$. Therefore the weighted absolute difference on the entire pre-small-cap interval tends to zero. This calculation replaces the selected-source identity involving $\alpha e^{\sigma t_{\text{ref}}}$.

On $[s_q, 0]$, the fixed-radius localization converts (23) into a whole fourth-moment difference with the same bound plus Gaussian errors. Since

$$\int_{s_q}^0 e^{-\sigma u/\beta} du \leq C\delta^{-1/\beta}(1 + L_q),$$

its weighted absolute difference tends to zero by (21). The negative canonical history on this segment has one fixed discounted norm. This proves a negative-history input bound consisting of a fixed reference majorant, $CC_{\text{past}} \delta^{1-1/\beta}$, and a vanishing reset contribution.

The controlled future. At reference the positive-envelope mass and fourth moment have fixed majorants. The own and descendant velocities have a fixed bounded sum by the individual scalar contractions and the convergent geometric velocity series. The older global radial drift is at most $C|y|/\beta$; inside the parent probe core its coefficient is $C(1 + C_s)\delta/\beta$, with Gaussian leaks. The parent-radius handoff supplies its initial exterior Gaussian tail. Keep the local radial coefficient $a = C(1 + C_s)\delta/\beta$ explicit in the proof of Proposition ??. Its unexited fourth moment has a prefactor determined only by the initial envelope moments and fixed source and descendant velocity bound, and a factor e^{4au} . All stop-dependent constants in the exited part are absorbed by its Gaussian after a later q threshold. The proposition and Corollary ?? thus yield on every shorter stopped interval

$$|B_q(u)| \leq C(1+u)^4 e^{C(1+C_s)\delta u/\beta} + o(q^P), \quad 0 \leq u \leq U_q,$$

for every fixed P . The prefactor is fixed by the reference-envelope majorants and bounded source and descendant speed. It does not multiply C_s . For each fixed trial C_s , a later threshold makes

$$C(1 + C_s)C_U \delta L \leq \log 2, \quad C(1 + C_s)\delta < \sigma/2.$$

Consequently the future absolute discounted input is bounded by a fixed multiple of $\int_0^\infty e^{-\sigma u/\beta} (1+u)^4 du$, and its tail from J is bounded by

$$C_{\beta,\sigma}(1+J)^4 e^{-\sigma J/(2\beta)} + o(q^P).$$

The errors remain negligible under all logarithmic-window and fixed q -power losses. Spatial differentiation of a retained fourth-harmonic stream field differentiates only its fixed smooth spatial basis, so the same argument supplies every fixed spatial derivative input norm. Together with the negative-history estimates, this gives a fixed input ball independent of the trial constants after their later thresholds. It is an upper bound only and requires neither a nonzero signed transform nor completion of a shorter density stop.

Once the complete prospective interval is available, every fixed window $[0, J]$ lies within it for small q . From (23), the actual reference state converges to the canonical reference state in X . On each such following window the canonical derivatives are bounded, viscosity tends to zero, and the actual older nonrotational and descendant fields on fixed support tend to zero. The same one-sided comparison therefore gives convergence in $C([0, J]; X)$. Positive-envelope radial localization on this fixed window, followed by conditional moment growth, gives uniformly vanishing fourth-moment tails outside a sufficiently large fixed ball. Whole fourth moments consequently converge on the window. All constants are uniform over the fixed prepared class.

The canonical positive history has its polynomial discounted tail. Truncate at J , apply fixed-window whole-moment convergence, and then let $J \rightarrow \infty$, using the uniform future tail just established. Combining this with the two negative-history pieces proves (22). The complex transform conclusion follows because $|e^{-\mu\beta u}| = e^{-\sigma u/\beta}$.

Every estimate used geometric normalized mass, amplitude or velocity sums, fixed spatial orders and the common clocks. Its thresholds depend on the fixed trial constants, not on depth or phase. Selecting those constants once and then taking the minimum of the finitely many thresholds yields the stated invariant-class quantifiers. If the force origin is instead chosen at the actual protected crossing of the fixed reference level, the same comparison and canonical positive slope change its time and constant phase only by $o(1)$; this preserves the history and transform conclusions. $\square$

Proposition 2.2 (Actual receiving-stage implication). *Consider actual colours w_l at scales $A_l = A_1 \beta^{l-1}$, $R_l = R_1 q^{l-1}$, in the common unforced Navier–Stokes velocity. Their initial conditions are the fixed smooth seeded profiles constructed in Section ??, with all seed amplitudes at most α ; every colour is real, fourfold symmetric and has zero circulation. Fix a receiving index i and its immediate child $c = i + 1$. Work in receiver units and remove the common solid rotation of all older cores. Assume $\kappa_c \leq q^{3/2}$ and hence $\kappa_i = \beta q^2 \kappa_c \leq \beta q^{7/2}$.*

The inductive child hypothesis is as follows. In child units it has an exact first cap τ_c , with $|\mathcal{A}(w_c(\tau_c))| = \delta$,

$$c_0 L \leq \tau_c \leq C_0 L, \quad \left\| w_c(t) - e^{\kappa_c t \Delta W} \right\|_X \leq C_{\text{past}} \delta e^{\sigma(t-\tau_c)} \quad (0 \leq t \leq \tau_c), \quad (24)$$

$$\inf_{\phi} \left\| w_c(\tau_c) - \mathcal{T}_{\phi} \Omega_{\delta_{\text{ref}}}(s_q) \right\|_X \leq C_{\text{reset}} \delta^{M+1} + q^{\chi} L^D, \quad (25)$$

where $C_{\text{past}}, C_{\text{reset}}$ are fixed finite trial constants, and $s_q < 0$ is the canonical exact-cap time $|\mathcal{A}(\Omega_{\delta_{\text{ref}}}(s_q))| = \delta$; the phase in (25) is chosen in the same exact protected-coordinate gauge. Here $\chi > 0$ is fixed. Choose the fixed finite order M so that $M + 1 > C_W/\sigma + 2$, where C_W is the one-sided cap-to-reference comparison constant. Then (21) holds and controls the discounted history

through the advance. The positive scalar envelope transported from $|w_c(0)|$ has at τ_c uniformly bounded moments of each fixed order and Gaussian tails outside a fixed ball, as supplied by the preparation localization lemma. All these constants are uniform in depth and in the allowed seed homotopies.

Choose fixed C_0, C_2 with $C_2 > C_0 > \max(K/\sigma, 6/\sigma)$, then β, D sufficiently large so that $C_0 > 6\beta/[\sigma(\beta - 1)]$, the unconditional inactive closure holds and

$$\epsilon_0 := C_2\sigma/\beta < K - 6, \quad C_2\lambda_B < K - 6, \quad C_2(\lambda_P + C_4/\beta) < 1, \quad \epsilon_0 < 1/2. \quad (26)$$

For small q , the actual receiver has a unique protected first cap $|\mathcal{A}(w_i(t_i))| = \delta$, with

$$t_i = t_0 + \sigma^{-1} \log \frac{\delta}{q^6 |C_*|} + o(1),$$

where t_0 is the child's fixed-reference time in receiver units. The reference phase ϕ_c used below is measured in the receiver's frame at t_0 . If θ_c is its phase in the child's frame with all older rotations removed, then $\phi_c = \theta_c + 4t_0/5$ modulo 2π : the parent core rotation is $-1/5$ in receiver time. Thus the source field has factor $e^{4i(t-t_0)/5} e^{i\phi_c}$, rather than counting the reference rotation a second time. It obeys an exponential preparation envelope

$$\|w_i(t) - e^{\kappa_i t \Delta} W\|_X \leq C q^6 e^{\sigma(t-t_0)} \quad (0 \leq t \leq t_i). \quad (27)$$

In its exact-cap phase gauge its state satisfies

$$\|w_i(t_i) - \mathcal{T}_{\phi_i} \Omega_{\delta_{\text{ref}}}(s_q)\|_X \leq C \delta^{M+1} + q^{\chi'} L^{D'}, \quad (28)$$

for some fixed $\chi' > 0, D' < \infty$. The exponent χ' depends only on the fixed stage parameters, and may be chosen independently of the incoming reset exponent χ . Advancing this cap to the fixed reference preserves an upper clock bound $C_0 L$, for small q . At the cap, and whenever the coordinate lies in $[3\delta/4, 5\delta/4]$ within its protected late window,

$$\frac{d}{dt} |\mathcal{A}(w_i(t))| \geq \sigma \delta / 2.$$

The corresponding upper derivative bound is $C\delta$. The positive receiver envelope has uniformly bounded fixed moments and Gaussian tails at the cap. The cap time and phase are continuous under all allowed seed homotopies. If the child's reference phase is ϕ_c , then the receiver cap phase is $\phi_c + \arg C_* + \tau S_q + o(1)$, uniformly, where $S_q = \sigma^{-1} \log[\delta/(q^6 |C_*|)]$.

Proof. Let $C_s \geq 1$ be a fixed finite trial density-stop constant and put

$$T_s = \inf\{t \geq 0 : \|w_i(t) - W_{\kappa_i}(t)\|_X = C_s \delta\}, \quad \inf \emptyset = +\infty.$$

For small q the initial density is $O(q^K) < C_s \delta$. While the child is current, Corollary ?? bounds the receiver independently of a signed child transfer. Its logarithmic horizon contains the child cap and its short reference advance by the strict clock slack. Lemma ?? therefore supplies the fixed-reference state, phase and positive-envelope moment reset before T_s . In receiver units,

$$t_0 \leq C_0 L / \beta + O(\log L).$$

A shift to the child's actual protected crossing of the fixed reference level changes this time and phase by $o(1)$ and preserves these inputs. Fix a prospective half-width $L_0 > 0$, to be enlarged below, and define

$$S_q^* = \sigma^{-1} \log \frac{\delta}{q^6 |C_*|}, \quad H_+ = t_0 + S_q^* + L_0, \quad \hat{H} = \min(H_+, T_s).$$

For small q , $H_+ \leq C_2 L$. Corollary ?? controls all older colours on $[0, \hat{H}]$. The receiver's nonradial mass is at most $2C_s \delta$ in normalized units, so its actual constant core follows from Proposition ??: $C_2 L \sqrt{C_s \delta} \rightarrow 0$ and $C_2 L \beta q \rightarrow 0$ for every fixed trial C_s . The associated harmonic and Gaussian-leak estimates hold on this entire density-stopped interval, including any portion after a hypothetical coordinate crossing.

Upper child-history bounds on shorter stops. Apply Lemma 2.1 with $U_s = \beta(\hat{H} - t_0) \leq C_2 \beta L$. The parent-radius Gaussian handoff supplies its initial exterior tail. In child units the own and descendant velocities have a fixed bounded sum, the global older radial drift is at most $C|y|/\beta$, and inside a fixed fraction of the parent core it is at most $C(1 + C_s)\delta|y|/\beta$ plus Gaussian leaks. These bounds persist to $\hat{H}$ because they require the density stop rather than a coordinate-cap restriction. The lemma gives

$$|B_c(u)| \leq C(1 + u)^4 e^{C(1 + C_s)\delta u/\beta} + o(q^P), \quad 0 \leq u \leq U_s,$$

for every fixed P , with the prefactor determined by the fixed reference moments and bounded source and descendant velocities, independently of C_s . For each fixed trial C_s , a later q threshold makes $C(1 + C_s)C_2 \delta L \leq \log 2$ and $C(1 + C_s)\delta < \sigma/2$. Hence the future discounted input is in a fixed ball independently of the trial constant. The pre-small-cap contribution is $O(C_{\text{past}} \delta^{1-1/\beta})$, and the weighted reset contribution tends to zero by (21). These upper bounds hold on every shorter density-stopped interval; no signed-transform lower bound or completion of the full interval has yet been used.

All other actual harmonic coefficients. In receiver units the inner packet has total mass $O(\beta q^2)$, L^p norm $O(\beta q^{2/p})$, and global velocity at most $C\beta q$. Inside the protected receiving core the older/current radial remainder has bound $C(1 + C_s)\delta r^3$ plus Gaussian leakage. Radial moment comparison and exit at a fixed core radius give, for every fixed n ,

$$\int |x|^{4n} |\text{inner packet}(t, x)| dx \leq C_n \beta q^{4n+2} (1 + H)^{4n} + o(q^P) \quad (0 \leq t \leq H \leq \hat{H}). \quad (29)$$

Indeed its radial moment root is bounded, before exit, by $C e^{C_n(1 + C_s)\delta t} [q + \beta q t + \sqrt{\kappa_i t}]$; $\kappa_i \leq \beta q^{7/2}$ makes the square-root term smaller than q up to logarithms. At exit the probability is $e^{-c/(\kappa_i H)}$, and the global radial growth amplifies each fixed moment by only $e^{C_n H}$, which is absorbed by this Gaussian. This calculation uses the positive aggregate, not its nonexistent summed L^∞ bound.

The descendants below the immediate child have mass $O(\beta q^2)$ in child units. The child's preparation envelope traps their positive aggregate in a fixed child ball up to its reference time; afterwards their fourth moment is bounded by

$$C\beta q^2 (1 + u)^4 e^{C(1 + C_s)\delta u/\beta} + o(q^P).$$

Thus their fourth moment adds only

$$O(q^{2-\epsilon_0} L^D)$$

to the weighted child-reference transform. The factor $q^{-\epsilon_0}$ comes from the finite negative-history endpoint, whose source clock is at most $C_2 L + O(\log L)$; its receiving discount is σ/β . This

argument controls the actual entire descendant packet and does not assert a separate small-density bound for its colours.

Retain exterior harmonics $4, 8, \dots, 4J$ of the inner packet and interior harmonics $3, 7, \dots, 4J-1$ of the older colours. Use their *actual* time coefficients, and fixed smooth stream cutoffs agreeing on the derivative collars; call the resulting global field b_J . The leading child fourth harmonic gives the first-order coordinate forcing

$$\beta q^6 \gamma_\epsilon e^{4i(t-t_0)/5} B_c(\beta(t-t_0)).$$

For each finite k , all retained higher inner harmonics have weighted C^k norm, divided by q^6 , at most $C_J q^{4-\epsilon_0} L^D$ by (29). The unconditional inactive theorem bounds the actual older harmonics on these collars by

$$O(q^{K+2-C_2\lambda_B}) + O(q^{8-C_2(\lambda_P+C_4/\beta)} L^D).$$

After source recentering, their weighted norm divided by q^6 is $o(1)$ by (26). The own initial seed contributes only $O(q^{K-6-\epsilon_0} L^D)$ to the first growing coefficient. All these strict powers are uniform in depth.

For the collar density leak, the inner positive aggregate has Gaussian L^1 tail outside a fixed protected radius and conserved finite L^p norm. Biot–Savart interpolation therefore gives a Gaussian velocity leak, despite its infinite sum of initial L^∞ amplitudes. Older leaks have the individual core estimates. The omitted harmonic remainder is $O(q^{4J+6} L^D)$ for the inner packet, and gains arbitrarily many q -powers for the older Taylor series. Choose J so large that, writing $P = 4J + 6$,

$$P > 6 + \epsilon_0 + C_2(C_1 - \sigma)_+ + \chi_1$$

for a fixed $\chi_1 > 0$; older omission orders are chosen to meet the same inequality. This is a relative envelope requirement, not only an absolute endpoint error bound: the first envelope is $a(0) = q^6 e^{-\sigma t_0} \gtrsim q^{6+\epsilon_0} L^{-D}$, and a q^P forcing error divided by $a(t)$ amplifies by at most $e^{(C_1-\sigma)+C_2L}$. The fixed-profile Kato exponent C_1 is selected before J .

The actual nonlinear density bootstrap. Set $\varepsilon = q^6$, $s = t - t_0$ and $a(t) = \varepsilon e^{\sigma s}$. Choose the finite orders first. The main child history and fixed stream basis determine a fixed derivative input ball. For every fixed triple $(C_{\text{past}}, C_{\text{reset}}, C_s)$ the preceding estimates place the actual input in this same ball after a later q threshold. Thus for every spatial derivative order needed in the finite recursion,

$$\int_0^H \varepsilon^{-1} e^{-\sigma(t-t_0)} \|b_J(t)\|_{C^k} dt \leq C_k. \quad (30)$$

This bound holds for every $H \leq \hat{H}$ and follows from Lemma 2.1 for the main harmonic and the strict power estimates for all other harmonics. Spatial differentiation acts on the fixed stream basis and leaves the actual time coefficients undifferentiated.

Apply Lemmas 1.1 and 1.2 to this actual run. In their notation take $\varepsilon = q^6$, $r = 6$, $\theta = \epsilon_0$, $C_H = C_2$ and $\gamma = 3$: $\kappa_i \leq \beta q^{7/2} \leq q^3$ for small q . For each retained coefficient use the explicit reserve $r_{j\ell} = k_* + 2(M-j) + 2(N-\ell)$ and $R = k_* + 2M + 2N + 3$, with $k_* = 4$ for the X conclusion. The finite stream field has polar sectors at most $4J$; its order- j coefficients therefore have sectors at most $4Jj$. The seed bound in (4) follows from $q^{K-6} e^{\sigma t_0} \|\Phi\|_{H^{r_{10}}} = o(1)$. The other input bound is exactly (30) for this finite derivative order R . Actual harmonics agree on the fixed broad annulus $\mathcal{U}$, with omission $m_q \leq q^P L^D$; Gaussian density leaks can be absorbed in that bound. The actual global radial bound is $|b \cdot e_r| \leq C(|x| + \beta q)$, uniformly in depth. Choose $(M+1)\sigma > C_1$ and a finite N so that $3(N+1) > C_2(C_1 - \sigma)_+ + \chi_2$. The harmonic

order was chosen to satisfy $P > 6 + \epsilon_0 + C_2(C_1 - \sigma)_+ + \chi_2$. These are precisely the relative-error inequalities (11). The coefficient recursion including every viscosity term is (5); its exact cancelled residual is (8), not a pointwise assumed forcing remainder. The actual foreign field stays in the divergence-free error transport. Consequently (12) gives

$$\|w_i - w_{\text{app}}\|_X \leq Ca(t)^{M+1} + q^{\chi_2} L^D a(t), \quad \|w_i - W_{\kappa_i}\|_X \leq Ca(t). \quad (31)$$

No signed growing-coefficient lower bound is used in this application.

The constant C_a is computed in the fixed input ball, before choosing C_s . Enlarge the fixed L_0 so that its half-width meets the early-margin requirement of Corollary 1.5, using the coefficient target lower bound $|C_*|/2$. Then choose

$$C_s > \max\{1, 2C_a e^{\sigma L_0} / |C_*|\}.$$

Only afterwards decrease q to validate the input estimates and core margins for this fixed choice, as well as the fixed child preparation and reset constants. Since $a(H_+) = \delta e^{\sigma L_0} / |C_*|$, (31) gives $\|w_i - W_{\kappa_i}\|_X < C_s \delta / 2$ throughout $[0, \widehat{H}]$. Continuity excludes $T_s \leq H_+$. Thus the full deterministic prospective interval has been reached using upper input bounds alone.

The complete actual child history and growing amplitude. Now $\beta t_0 \rightarrow \infty$ and $U_+ = \beta(H_+ - t_0) \rightarrow \infty$. Extend the actual phase-measured child moment $B_c(u)$ by zero outside $[-\beta t_0, U_+]$. Lemma 2.1 proves weighted absolute convergence on this complete interval and therefore

$$\int_{\mathbb{R}} e^{-\mu \beta u} B_c(u) du = e^{i\phi_c} \mathcal{F}_{\text{ref}}(\mu \beta) + o(1). \quad (32)$$

The zero extension is only of the controlled finite history; no estimate of the actual infinite future after H_+ is used. Extend the retained field b_J by zero after H_+ , and define its *full actual* first growing amplitude by

$$C_q = \frac{\alpha}{\varepsilon} e^{\lambda t_0} \mathcal{A}(\Phi) - \frac{1}{\varepsilon} \int_0^{H_+} e^{-\lambda(t-t_0)} \left[\int T b_J(t) \cdot \nabla W dx \right] dt. \quad (33)$$

This includes the own seed and every retained actual older, descendant and higher-sector input. Equivalently it is $\lim_{s \rightarrow \infty} e^{-\lambda s} \mathcal{A}(g_{10}(s + t_0))$ for this zero-extended first coefficient. Retained sectors other than ± 4 have zero direct growing projection, but remain in its full spatial coefficient and in the nonlinear recursion. Its difference from the actual main-child discounted transform is $O(q^{\chi_4} L^D)$ for some fixed $\chi_4 > 0$, by the preceding power estimates. The main-child transform converges to the canonical value by (32); that convergence is only $o(1)$ here, since the finite ancient-truncation error may be logarithmic. Hence

$$C_q = e^{i\phi_c} C_* + o(1), \quad |C_q| \geq |C_*|/2.$$

Universal coefficient reset. Use Lemma 1.3 with main field equal to the actual immediate-child fourth harmonic. Its post-reference C^R norm is $C(1+s)^4 e^{C(1+C_s)\delta s}$, so $h_* = \sigma/2$ is admissible for the final small q . Every other retained field has weighted remainder norm $\eta_q \leq q^{\chi_3} L^D$ by the preceding strict power estimates. The pattern rotation is the vector pushforward/scalar rotation of Section 1; all angular sectors are rotated separately. The full coefficient in (33) equals (14), with the field zero-extended at H_+ . Choose a fixed $0 < d < \min(\sigma - 2\eta, \sigma/2)$. Lemma 1.3 proves, with the explicit derivative reserve, convergence to $F_j(C_q)$ for each pure coefficient. At $s = S_q + O(1) \asymp L$ its transient is a positive q -power. The actual main waveform need not be

close to a canonical waveform for this statement: only its finite input ball and future envelope are used. Its canonical closeness was used for the lower bound on C_q . Formula (19), with the positive-viscosity correctors and $\|W_{\kappa_i} - W\|_X \leq C\kappa_i H_+$, gives a state error $C\delta^{M+1} + q^{\chi'} L^{D'}$ against the universal polynomial. Lemma 1.4 matches its *full* protected coordinate by the two-real-variable inverse, and identifies the matching all-finite-jet ancient state. This proves (28) once the exact first cap is found. Neither the incoming waveform error nor an incoming reset exponent enters this intrinsic transient exponent.

Protected first crossing. The global density envelope excludes a cap before $H_+ - 2L_0$ for the fixed L_0 chosen above. In the ensuing fixed window the settled first growing coordinate is $\varepsilon e^{\lambda s} C_q + O(a^2) + o(a)$. The polynomial future tail and the small weighted remainders make the error uniform there. Its moduli at the two ends straddle δ . The actual adjoint-coordinate identity gives, when the coordinate is in $[3\delta/4, 5\delta/4]$ within this protected late window,

$$\mathcal{A}' = \lambda \mathcal{A} + O(\delta^2) + o(\delta).$$

For the instantaneous radial-base forcing, use (29), not merely the weighted history: the collar field is $O(q^6 L^D)$ plus the small older field. For the exact foreign cross-advection pairing use Lemma ???:

$$\int |f| |b| |\nabla T| \leq C \chi_q \|f\|_1 = o(\delta), \quad \chi_q = C \left(\beta q + \frac{\delta q^2 + q^4 + \varepsilon_{\text{core}}}{\beta} \right).$$

This uses the global test weight and the actual older cores; it requires no claim that the receiver's own heat-base defect is Gaussian in its core. The heat-profile foreign correction is $O(\kappa_i H)$, using the global radial bound on b and the weighted smooth heat Taylor bound. It is $o(\delta)$; the remaining heat errors and the quadratic self error have the stated orders too. Thus the three separate pointwise criteria of (20) hold with $\rho_q \rightarrow 0$. Corollary 1.5, applied with half-width $L_0/2$, proves the protected first crossing and its lower and upper slopes. Its center uses $|C_q|$; replacing that by $|C_*|$ changes the center by $o(1)$. Consequently this narrower window is contained in the controlled prospective interval for small q ; no forcing beyond H_+ is used. This gives the asserted time and phase asymptotics. Uniqueness of recrossing is asserted in the protected window; no later Euler or Navier–Stokes return to the level is excluded.

For the universal polynomial, let $a = |z|$ denote the radius of its parameter in the growing phase gauge. Its coordinate modulus is $R(a) = a[1 + O(a)]$. $dR/da = 1 + O(a) > 0$. Since $a' = \sigma a$, its time derivative is $dR/dt = \sigma a[1 + O(a)]$. Matching the actual first coordinate therefore selects its time/amplitude and phase gauges uniquely; the matching error has the same orders as (28). This step uses the exact cap, rather than identifying it with the linear preparation time.

Finally, (27) gives integrated receiver nonradial drift $O(\delta)$; the younger global velocity gives $O(\beta q H)$, and the older field is smaller. The positive receiver envelope has Gaussian tails outside a fixed ball and bounded fixed moments at the cap by the radial exit lemma and global moment growth. The same uniform preparation and crossing estimates hold under the stated homotopies. Positive-viscosity continuous dependence, followed by the strictly positive crossing slope, proves cap-time and cap-phase continuity. Its own cap clock is $[6/\sigma + C_0/\beta]L + O(\log L) < C_0 L$; the additional reference advance is only $O(\log L)$, so the same strict clock margin remains. The lower clock bound persists since $t_i \geq (6/\sigma)L - O(\log L)$. This completes the actual receiving-stage implication. $\square$