

1 Material colours, exact moments, and interaction estimates

Identify the square torus with $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and use $u = \nabla^\perp \psi$, $\Delta \psi = \omega$, where $\nabla^\perp = (-\partial_2, \partial_1)$. Fix

$$\rho^2 = 3/5, \quad W = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}, \quad A_j = \beta^j, \quad R_j = q^j, \quad 4 \leq \beta \leq 5.$$

An overall common amplitude factor does not change any estimate below. Constants are independent of the number N of colours and of sufficiently small q . Unless specified otherwise, their other dependencies are the fixed exponent $0 < \gamma < 1$ and fixed contour neighbourhoods.

For a finite transported stack let

$$w_j(t) = A_j[-\mathbf{1}_{D_{j,1}(t)} + \rho^2 \mathbf{1}_{D_{j,2}(t)}]. \quad (1)$$

The two domains are transported by the same area-preserving flow as all other colours. Initially they have areas $|D_{j,1}| = \pi \rho^2 R_j^2$, $|D_{j,2}| = \pi R_j^2$, they are invariant under $z \mapsto iz$, and $D_{j,1} \subset D_{j,2}$. We also require the initial cross-colour inclusions $D_{j+1,2} \subset D_{j,1}$ whenever they are used. A common flow preserves these inclusions. The finite Euler solution is quarter-turn invariant by uniqueness, so its flow commutes with the quarter turn and fixes the origin. Thus every colour retains its symmetry, both areas, zero circulation, and

$$\int w_j = 0, \quad \|w_j\|_\infty = \frac{3}{5} A_j, \quad \|w_j\|_1 = \frac{12\pi}{25} A_j R_j^2. \quad (2)$$

The last equality uses $D_{j,1} \subset D_{j,2}$; its upper bound alone would suffice. We will use the material packet $\mathcal{P}_k = \bigcup_{l \geq k} \overline{D_{l,2}}$ and its radius, so that an accidental cancellation in the signed sum cannot invalidate a support or separation assertion.

For a real, integrable vorticity w supported in a coordinate disk put

$$M_n(w) = \int \zeta^n w(\zeta) dA(\zeta), \quad B_n(w) = \int \bar{\zeta}^n w(\zeta) dA(\zeta) = \overline{M_n(w)}.$$

Quarter-turn invariance gives $M_n = i^n M_n$, and therefore

$$M_n = B_n = 0 \quad (n \not\equiv 0 \pmod{4}). \quad (3)$$

Circulation gives $M_0 = 0$ separately. The moments with degree a positive multiple of four are generally *not* conserved by transport. For a positively oriented boundary X the exact Stokes identities are

$$\int_D \bar{z}^n dA = \frac{1}{2i(n+1)} \int_{S^1} \bar{X}^{n+1} X' d\theta, \quad \int_D z^n dA = -\frac{1}{2i(n+1)} \int_{S^1} X^{n+1} \bar{X}' d\theta. \quad (4)$$

They follow by differentiating $\bar{z}^{n+1} dz$ and $z^{n+1} d\bar{z}$. These formulas show directly that the moments are bounded polynomial functionals on the full material $E = c^{1,\gamma}(S^1; \mathbb{R}^2)^2$, independent of relabelling.

Write a local colour's boundaries as $R_l X_{l,a}$, where $X_{l,a} = r_a e^{i\theta} + h_{l,a}$, $r_1 = \rho$, $r_2 = 1$, and $x_l = \|h_l\|_E$. Choose once and for all a sufficiently small neighbourhood with uniformly bounded E norms, chord-arc constants, support in B_{bR_l} , and $B_{aR_l} \subset D_{l,1}$, for fixed $0 < a < \rho < b$. Put $\delta w_l = w_l - A_l W(\cdot/R_l)$. The boundary is within $CR_l x_l$ of its reference circle, so its symmetric difference with the reference disk has area at most $CR_l^2 x_l$. For $n \geq 1$ this gives

$$|M_n(w_l)| = |M_n(\delta w_l)| \leq CA_l R_l^{n+2} b^n x_l. \quad (5)$$

Here all positive holomorphic moments of each reference disk vanish. The same bound follows from (4). The domains and δw_l have the stated symmetry; δw_l vanishes on B_{aR_l} and outside B_{bR_l} .

1.1 The two planar multipole expansions

The complex velocity $U = u_1 + iu_2$ is exactly

$$U(z) = \frac{i}{2\pi} \int \frac{w(\zeta)}{\bar{z} - \bar{\zeta}} dA(\zeta). \quad (6)$$

It is $2i\partial_{\bar{z}}\psi$ for $\psi = (2\pi)^{-1} \int \log|z - \zeta| w(\zeta) dA$. If the source is supported in B_r and $|z| > r$, the convergent geometric series in (6) gives

$$U(z) = \frac{i}{2\pi} \sum_{n \geq 0} \frac{B_n(w)}{\bar{z}^{n+1}} = \frac{i}{2\pi} \sum_{m \geq 1} \frac{B_{4m}(w)}{\bar{z}^{4m+1}}. \quad (7)$$

The second equality requires both zero circulation and quarter-turn symmetry. This also checks all complex conjugations in the formulas. For a local inner source $l > i$, assume $bR_l \leq \eta aR_i$ with one fixed $\eta < 1$, and that both receiving contours stay in the chosen local neighbourhood. Normalizing the velocity by R_i , the leading moment is of size $A_l R_l^6 x_l / R_i^6$. Termwise differentiation of (7) through two spatial derivatives, followed by a uniform geometric-series estimate, gives

$$\sup_{aR_i \leq |z| \leq bR_i} |D_z^m U_l(z)| \leq C_m A_l R_l^6 x_l R_i^{-5-m}, \quad m = 0, 1, 2, 3.$$

Consequently

$$\left\| R_i^{-1} U_l(R_i X_i) \right\|_E \leq C A_l (R_l / R_i)^6 x_l = C A_l q^{6(l-i)} x_l. \quad (8)$$

More precisely, the first term in the normalized velocity is $iB_4(w_l)/(2\pi R_i^6 \bar{X}_i^5)$; the E norm of the remaining terms is at most $C A_l (R_l / R_i)^{10} x_l$.

For $|z| < aR_l$ the reference radial colour is exactly $U_l^0(z) = -iA_l z/5$. The perturbation has the nonsingular expansion

$$\begin{aligned} U_l(z) + iA_l z/5 &= -\frac{i}{2\pi} \sum_{n \geq 0} \bar{z}^n \int \frac{\delta w_l(\zeta)}{\bar{\zeta}^{n+1}} dA \\ &= -\frac{i}{2\pi} \sum_{m \geq 1} \bar{z}^{4m-1} I_{4m,l}, \quad I_{4m,l} = \int \frac{\delta w_l(\zeta)}{\bar{\zeta}^{4m}} dA. \end{aligned} \quad (9)$$

Symmetry kills every index except $n+1 \equiv 0 \pmod{4}$. The symmetric-difference estimate gives $|I_{4m,l}| \leq C A_l R_l^{2-4m} a^{-4m} x_l$. For a target core $i > l$, require $bR_i \leq \eta aR_l$. Evaluating (9) and differentiating its series yields

$$\sup_{|z| \leq bR_i} |D_z^m [U_l(z) + iA_l z/5]| \leq C_m A_l x_l R_i^{3-m} / R_l^2, \quad m = 0, 1, 2, 3,$$

and hence

$$\left\| R_i^{-1} [U_l(R_i X_i) + iA_l R_i X_i/5] \right\|_E \leq C A_l (R_i / R_l)^2 x_l = C A_l q^{2(i-l)} x_l. \quad (10)$$

The subtracted field is the *actual exact* solid rotation in the constant-vorticity plateau, not a linearized rotation. Quarter-turn symmetry removes harmonic constant, linear, and quadratic velocities; the first remaining harmonic velocity is cubic.

For completeness, these are full material estimates. If a smooth spatial field F on a fixed neighbourhood of the two normalized target curves has $\sup_{m \leq 2} \|D^m F\| \leq M$, the chain rule gives

$$\|F \circ X\|_{c^1, \gamma} \leq CM \quad (11)$$

uniformly for their bounded E norms. Indeed $(F \circ X)' = DF(X)X'$, and its Hölder seminorm is bounded by $\|DF\|_\infty[X']_\gamma + \|D^2F\|_\infty\|X'\|_\infty^{1+\gamma}(2\pi)^{1-\gamma}$. The estimates include both vector components and all tangential material velocities. The resulting composition is in the little space: approximate X in that space by smooth curves and use the same bound. The displayed third spatial derivative also bounds the full normalized foreign field in $C^{2,\gamma}$ on a fixed collar of the target curves. All these series can be differentiated to any prescribed fixed order, with a corresponding constant depending on that order; separation is maintained on the slightly larger collar used for the derivative bound. Separated cross-contour integral formulas (or the uniformly convergent moment series above) also show analytic dependence on local source and target embeddings. No derivative of a radial-graph parametrization is used. The numerical gains are asserted on the actual area constraints and quarter-turn symmetry, not for arbitrary ambient area variations. A perturbation of the target alone cannot create a missing source term: a circular complete colour has exactly zero exterior field and exactly the stated interior solid rotation.

At a receiving circle $z = re^{i\theta}$, the first exterior term gives

$$\frac{u_r}{r} = \Re \left(\frac{iB_4(w)}{2\pi r^6} e^{4i\theta} \right), \quad (u_r/r)_4 = \frac{iB_4(w)}{4\pi r^6}.$$

The last expression is the positive complex Fourier coefficient, not the cosine amplitude. With receiver radii $(\rho R_i, R_i)$ and the specified left dual $\ell = (1/2, i/(2\rho))$, its projection is

$$\ell f_4 = D_4 \frac{B_4(w)}{R_i^6}, \quad D_4 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0. \quad (12)$$

In receiver time $\theta = A_i t$ divide the right side by A_i . If $a(X) = \ell Q(X)$ is the geometric moment coordinate with $Q_a = (2\pi r_a^6)^{-1} \int_{D_a(X)} \bar{z}^4 dA$, then (4) gives $Da(X^0)$ equal to that normal left dual and $\|Da(X) - Da(X^0)\| \leq Cx_i$. Thus an actual perturbed recipient has leading scalar forcing (12) plus a planar remainder bounded by

$$CA_l(R_l/R_i)^6 x_l x_i + CA_l(R_l/R_i)^{10} x_l \quad (13)$$

for an inner local source $l > i$. The first term includes both evaluation on the perturbed material curve and the changed moment derivative. For a packet of mass μ and radius r , its analog is $C\mu r^4 R_i^{-6} x_i + C\mu r^8 R_i^{-10}$. Image terms are bounded separately by (18) or (19). The leading coefficient is exact on the reference circles; it is not asserted to be exact on a finite perturbed contour.

1.2 An arbitrary spent packet

Let w_P be any real zero-circulation quarter-turn invariant source, with $\|w_P\|_1 \leq \mu$ and support in B_r . No boundary regularity is required. Then $|B_{4m}(w_P)| \leq \mu r^{4m}$. If $r \leq \eta a R_i$, the same differentiated series proves

$$\left\| R_i^{-1} U_P(R_i X_i) \right\|_E \leq C \frac{\mu r^4}{R_i^6}, \quad |B_4(w_P)| \leq \mu r^4. \quad (14)$$

These bounds replace (8) once the inner packet has left every shape neighbourhood. The separation condition is essential; a conserved mass bound without a support/separation bound does not control the field at a receiving interface.

1.3 Square-torus regular kernel and the exact image cancellation

Let G be the mean-zero Green function satisfying $\Delta G = \delta_0 - 1$ on the unit square torus. Its logarithmic singularity has coefficient $(2\pi)^{-1}$. On a fixed disk B_{r_0} containing no nonzero lattice point write $G(z) = (2\pi)^{-1} \log |z| + H(z)$. Then $\Delta H = -1$, so $H + |z|^2/4$ is harmonic and real analytic. Uniqueness of G gives $G(iz) = G(z)$ and hence

$$H(z) = c - \frac{1}{4}|z|^2 + \Re \sum_{n \geq 1} c_{4n} z^{4n}. \quad (15)$$

To verify the allowed powers, represent the harmonic part as the real part of a holomorphic power series. Quarter-turn invariance kills every nonconstant power not divisible by four. Convergence and all derivative bounds hold on every smaller fixed disk. This derivation requires no formula for the numerical coefficients c_{4n} .

Convolve (15) with a *complete* constrained colour, or a sum of complete colours, w . All differences $z - \zeta$ must remain in that fixed disk. The quadratic polynomial becomes

$$\int |z - \zeta|^2 w(\zeta) dA = |z|^2 M_0 - 2\Re(z \overline{M_1}) + \int |\zeta|^2 w(\zeta) dA.$$

It is constant in z because $M_0 = M_1 = 0$. For the harmonic part,

$$\int (z - \zeta)^{4n} w(\zeta) dA = \sum_{m=0}^n \binom{4n}{4m} z^{4n-4m} M_{4m}. \quad (16)$$

The signs are positive since $4m$ is even. The $m = 0$ term vanishes by circulation and the $m = n$ term is constant in z . Thus the degree-four term is only M_4 , whereas the degree-eight term is $70z^4 M_4 + M_8$. For a potential $\Re P(z)$ its complex velocity is $i\overline{P'(z)}$. Consequently the first *possible* nonzero image velocity is

$$U^{\text{im}}(z) = 280i c_8 \overline{M_4(w)} \bar{z}^3 + \text{terms of higher total degree}. \quad (17)$$

In particular there is no uncanceled degree-four velocity. The conjugation in (17) agrees with the exterior moment $B_4 = \overline{M_4}$ in (7).

Choose source and target radii small enough that their sum lies in a strictly smaller Green disk. Differentiating the normally convergent series (16), and using (5), gives for local source l and local target i , including $i = l$,

$$\sup_{|z| \leq bR_i} |D_z^m U_l^{\text{im}}(z)| \leq C_m A_l R_l^6 x_l R_i^{3-m}, \quad m = 0, 1, 2, 3,$$

and therefore

$$\left\| R_i^{-1} U_l^{\text{im}}(R_i X_i) \right\|_E \leq C A_l R_l^6 R_i^2 x_l. \quad (18)$$

Every nonconstant surviving term has source degree at least four and potential target degree at least four. The higher powers are summed uniformly on the smaller disk; (11) then applies. For an arbitrary packet of mass μ and radius r the corresponding bound is

$$\left\| R_i^{-1} U_P^{\text{im}}(R_i X_i) \right\|_E \leq C \mu r^4 R_i^2. \quad (19)$$

Thus the ratios to the planar estimates are respectively

$$\begin{array}{l|l} \text{inner local source } l > i & R_i^8 \\ \text{outer local source } l < i & R_l^8 \\ \text{local self interaction } l = i & R_i^8 \\ \text{arbitrary separated inner packet} & R_i^8. \end{array}$$

Since $R_j \leq q$, each gain is at most q^8 . These statements apply to the two disks *only after their signed strengths are combined*. A single disk has nonzero circulation and does not satisfy these cancellations. For a complete circular radial colour all positive holomorphic moments vanish as well, so its entire image velocity in this coordinate neighbourhood is exactly zero.

2 Uniform confinement of an aggregate material packet

The torus kernel satisfies $|K_{\mathbb{T}}(z)| \leq C/|z| + C$ in coordinate distance. Splitting its convolution at distance h gives

$$\|K_{\mathbb{T}}w\|_{\infty} \leq C\|w\|_{\infty}h + C\|w\|_1/h + C\|w\|_1.$$

Choose h proportional to $\min\{1, \sqrt{\|w\|_1/\|w\|_{\infty}}\}$, with the zero case understood by continuity. Hence

$$\|K_{\mathbb{T}}w\|_{\infty} \leq C\sqrt{\|w\|_1\|w\|_{\infty}} + C\|w\|_1, \quad \|K_{\mathbb{T}}w_l\|_{\infty} \leq CA_lR_l. \quad (20)$$

For $\beta q < 1/2$ and $\beta q^2 < 1/2$ this yields

$$\sum_{l \geq k} \|K_{\mathbb{T}}w_l\|_{\infty} \leq \frac{CA_kR_k}{1 - \beta q}, \quad \left\| \sum_{l \geq k} w_l \right\|_1 \leq \frac{CA_kR_k^2}{1 - \beta q^2}. \quad (21)$$

Neither constant depends on N or the geometry of the spent colours.

Suppose all colours $l < k$ stay in the fixed local neighbourhood, and $\mathcal{P}_k$ lies in $B_{\eta a R_l}$ for every $l < k$. This is an explicit central-plateau assumption, to be improved by the radius bootstrap. For all such l , (9) and (18) give on that ball

$$U_l(z) = -iA_lz/5 + O(A_lx_l|z|^3/R_l^2).$$

The image correction is included; its cubic coefficient is smaller by R_l^8 . The first term is perpendicular to the radius exactly. Let $s = A_k(t - t_0)$ and $d(s) = R_k^{-1} \sup_{z \in \mathcal{P}_k(t)} |z|$. Taking the upper right Dini derivative of the supremum over material particles gives

$$\begin{aligned} D^+d(s) &\leq \frac{C}{1 - \beta q} + Cd(s)^3 \sum_{l < k} \frac{A_l}{A_k} \left(\frac{R_k}{R_l} \right)^2 x_l \\ &\leq C + C\beta^{-1}q^2d(s)^3 \leq C + Cq^2d(s)^3. \end{aligned} \quad (22)$$

In the second line x_l is bounded by the fixed neighbourhood radius and the geometric row is $\sum_{m \geq 1} (q^2/\beta)^m$. For finite N , the material particle paths are continuously differentiable in time; the radius is locally Lipschitz, and the usual maximum-of-paths argument proves this Dini inequality. It is valid also when the maximizing particle changes.

Fix $H \leq C_{\beta}L$, $L = \log(1/q) \geq 1$, and $d(0) \leq d_0$ with d_0 fixed. Choose $D \geq 4(d_0 + C)$ first. Stop if $d(s) = D(1 + s)$ or a central separation fails. On the stopped interval (22) implies

$$d(s) \leq d_0 + Cs + \frac{Cq^2D^3}{4}[(1 + s)^4 - 1].$$

For small q the ratio of this bound to $1 + s$ is at most $D/2$, uniformly up to H , since $q^2(1 + H)^3 \rightarrow 0$. Moreover require $qD(1 + H) < \eta a/2$; then $D(1 + s)R_k < \eta a R_{k-1}/2$, and hence the same strict

inclusion holds in every larger local plateau. This simultaneously improves both stopping margins. Consequently

$$d(s) \leq C(1+s), \quad \left| B_4 \left(\sum_{l \geq k} w_l \right) \right| \leq C A_k R_k^6 (1+s)^4 \quad (0 \leq s \leq H). \quad (23)$$

If a different part of the induction has its own earlier exit, these are conclusions on that stopped interval; this radius argument excludes radius and central-separation exits independently of it. All material supports must also stay in the common coordinate disk; $qD(1+H) \ll 1$ provides the required margin.

For any fixed $c > 0$, (23) gives the explicit tail estimate

$$\int_S^H e^{-cs} \frac{|B_4(\sum_{l \geq k} w_l)|}{A_k R_k^6} ds \leq C_c e^{-cS} (1+S)^4. \quad (24)$$

At a following cap the packet's radius relative to the next larger scale is $O(q(1+L))$. The assertion that this is also the deeper packet's radius at an already completed source cap is induction data: it follows by applying the preceding estimate at the completed stage, not by assuming the next recipient cap exists.

3 A causal deeper-packet bound on a compact source window

Here is the support lemma needed when the current source ceases to be close to a circle. Fix a compact regular planar material trajectory $X_*(s)$, $0 \leq s \leq S$, with the reference two areas and quarter-turn symmetry. Require uniformly separated $c^{1,\gamma}$ interfaces, bounded contour norms and chord-arc constants, support in $B_{b_S R_k}$ after scaling, and an origin plateau $B_{h_S R_k} \subset D_{k,1}^*(s)$ with $h_S > 0$. The last requirement follows, for a regular flow that fixes the origin, from an initially interior origin and compactness in time; it does not follow merely from a conserved area. Take a fixed small material E tube of this trajectory such that every actual source in the tube has the same uniform geometric bounds and contains $B_{h_S R_k/2}$.

In the source's origin plateau its vorticity is exactly $-2A_k/5$. Even without a small shape norm, its planar velocity has the form

$$U_k(z) = -iA_k z/5 + \mathcal{H}_k(z), \quad |\mathcal{H}_k(z)| \leq C_S A_k |z|^3 / R_k^2 \quad (|z| \leq h_S R_k/8). \quad (25)$$

One elementary proof is to subtract the circular patch $(-2A_k/5)\mathbf{1}_{B_{h_S R_k/4}}$. The difference vanishes near zero and has mass at most $C_S A_k R_k^2$. Its inverse-moment series as in (9) is convergent on $B_{h_S R_k/8}$; quarter-turn symmetry makes the first term cubic and bounds its coefficient by $C_S A_k / R_k^2$. The subtracted patch produces the displayed solid rotation. Complete-colour torus images are cubic as well by (16), and their coefficient obeys the same bound after increasing C_S .

Let the actual completed cap at $s = 0$ have full material distance $\eta_0(q) \rightarrow 0$ from a rotated canonical cap and let the deeper material packet initially obey

$$e(0) := \text{rad}(\mathcal{P}_{k+1}(0))/R_k \leq C_d q(1+L).$$

The larger colours are assumed to remain in their fixed local neighbourhoods on the receiving stopped interval. Remove their exact solid rotations, which has no effect on any radius or any absolute moment bound. The joint stopping conditions are: leaving the source tube, reaching the source's origin-plateau margin, reaching the deeper-packet barrier below, or any independently

specified larger contour exit. On this joint stopped interval, while $e \leq h_S/8$, (20) and (25) imply in source time

$$D^+e(s) \leq \frac{C\beta q}{1-\beta q} + C_S e(s)^3 \leq C\beta q + C_S e(s)^3. \quad (26)$$

The larger local colours contribute only another bounded cubic coefficient, with row

$$\sum_{m \geq 1} \beta^{-m} q^{2m} x_{k-m}.$$

The source's linear rotation has zero radial component. The entire deeper packet's own velocity is bounded by $CA_{k+1}R_{k+1}/(1-\beta q)$, independently of its shape or N .

Choose $M \geq 4(C_d + C)$ and impose the barrier $e(s) \leq Mq(1+L+\beta s)$. On $0 \leq s \leq S$, integration yields

$$e(s) \leq C_d q(1+L) + C\beta q s + C_S M^3 q^3 (1+L+\beta S)^3 S \leq \frac{1}{2} M q(1+L+\beta s)$$

for sufficiently small q . This improves the barrier strictly, and the improved bound lies strictly inside $h_S/8$. The tube's origin plateau therefore contains the deeper packet with a fixed separation margin; this fact was proved on the joint stop rather than assumed from future source shadowing.

The deeper mass is at most $CA_k R_k^2 \beta q^2 / (1-\beta q^2)$. Applying (14) on the separated actual source interfaces therefore bounds its full normalized material forcing, in source time, by

$$\frac{C_S \beta q^2}{1-\beta q^2} e(s)^4 \leq C_S q^6 (1+L)^4. \quad (27)$$

The compact source's own image forcing is $C_S R_k^8 \leq C_S q^8$ in that time scale, using its conserved mass, bounded radius, and complete-colour cancellation. For the local larger colours the nonradial forcing is bounded by

$$J_k(s) = C_S \sum_{l < k} \frac{A_l}{A_k} (R_k/R_l)^2 x_l(s); \quad (28)$$

their image row is smaller by at least q^8 . In particular, if the adjacent receiver has $x_{k-1} = O(q^6(1+L)^6)$ and the remaining prefix gives a no-larger geometric row, then $J_k = O(q^8(1+L)^6)$.

The local contour-flow comparison theorem on this fixed regular tube gives, by bounded-time Grönwall,

$$\sup_{0 \leq s \leq S} \|X_k(s) - X_*^{\text{rot}}(s)\|_E \leq C_S \left[\eta_0(q) + \int_0^S J_k(s) ds + q^6(1+L)^4 + q^8 \right]. \quad (29)$$

This is the only contour regularity input here: the autonomous material contour vector field is locally Lipschitz on the specified compact chord-arc tube, including both signed self-interface terms, and forced solutions satisfy the corresponding mild comparison. That theorem must be established in the contour section; smooth kernel estimates alone would not establish its self-interface part. If the bracket tends to zero, the estimate strictly improves the tube and its origin-hole margins. Together with the independent radius improvement this rules out every source-tube, deeper-radius, or source-hole exit before S . Any independent larger-contour exit is excluded by its own induction estimate. No future receiving cap or long-time shadowing of the source is invoked. By (4), the source moment is Lipschitz on this compact tube and satisfies the same error bound. The negative half of a two-sided compact window, when required, comes from a previously completed source stage and bounded-time backward comparison from the already known cap; it is distinct from this forward support argument.

Hypotheses that cannot be dropped. The q^6 outward gain needs complete-colour zero circulation and quarter-turn symmetry, plus separated supports. The q^2 inward gain needs an actual constant-vorticity origin plateau, with its solid rotation removed. The q^8 image gain needs the square torus and complete constrained colours; neither an unconstrained individual disk nor a mere signed mass bound supplies it. The compact source-window argument needs an independently regular source trajectory with an origin plateau and an initial $O(q(1+L))$ deeper-packet radius. Within those hypotheses the interaction and support estimates above are uniform in N .