1 Fixed seeds: separate initial and forced prefix contributions

The result in this section is a majorant lemma. Its coupling inequality is an explicit hypothesis; no assertion that diffusive colours already satisfy that inequality is made.

Lemma 1.1 (Two prefix norms)**.** *Let $\beta \geq 4$, $A_i = \beta^{i-1}$, and fix a receiving index $j \geq 2$. Suppose nonnegative continuous functions x_i, $1 \leq i < j$, satisfy, on $0 \leq \theta \leq S$,*

$$x_i(\theta) \leq Mbe^{\lambda_0 \theta} + M \int_0^\theta e^{\lambda_0(\theta - s)} \left(\sum_{l<j} K_{il} x_l(s) + f_i(s) \right) \mathrm{d}s, \qquad \lambda_0 = \frac{\sigma + \eta}{\beta}, \tag{1}$$

where $M, C, \sigma > 0$, $\eta, r_* > 0$, $f_i \geq 0$ are integrable, and the nonnegative matrix K obeys

$$K_{ii} \leq C r_* / \beta, \qquad K_{il} \leq \begin{cases} C\beta^{l-j} q^{2(i-l)}, & l < i, \\ C\beta^{l-j} q^{6(l-i)}, & i < l < j. \end{cases}$$

Assume $\beta q^6 < 1$. Define

$$r_B = \frac{C}{\beta} \left(r_* + \frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right), \qquad \lambda_B = \lambda_0 + M r_B.$$

Then $x_i \leq h_i + p_i$, where the initial contribution satisfies

$$\max_{i<j} h_i(\theta) \leq Mbe^{\lambda_B \theta}. \tag{2}$$

For $D > \beta$ with $Dq^8 < \beta$, put $w_i = (Dq^6)^{j-i}$ and

$$r_P = \frac{C}{\beta} \left(r_* + \frac{\beta}{D - \beta} + \frac{Dq^8}{\beta - Dq^8} \right), \qquad \lambda_P = \lambda_0 + M r_P.$$

The forced contribution satisfies

$$P(\theta) := \max_{i<j} \frac{p_i(\theta)}{w_i} \leq M \int_0^\theta e^{\lambda_P(\theta - s)} \max_{i<j} \frac{f_i(s)}{w_i} \, \mathrm{d}s. \tag{3}$$

All constants are independent of j. One can choose η, r_* small, then D large and q small, so both λ_B and λ_P are strictly smaller than $\sigma/2$.

Proof. The row sums of K are bounded by r_B: for $l = i + d$ the inner coefficient is at most $C\beta^{i-j}(\beta q^6)^d$, and for $l = i - d$ the outer coefficient is at most $C\beta^{i-j}(q^2/\beta)^d$. Use $\beta^{i-j} \leq 1/\beta$ and sum the geometric series.

Solve the positive comparison system

$$Y' = (\lambda_0 I + MK)Y + Mf, \qquad Y(0) = Mb\mathbf{1}.$$

Iteration of (1), or its integral comparison principle, gives $x \leq Y$. The variation-of-constants formula splits $Y = h + p$, with

$$h = e^{(\lambda_0 I + MK)\theta} Mb\mathbf{1}, \qquad p = M \int_0^\theta e^{(\lambda_0 I + MK)(\theta - s)} f(s) \, \mathrm{d}s.$$

The unweighted maximum norm gives (2). For the weighted maximum norm, the inner row acquires $(\beta/D)^d$ and the outer row acquires $(Dq^8/\beta)^d$. Their geometric sums give r_P and hence (3). Since λ_0 is close to $\sigma/\beta \leq \sigma/4$, first make the diagonal term small. Increasing D makes the remaining q-independent term in r_P small; subsequently decreasing q makes all other terms small. □

Corollary 1.2 (The fixed-seed baseline is small in its actual field)**.** *Assume the hypotheses of Lemma 1.1, with $\lambda_B \le \lambda_* < \sigma/2$, in the clock $\theta = A_j t$. Suppose the completed cap clocks satisfy*

$$T_{j,N} \le S/A_N + H \sum_{m=j}^{N-1} A_m^{-1}, \qquad N \ge j, \qquad S \ge H\beta/(\beta-1).$$

Let $b = \delta e^{-\sigma S}$. Then through $T_{j,N}$ the initial contribution is bounded uniformly by

$$B = M\delta e^{-(\sigma-\lambda_*)S}.$$

Its normalized inward feedback on the receiver j is at most

$$C \sum_{i<j} \beta^{i-j} q^{2(j-i)} h_i \le CB \frac{q^2/\beta}{1 - q^2/\beta}. \tag{4}$$

The forced contribution has feedback at most

$$CP \frac{Dq^8/\beta}{1 - Dq^8/\beta}.$$

In particular, with $H = h_0 \log(1/q)$ one can take $S = C_0 \log(1/q)$ so that $B = O(q^6)$ and its feedback is $O(q^8)$, with constants independent of depth.

Proof. Writing $a = \beta^{j-N} \in (0, 1]$ gives

$$A_j T_{j,N} \le Sa + \frac{H\beta}{\beta - 1}(1 - a) \le S.$$

Apply (2). The actual inward row in (4) is a geometric series with ratio q^2/β. For $p_i \le w_i P$ its ratio is Dq^8/β. Choose $C_0 \ge h_0\beta/(\beta-1)$ and $C_0(\sigma - \lambda_*) > 6$. No weighted norm of the initial seeds is used. If $Dq^6 < 1$, the local chart bound follows from $x_i \le B + w_i P \le B + Dq^6 P$ whenever the latter quantity is below r_*. □

The corollary repairs a specific scalar obstruction. To use it for Navier–Stokes one must prove (1), including its spatial q^2 and q^6 rows for diffusive colours, and bound the forced term P. Those remain coupled PDE hypotheses.

2 A fixed passive colour is determined by the total Euler path

The next result allows signed smooth colours as well as indicator colours. It therefore does not require a patch reference profile.

Proposition 2.1 (Recovery of fixed passive colours)**.** *Let u be bounded and divergence free on $[0,T] \times \mathbb{R}^2$, with $u \in L^1([0,T]; W^{1,p}_{\mathrm{loc}})$ for some $p > 1$. For each compactly supported $f_0 \in L^1 \cap L^\infty$ there is at most one solution of*

$$\partial_t f + u \cdot \nabla f = 0, \qquad f(0) = f_0,$$

in $L^\infty([0,T]; L^1 \cap L^\infty)$ with its weak initial trace. It is renormalized and belongs to $C([0,T]; L^2)$.

Suppose $u_n \to u$ uniformly, $\sup_n \|u_n\|_\infty \le V$, $\nu_n \to 0$, and $0 \le \nu_n \le 1$. Let f_n solve

$$\partial_t f_n + u_n \cdot \nabla f_n = \nu_n \Delta f_n, \qquad f_n(0) = f_0,$$

with the L^1, L^2, and L^∞ contraction bounds and the weak-form equation. Then their transport limit f is unique and

$$f_n \longrightarrow f \quad \text{in } C([0,T]; L^1 \cap L^2).$$

The same conclusion holds for transported colours with $\nu_n = 0$.

Proof. We give the renormalization argument needed for uniqueness. Let $f_\varepsilon = f * \eta_\varepsilon$ be a spatial mollification. It satisfies

$$\partial_t f_\varepsilon + u \cdot \nabla f_\varepsilon = r_\varepsilon, \qquad r_\varepsilon(x) = \int \nabla \eta_\varepsilon(y) \cdot [u(x) - u(x-y)] f(x-y)\,\mathrm{d}y.$$

On a compact spatial set the difference-quotient estimate bounds its L^1 norm by $C\|\nabla u\|_p \|f\|_{p'}$, on a slightly larger set. This follows by writing the velocity difference as the integral of its gradient along the segment and then applying Hölder's inequality. Approximate u in $L^1_t W^{1,p}_{\mathrm{loc}}$ by divergence-free spatial convolutions. For a smooth approximating velocity, divergence freedom allows subtraction of $f(x)$ in the commutator integral. Translation continuity of f then makes the commutator tend to zero in $L^1_{t,x}$. The uniform difference-quotient bound handles the velocity approximation. Thus $r_\varepsilon \to 0$ locally in L^1.

Multiplication by a bounded derivative of a C^1 function and passage to the limit shows that bounded renormalizations satisfy the transport equation, including their initial trace. For the difference of two bounded solutions, use a renormalization equal to s^2 on their bounded value range. Integration against expanding cutoffs has a flux bounded by $C\|u\|_\infty R^{-1} \int |f|^2$. Since bounded integrable functions are square integrable, that flux tends to zero. The initial square norm is zero, which proves uniqueness. The same argument preserves the square norm of one solution. Weak time continuity and preservation of that norm give strong L^2 time continuity.

For convergence, the weak-form identity gives, for compact smooth φ,

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int \varphi f_n \right| \le \|f_0\|_1 (V \|\nabla \varphi\|_\infty + \|\Delta \varphi\|_\infty).$$

The contraction bounds and this equicontinuity give weak L^2 convergence uniformly in time along subsequences. The Kato inequality and a radial cutoff also give uniform tightness:

$$\sup_{t \le T} \int_{|x|>2R} |f_n(t,x)|\,\mathrm{d}x \le \int_{|x|>R} |f_0(x)|\,\mathrm{d}x + CT\|f_0\|_1 (V/R + 1/R^2).$$

Passing to the distributional equation identifies every limit as a bounded integrable transported solution with the given initial trace. Uniqueness identifies all subsequences.

To prove uniform strong L^2 convergence, suppose otherwise and choose bad times $t_n \to t$. Weak time equicontinuity gives $f_n(t_n) \rightharpoonup f(t)$ in L^2, while contraction and renormalization give

$$\limsup_n \|f_n(t_n)\|_2 \le \|f_0\|_2 = \|f(t)\|_2.$$

Hilbert-space uniform convexity gives strong convergence, a contradiction. Local L^1 convergence follows from L^2, and tightness gives global uniform L^1 convergence. □

For $p > 2$ the kernel split gives the explicit useful bound

$$\|K * h\|_\infty \le C_p(\|h\|_1 + \|h\|_p) : \tag{5}$$

the near kernel belongs to $L^{p'}(B_1)$ because $p' < 2$, and the far kernel is bounded. Thus strong convergence of total vorticity paths in $C_t(L^1 \cap L^p)$ implies uniform convergence of their velocities. Whenever the limiting velocities have the Sobolev regularity in Proposition 2.1, the proposition applies to every fixed bounded initial colour. In particular, if two total Euler paths coincide, their recovered fixed colours coincide. No uniqueness of the total nonlinear Euler equation is used.

Let $f_{1,0}, \dots, f_{k,0}$ be finitely many fixed bounded compact colours and choose complex constants c_l. A suitable observable is

$$a_u(t) = \sum_{l=1}^{k} c_l \int \Phi_l(x) f_l[u](t,x)\,\mathrm{d}x, \qquad \Phi_l(x) = \chi_l(|x|)\bar{z}^4, \tag{6}$$

where each χ_l is a fixed smooth radial compact cutoff. The weak-form estimate makes a_u uniformly Lipschitz for a bounded family of drifts and $\nu \le 1$, and the proposition makes it continuous under the strong total-path convergence just specified. These tests cover one smooth outer colour and also a moment coordinate made from several patch colours. If uncut moments are wanted in an Euler limit, the cutoffs can equal one on a common support ball: bounded velocity gives finite propagation. Indeed test $|f|$ against a nondecreasing smooth function of $|x| - R_0 - Vt$; its material derivative is nonpositive and its initial pairing is zero.

3 Protected first caps and phase shooting

A phase evaluated at an arbitrary cap time need not distinguish trajectories. The following first-cap functional does distinguish them and requires no common cap clock.

Lemma 3.1 (Continuity of a protected first-cap phase)**.** *Fix $\delta, c, e, h, L > 0$, with $2e < T$. Let $a : [0,T] \to \mathbb{C}$ be L-Lipschitz, let*

$$\tau(a) = \inf\{t \ge 0 : |a(t)| \ge \delta\}, \qquad \mathcal{P}(a) = a(\tau(a))/\delta,$$

and suppose

$$\begin{aligned} |a(0)| &\le \delta - h, \qquad e \le \tau(a) \le T - e, \\ |a(t)| &\le \delta - h \quad (0 \le t \le \tau(a) - e), \\ |a(t)| - |a(s)| &\ge c(t-s) \quad (\tau(a) - e \le s < t \le \tau(a) + e). \end{aligned}$$

For any continuous b with $\varepsilon = \|a - b\|_\infty < \min(h/2, ce/4)$, its first cap exists,

$$|\tau(a) - \tau(b)| \le 2\varepsilon/c, \qquad |\mathcal{P}(a) - \mathcal{P}(b)| \le \delta^{-1}(1 + 2L/c)\varepsilon.$$

Families satisfying the displayed conditions with common constants retain a protected first cap under uniform convergence. Hence $\mathcal{P}$ is a continuous functional on that protected class.

Proof. Before $\tau(a) - e$ the strict margin excludes any cap of b. On $[\tau(a) - e, \tau(a)]$ the radial lower-slope inequality gives $|a(t)| \le \delta - c(\tau(a) - t)$. Thus b cannot cap before $\tau(a) - 2\varepsilon/c$. At $\tau(a) + 2\varepsilon/c$, its modulus is greater than or equal to $\delta + \varepsilon$, so it has capped by that time. This proves the time estimate. Since both first-cap values have modulus δ, the phase estimate follows from

$$|a(\tau(a)) - b(\tau(b))| \le L|\tau(a) - \tau(b)| + \|a - b\|_\infty.$$

For a uniformly convergent protected sequence, take a convergent subsequence of its cap times. The common Lipschitz bound, the precap margin, and the integrated radial lower-slope inequality pass to the limit. They identify the limiting time as the unique first cap. This argument applies to every subsequence and proves the final assertion. □

Corollary 3.2 (Separation of total inviscid paths)**.** *Consider one fixed total initial datum and fixed bounded initial colours. Suppose two Navier–Stokes subsequences converge strongly in $C_t(L^1 \cap L^p)$, their limits satisfy the velocity regularity of Proposition 2.1, and their observables* (6) *have uniformly protected first caps. If their limiting first-cap phases are different, the two total Euler paths are different. The cap times may differ.*

Proof. Proposition 2.1 and Lemma 3.1 pass the colours, observables, and phases to the limit. Equal total paths would have equal velocities and recovered colours, hence equal first-cap phases, a contradiction. □

Proposition 3.3 (Conditional degree shooting)**.** *Fix $\nu > 0$ and an active index $N \ge 2$. Suppose there is a continuous initial-data family indexed by $(r, \theta) \in [0, 1] \times \mathbb{S}^1$: the active seed at N has phase θ, and all other seed amplitudes are multiplied by r. Assume the total and passive-colour solution paths at this fixed viscosity depend continuously on these parameters. Assume their outer observables have first caps protected by the common constants of Lemma 3.1.*

If at $r = 0$ all other colours are radial and the active seed family is obtained by spatial rotation through angle $-\theta/4$, then the output phase map at $r = 1$ is surjective onto $\mathbb{S}^1$.

Proof. The first-cap map is continuous by Lemma 3.1. At $r = 0$, rotational covariance and the radial cutoffs in (6) give

$$\mathcal{P}(a_{0,\theta}) = e^{i\theta}\mathcal{P}(a_{0,0}).$$

Its degree is one. The degree is constant along the homotopy in r: a continuous argument lifted on the rectangle $[0, 1] \times [0, 2\pi]$ has endpoint difference $2\pi d(r)$, and the continuous integer $d(r)$ is constant. A circle map omitting one point admits a periodic continuous argument and has degree zero. The map at $r = 1$ therefore cannot omit a point. □

The continuity hypothesis at fixed positive viscosity is routine and does not hide a uniform-in-viscosity estimate. For completeness, in $X = L^1 \cap L^p$, (5) makes $(K * w)w$ locally Lipschitz on bounded X balls. The mild vorticity equation gives

$$d(t) \le d(0) + \frac{C_p M}{\sqrt{\nu}} \int_0^t (t - s)^{-1/2} d(s)\, \mathrm{d}s.$$

Short intervals on which the integral coefficient is at most $1/2$, followed by iteration, prove continuity for each fixed ν. The passive mild equation has the same kernel and an additional source controlled by the uniform drift difference. It gives continuity of colours, including when their initial bounded domains vary continuously. Uniform cap protection across the homotopy, by contrast, is a substantive coupled PDE hypothesis.

Proposition 3.4 (Conditional sparse-phase construction)**.** *Fix $p > 2$ and $\beta q^{2/p} < 1$. Let a family of infinitely seeded, zero-circulation compact initial data be indexed by a phase at every depth. Assume its series converges absolutely in $X = L^1 \cap L^p$, that changing only the phase at depth N changes the datum by at most $\eta_N \to 0$, uniformly in all phases, and that depths $N \geq 2$ leave the fixed outer initial colours unchanged.*

Suppose $\nu_N \downarrow 0$, and for every depth and every choice of all other phases, Proposition 3.3 applies with common cap-protection constants. Finally suppose that for every datum in this family its Navier–Stokes family is relatively compact in $C_t X$, and every limit satisfies the velocity hypothesis of Proposition 2.1. Then some single datum in this family has two distinct vanishing-viscosity accumulation paths.

Proof. Start with arbitrary default phases at all depths. At step k, choose a new depth $N_k > N_{k-1}$ with $\nu_{N_k} < \nu_{N_{k-1}}/2$ and

$$\eta_{N_k} < 2^{-k} \min(1, r_1, \dots, r_{k-1}).$$

Shoot its phase to give first-cap output $(-1)^k$ at ν_{N_k}, using Proposition 3.3, and call the resulting datum g_k. Fixed-positive-viscosity continuity gives $r_k > 0$ such that an admissible datum within r_k of g_k changes that output by less than $1/4$. All admissible future phase choices already have protected caps by hypothesis; openness of a derivative lower bound in the uniform topology is not asserted.

The summable phase changes give one limiting datum g_∞; each coordinate is changed at most once and the absolutely convergent series identifies the limit with that phase sequence. For each k,

$$\|g_\infty - g_k\|_X \leq \sum_{l>k} 2^{-l} r_k < r_k.$$

Hence its selected-viscosity output lies within $1/4$ of $(-1)^k$. Relative compactness supplies limiting paths along each parity. Their first-cap phases lie in the disjoint closed $1/4$-neighbourhoods of $+1$ and -1. Corollary 3.2 proves that the total paths are different. Zero circulation and compact support pass by strong L^1 convergence. □

The proposition is an implication with explicit uniform cap and coupling hypotheses. It is not an unconditional Navier–Stokes counterexample. Every intermediate datum retains the full unbounded infinite background; the proof does not assert arbitrary-small macroscopic controllability around a bounded finite prefix.

4 Compactness for one fixed compact zero-circulation datum

The compactness needed for phase separation does not require a new cascade hypothesis. We prove it for the initial-data class used in the construction. Only standard existence of the planar Navier–Stokes solution, its vorticity contraction bounds, and its finite-energy inequality are inputs. These inputs are obtained by smooth approximation of the initial datum; no Euler uniqueness or viscosity-selection principle is used.

Lemma 4.1 (The velocity estimates used below)**.** *Let $p > 2$, $\alpha = 1 - 2/p$, and $w \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Then*

$$\|K * w\|_\infty + \|K * w\|_p \leq C_p(\|w\|_1 + \|w\|_p), \qquad [K * w]_{C^\alpha} \leq C_p \|w\|_p, \qquad \|\nabla K * w\|_p \leq C_p \|w\|_p.$$

If w has compact support and zero integral, then $K * w \in L^2$.

Proof. For the supremum norm split K at radius one: its near part is in $L^{p'}$ since $p' < 2$, and its far part is bounded. For the L^p norm the near part is in L^1 and the far part is in L^p; apply Young's inequality. For a velocity difference at points a distance d apart, the two radius-$2d$ disks contribute at most $C_p\|w\|_p d^\alpha$. On their complement use $|K(x-z) - K(y-z)| \le Cd|x-z|^{-2}$ and Hölder's inequality; the same power d^α follows.

We recall a direct proof of the derivative bound to specify the analytic input. Each component T of $\nabla K*$ has a Fourier multiplier bounded in absolute value, and hence is bounded on L^2 by Plancherel. Its off-origin kernel satisfies $|t(x)| \le C|x|^{-2}$, $|\nabla t(x)| \le C|x|^{-3}$ and has zero angular average; a possible multiple of the identity is harmless. For $f \in L^1 \cap L^2$ decompose at height λ into $g + \sum b_Q$ using maximal dyadic squares with average of $|f|$ larger than λ. The squares are disjoint, and

$$\|g\|_\infty \le 4\lambda, \quad \|g\|_1 \le \|f\|_1, \quad \sum |Q| \le \|f\|_1/\lambda, \quad \int b_Q = 0, \quad \sum \|b_Q\|_1 \le 2\|f\|_1.$$

The L^2 bound controls the level set of Tg by $C\|f\|_1/\lambda$. Cancellation and the kernel derivative estimate give

$$\int_{\mathbb{R}^2\setminus 5Q} |Tb_Q(x)|\,\mathrm{d}x \le \int_Q |b_Q(y)| \int_{\mathbb{R}^2\setminus 5Q} |t(x-y) - t(x-c_Q)|\,\mathrm{d}x\,\mathrm{d}y \le C\|b_Q\|_1.$$

The last inner integral is bounded by $C|y-c_Q| \int_{C\ell(Q)}^\infty r^{-3} r\,\mathrm{d}r$. This proves weak type $(1,1)$, first for finite sums and then by their L^2 convergence. Splitting f above and below a height proportional to λ consequently gives

$$|\{|Tf| > \lambda\}| \le C\left[\lambda^{-1}\int_{|f|>c\lambda} |f| + \lambda^{-2}\int_{|f|\le c\lambda} |f|^2\right].$$

Integration against $s\lambda^{s-1}\,\mathrm{d}\lambda$, with $1 < s < 2$, and Fubini bound the L^s norm: the two integrals produce finite factors $(s-1)^{-1}$ and $(2-s)^{-1}$. The adjoint has the same bounds, so duality with $s = p'$ proves the required L^p estimate. Density identifies the result with the distributional derivative of $K * w$.

Finally, compact support gives a finite first absolute moment. At low frequencies zero circulation implies $|\widehat{w}(\xi)| \le |\xi| \int |x||w(x)|\,\mathrm{d}x$. At high frequencies use $w \in L^2$, which follows by interpolation between L^1 and L^p. The Fourier multiplier $i\xi^\perp/|\xi|^2$ and Plancherel prove $K * w \in L^2$. □

Theorem 4.2 (Strong vanishing-viscosity precompactness)**.** *Let $2 < p < \infty$ and let one fixed real initial vorticity satisfy*

$$\omega_0 \in L^1 \cap L^p, \qquad \operatorname{supp}\omega_0 \subset B_{R_0}, \qquad \int \omega_0 = 0.$$

For every sequence $\nu_n \downarrow 0$, the unforced planar Navier–Stokes vorticities with this same initial datum have a subsequence converging strongly in $C([0,T]; L^1 \cap L^p)$ for each fixed T. Every limit is a renormalized Euler solution with initial vorticity ω_0. Its velocity is bounded, belongs to $L^\infty_t W^{1,p}$, and its vorticity preserves the L^1 and L^p norms and circulation at every time. In particular the Sobolev-velocity and compactness hypotheses needed for fixed-colour first-cap separation hold for this class.

Proof. Write $m = \|\omega_0\|_1$, $P = \|\omega_0\|_p$ and $E = \|K * \omega_0\|_2$. The last quantity is finite by Lemma 4.1. Contraction and the kinetic-energy inequality give uniformly, for $0 < \nu_n \le 1$,

$$\|\omega_n(t)\|_1 \le m, \quad \|\omega_n(t)\|_p \le P, \quad \|u_n(t)\|_2 \le E, \quad \|u_n(t)\|_\infty \le V, \qquad V = C_p(m+P). \tag{7}$$

The velocity lemma also bounds $[u_n(t)]_{C^\alpha}$ and $\|\nabla u_n(t)\|_p$, where $\alpha = 1 - 2/p$.

Local compactness of velocity. Apply the Fourier projection onto divergence-free fields to the velocity equation. It is bounded on every H^s by its bounded matrix multiplier. Since $\|u_n \otimes u_n\|_2 \le VE$, the equation gives

$$\|\partial_t u_n\|_{H^{-2}} \le CE(V+1).$$

For a spatial mollifier η_h, $0 < h \le 1$, $\|\eta_h\|_{H^2} \le Ch^{-3}$ in two dimensions. Hence

$$\|u_n(t) - u_n(s)\|_\infty \le Ch^\alpha + Ch^{-3}|t-s|.$$

Here the first term comes from the spatial Hölder bound and the second from duality of H^{-2} and H^2 after convolution. Taking $h = |t-s|^{1/(3+\alpha)}$ proves uniform time equicontinuity. Arzelà–Ascoli on compact cylinders and a diagonal extraction give

$$u_n \longrightarrow u \quad \text{locally uniformly on } [0,T] \times \mathbb{R}^2.$$

The uniform gradient bound passes weakly to $\nabla u \in L^\infty_t L^p$. Also u is bounded and divergence free. The L^p velocity bound in Lemma 4.1 passes by lower semicontinuity, so $u \in L^\infty_t W^{1,p}$.

Weak vorticity compactness and its tightness. For compact smooth φ the scalar equation gives

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \omega_n \right| \le m(V\|\nabla\varphi\|_\infty + \|\Delta\varphi\|_\infty).$$

A countable dense set of tests in $L^{p'}$, the uniform L^p bound, and Arzelà–Ascoli provide a weakly continuous L^p path ω such that

$$\sup_{t \le T} \left| \int \varphi(\omega_n(t) - \omega(t)) \right| \to 0 \quad \text{for every } \varphi \in L^{p'}. \tag{8}$$

Its initial value is ω_0. Each weak value also has L^1 norm at most m, by testing on finite balls and then enlarging them.

Let χ_R be zero on B_R, one outside B_{2R}, with first and second derivatives bounded by C/R and C/R^2. Kato's inequality and (7) give

$$\sup_{t \le T} \int_{|x|>2R} |\omega_n(t,x)| \,\mathrm{d}x \le \int_{|x|>R} |\omega_0(x)| \,\mathrm{d}x + CTm(V/R + 1/R^2). \tag{9}$$

The same bound passes to $\omega(t)$ by local lower semicontinuity and monotone enlargement of the outer integration region.

The local uniform convergence of u_n and weak vorticity convergence pass the nonlinear drift product in compactly supported tests. The diffusion term is bounded by $\nu_n P \|\Delta\varphi\|_{p'}$ and vanishes. Thus

$$\partial_t \omega + u \cdot \nabla \omega = 0, \qquad \omega(0) = \omega_0.$$

We identify $u = K * \omega$ after establishing strong convergence below; this avoids requiring a preliminary global compactness argument for velocities.

Renormalization and traces at every time. Spatial mollification gives

$$\partial_t \omega_\varepsilon + u \cdot \nabla \omega_\varepsilon = r_\varepsilon,$$

where, using divergence freedom to subtract $\omega(x)$,

$$r_\varepsilon(x) = \int \nabla \eta_\varepsilon(y) \cdot [u(x) - u(x-y)] \, [\omega(x-y) - \omega(x)] \, \mathrm{d}y.$$

The Sobolev difference estimate and Hölder's inequality yield

$$\|r_\varepsilon(t)\|_{p/2} \le C \|\nabla u(t)\|_p \sup_{|y| \le \varepsilon} \|\omega(t, \cdot - y) - \omega(t)\|_p.$$

Translation continuity and dominated convergence imply $r_\varepsilon \to 0$ in $L^1_t L^{p/2}_x$. Since $p/2 > 1$, this gives local space-time L^1 convergence. For every C^1 function β with bounded derivative, the chain rule therefore passes to the renormalized transport equation.

There is no ambiguity about endpoint traces in this argument. For fixed ε the mollified equation is absolutely continuous in time in local smooth spatial norms. Integrate its renormalized equation between *any fixed* times s, t. At each of these fixed times, $\omega_\varepsilon(s) = \eta_\varepsilon * \omega(s) \to \omega(s)$ and $\omega_\varepsilon(t) = \eta_\varepsilon * \omega(t) \to \omega(t)$ strongly in L^p, by the spatial approximate-identity property. The Lipschitz function β passes these endpoint terms in local L^1, while the commutator error vanishes. Consequently the renormalized identity has traces $\beta(\omega(s))$ and $\beta(\omega(t))$ for every s, t, including zero.

For $\beta(0) = 0$, integrate against cutoffs equal to one on B_R and vanishing outside B_{2R}. The flux error between any two times is at most $CTVm\|\beta'\|_\infty / R$. Letting $R \to \infty$ proves

$$\int \beta(\omega(t)) = \int \beta(\omega_0) \quad (0 \le t \le T).$$

To recover the L^p norm, use the convex tangent truncations

$$\beta_M(s) = \begin{cases} |s|^p, & |s| \le M, \\ pM^{p-1}|s| - (p-1)M^p, & |s| > M. \end{cases}$$

They are C^1 with bounded derivative, nonnegative, and increase to $|s|^p$. For $|s| > M$ their derivative with respect to M is $p(p-1)M^{p-2}(|s| - M) \ge 0$. Monotone convergence therefore proves

$$\|\omega(t)\|_p = \|\omega_0\|_p \quad \text{at every } t. \tag{10}$$

Using $\sqrt{s^2 + \varepsilon^2} - \varepsilon$ and then $\varepsilon \downarrow 0$ proves conservation of the L^1 norm. Using $\beta(s) = s$ preserves circulation.

Strong convergence, uniformly in time. For $p \ge 2$ the pointwise Clarkson inequality gives

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \le \frac{\|f\|_p^p + \|g\|_p^p}{2}.$$

Weak time continuity and the constant norm (10) imply $\omega \in C_t L^p$ by this inequality and lower semicontinuity. If $\omega_n \to \omega$ were not uniform strongly in L^p, choose bad times $t_n \to t$. Equation (8) gives $\omega_n(t_n) \rightharpoonup \omega(t)$, while contraction gives

$$\limsup_n \|\omega_n(t_n)\|_p \le P = \|\omega(t)\|_p.$$

The same inequality proves strong convergence to $\omega(t)$; continuity of ω contradicts the bad times. Thus convergence is strong in $C_t L^p$. It is strong in $C_t L^1$ as well: on a fixed ball use Hölder's inequality, and outside it use (9) for both paths.

Finally Lemma 4.1 gives uniform global convergence $u_n = K * \omega_n \to K * \omega$. Their already obtained local limit is therefore exactly $u = K * \omega$. This proves the Euler equation and all claimed regularity and compactness statements. □

Corollary 4.3. *For the compact zero-circulation initial-data class of Theorem 4.2, the relative compactness as $\nu \downarrow 0$ and Sobolev-velocity assumptions in Proposition 3.4 are automatic. The substantive remaining hypotheses there are uniform protected caps and phase shooting for the actual coupled system.*