1 Actual inactive densities and a large amplitude ratio

Fix the smooth radial reference W and $p > 2$. The estimate below uses the actual coloured Navier–Stokes equation and requires no derivative bound on an inactive colour. Its input is a weighted history bound for the current and deeper packet. That input remains part of the interacting active theorem, and is not proved in this section.

For clarity, the amplitude ratio can be chosen large after W and p are fixed. The interval $[4, 5]$ used in the earlier parameter discussion is not imposed here. A large fixed ratio compensates for the ordinary transport-stability constant of the smooth reference; it is not asserted that this constant equals its unstable eigenvalue.

Let P denote angular averaging. Use

$$\begin{aligned} A_l &= A_1\beta^{l-1}, & R_l &= R_1 q^{l-1}, \\ G_l(t,x) &= A_l\left[e^{\nu t R_l^{-2}\Delta}W\right](x/R_l), & & \\ \zeta_l &= w_l - G_l, & h_l &= (I-P)w_l = (I-P)\zeta_l. \end{aligned}$$

For any density v put

$$\|v\|_{X_l} = \frac{\|v\|_1}{A_l R_l^2} + \frac{\|v\|_p}{A_l R_l^{2/p}}, \qquad x_l(t) = \|\zeta_l(t)\|_{X_l}.$$

Assume the initial colours are real, fourfold symmetric, have zero circulation, are supported in $B_{c_0 R_l}$, equal a constant in $B_{a_0 R_l}$, and satisfy

$$\|w_l(0)\|_1 \le C_0 A_l R_l^2, \qquad \|w_l(0)\|_\infty \le C_0 A_l.$$

Here a_0, c_0, C_0 are fixed. For the smooth seeded family these facts follow from the fixed compact derivative collars. The series of colours is absolutely convergent in $L^1 \cap L^p$, and $\beta q < 1$, so its common velocity and coloured evolution are well defined as before.

Choose a fixed broad annulus $\mathcal{A}_l$, containing both derivative collars of $W(x/R_l)$, whose two radii are fixed multiples of R_l.

Proposition 1.1 (Stopped inactive row with explicit parabolic errors)**.** *There are constants $\gamma, C, c > 0$, depending only on the fixed smooth profile, p, and the displayed initial prefactors, with the following property. Fix $j \ge 2$ and*

$$T \le S/A_j, \qquad \kappa_j = \frac{\nu}{A_j R_j^2} \le q^{3/2}, \qquad 1 \le S \le C_1 \log(1/q).$$

Stop at the first time any $i < j$ reaches $x_i = \delta$. Assume $\beta q \le 1/2$, q sufficiently small, and

$$CS(\sqrt{\delta} + \beta q) \le c. \tag{1}$$

In the clock $\theta = A_j t$, every inactive $i < j$ obeys

$$\frac{\mathrm{d}^+ x_i}{\mathrm{d}\theta} \le \gamma\beta^{i-j}x_i + C\sum_{l<i}\beta^{l-j}q^{2(i-l)}x_l + C\sum_{i<l<j}\beta^{l-j}q^{6(l-i)}x_l + f_i(\theta) + E_{j-i}, \tag{2}$$

where

$$f_i(\theta) = \frac{1}{A_j} \left\| \mathbf{1}_{\mathcal{A}_i} \left(\sum_{l \geq j} K * h_l \right) \cdot \nabla G_i \right\|_{X_i} \qquad at\ t = \theta / A_j$$

is the actual current/deep nonradial forcing. With

$$e_m = C e^{CS} \exp\left[-\frac{c q^{-2m}}{\kappa_j S} \right], \qquad m \geq 1,$$

one can take

$$E_k = C\Big[\beta^{-k} e_k + \sum_{r>k} \beta^{-r} e_r + \sum_{1 \leq r < k} \beta^{-r} q^{6(k-r)} e_r + e_k \sum_{1 \leq r < k} \beta^{-r} q^{k-r} \Big]. \tag{3}$$

The sums may be enlarged to infinite sums of nonnegative terms. No small nonradial-density assumption is imposed on colours $l \geq j$.

Proof. The exact radial heat profile is an isolated Navier–Stokes solution. For the actual common velocity u, subtraction gives

$$(\partial_t + u \cdot \nabla - \nu \Delta) \zeta_i = - \left[K * \zeta_i + \sum_{l \neq i} K * w_l \right] \cdot \nabla G_i. \tag{4}$$

Every radial foreign velocity is tangential and annihilates ∇G_i, so the foreign fields on the right can be replaced by $K * h_l$. In particular diffusion produces no separate angular source. If every initial colour is radial, $\zeta_i = h_i = 0$ solves this system.

The scaled Biot–Savart split gives $\|K * \zeta_i\|_\infty \leq C_p A_i R_i x_i$. Heat contraction of the fixed reference gradient in L^1 and L^p, and the scalar Kato inequalities, bound the self term by $\gamma A_i x_i$. For example γ can be chosen as a constant multiple of $C_p(\|\nabla W\|_1 + \|\nabla W\|_p)$. No derivative of ζ_i enters this estimate; the actual nonlinear transport term disappears from the norm inequalities.

We verify the localized foreign-field estimates under the stop. For any fixed index l, the global radial component obeys

$$|u(t,x) \cdot e_r| \leq C A_l |x| + C A_l R_l \beta q. \tag{5}$$

Indeed every radial average has tangential velocity. For $k \leq l$, the fourfold origin-growth bound gives $|K * h_k(x)| \leq C \|h_k\|_\infty |x| \leq C A_k |x|$, and $\sum_{k \leq l} A_k \leq C A_l$. For $k > l$ use $\|K * w_k\|_\infty \leq C A_k R_k$ and $\sum_{k>l} A_k R_k \leq C A_l R_l \beta q$. The same estimate holds for only the foreign radial field. Thus no divergent sum of amplitudes is used, even when κ_j is much smaller than $q^{3/2}$.

Apply the actual outer-core bootstrap to the stopped finite prefix $l \leq j-1$. Its nonradial-density hypothesis follows from $\|h_l\|_1 \leq 2 A_l R_l^2 x_l \leq 2\delta A_l R_l^2$. All colours $k \geq j$ enter that bootstrap through the unrestricted, convergent velocity sum $\sum_{k>l} A_k R_k$. The parameter margin (1) and small q therefore give the individual core estimate

$$\|w_l(t) - c_l\|_{L^\infty(B_{aR_l})} \leq C A_l e^{-c R_l^2/(\nu t)}, \qquad a = a_0/2. \tag{6}$$

The bootstrap error threshold is of order S^{-1}; its improvement follows from $\kappa_j S \ll 1$ just as in the outer-core proof. Retaining the individual radius in that proof, rather than its maximum over l, gives (6).

Fix $c_2 > c_0$. On $B_{c_2R_l}$ the own nonradial velocity is bounded by $C\sqrt{\delta}A_lR_l$. Older constant cores contribute only rotation; their harmonic remainders contribute $C\delta A_lR_lq^2/\beta$ and their core leaks are exponentially small. All inner velocities contribute at most $CA_lR_l\beta q$. Choose q so that this ball is strictly inside every older core probe disk. By (1), the integrated radial drift is less than a fixed fraction of $(c_2-c_0)R_l$. The radial exit estimate for the positive scalar evolution from $|w_l(0)|$ gives

$$\int_{|x|>c_2R_l} |w_l(t,x)|\,\mathrm{d}x \le CA_lR_l^2e^{-cR_l^2/(\nu t)}. \tag{7}$$

For its post-exit fourth moment use $\phi(x) = (R_l^2+|x|^2)^2$. Equation (5) gives $u\cdot\nabla\phi \le CA_l\phi$, while $\nu\Delta\phi \le C\nu R_l^{-2}\phi$. Conditioning at exit from $B_{c_2R_l}$ consequently gives $\mathbb{E}[\phi(X_t)\mid\mathcal{F}_{\text{exit}}] \le CR_l^4e^{C(A_l+\nu/R_l^2)T} \le CR_l^4e^{CS}$. Multiplication by the exit probability proves

$$\int_{|x|>c_2R_l} |x|^4|w_l(t,x)|\,\mathrm{d}x \le CA_lR_l^6e^{CS}e^{-cR_l^2/(\nu t)}. \tag{8}$$

Angular averaging preserves radial integrals. Inside $B_{c_2R_l}$ use $\|h_l\|_1 \le 2A_lR_l^2x_l$; outside use (8). It follows that

$$\int |x|^4|h_l(t,x)|\,\mathrm{d}x \le CA_lR_l^6[x_l(t)+e_{j-l}]. \tag{9}$$

For an inner colour $i<l<j$, take a fixed small $\eta>0$ such that $B_{\eta R_i}$ is inside the initial constant disk of colour i and $c_0q<\eta/4$. In this ball the radial velocity estimate based on index i has the same small integrated bound: the own nonradial term is $C\sqrt{\delta}A_iR_i$, older remainders are controlled by (6), and all inner colours enter by $CA_iR_i\beta q$. Starting the lth positive colour in $B_{c_0R_l}$, radial exit from $B_{\eta R_i}$ therefore gives

$$\int_{|x|>\eta R_i} |w_l(t,x)|\,\mathrm{d}x \le CA_lR_l^2e^{-cR_i^2/(\nu t)}. \tag{10}$$

This receiver-radius estimate is essential.

On $\mathcal{A}_i$, for $l<i$, split h_l into its core and exterior. Equation (6) bounds the core density by $CA_le^{-cR_l^2/(\nu t)}$; the origin-growth estimate bounds its velocity by $CA_l|x|e^{-cR_l^2/(\nu t)}$. The exterior part has mass at most $2A_lR_l^2x_l$ and a fourfold inverse-moment expansion starting at degree three. Thus

$$\frac{|K*h_l(x)\cdot e_r|}{R_i} \le CA_l[q^{2(i-l)}x_l+e_{j-l}]. \tag{11}$$

For $i<l<j$, truncate h_l at radius ηR_i. Every radial truncation has zero circulation because $Ph_l=0$, and fourfold symmetry cancels harmonic moments of degrees one to three. Its exterior field is bounded by $CR_i^{-5}\int|y|^4|h_l(y)|\,\mathrm{d}y$. For the remaining density use (10), $\|h_l\|_\infty \le CA_l$, and Biot–Savart interpolation. This gives

$$\frac{|K*h_l(x)\cdot e_r|}{R_i} \le CA_l\left[q^{6(l-i)}(x_l+e_{j-l})+q^{l-i}e_{j-i}\right]. \tag{12}$$

The fixed exponent c in e_m has been reduced to absorb square roots and polynomial factors.

Away from $\mathcal{A}_i$, ∇G_i is the Gaussian convolution of the compactly supported reference gradient. Its scaled L^1 and L^p tails after multiplication by $|x|+R_i$ are bounded by $Ce^{-cR_i^2/(\nu t)}$. The foreign version of (5) consequently bounds the total omitted forcing by CA_ie_{j-i}. Combining this with the self estimate, (11), (12), and the explicitly retained f_i proves (2)–(3). □

Corollary 1.2 (Inactive density closure under packet history)**.** *Let $\sigma > 0$ be the growth rate of the fixed smooth reference. Choose β sufficiently large that $\gamma/\beta < \sigma/4$, then a fixed $D > \beta$ sufficiently large, then q sufficiently small. Set $w_i = (Dq^6)^{j-i}$ and $b = \delta e^{-\sigma S}$. Assume $x_i(0) \le Mb$, and the packet-history bound*

$$\sup_{\theta \le A_j T} \int_0^\theta e^{\lambda_P(\theta - s)} \max_{i<j} \frac{f_i(s)}{w_i} \, \mathrm{d}s \le C_* \delta, \tag{13}$$

where $\lambda_P < \sigma/2$ is the weighted comparison rate below. Take $S = C_2 \log(1/q)$, with C_2 sufficiently large and fixed, and δ small enough for (1)*. Then the stopped solution remains strictly below its stopping threshold:*

$$x_i(t) \le B + w_i \left[C C_* \delta + o(q^K) \right] < \delta/4, \qquad B \le M \delta e^{-(\sigma - \lambda_B)S},$$

for every $i < j$ and $t \le T$. Here K is any fixed prescribed power, and the constants are independent of depth. This is an actual inactive-density conclusion conditional on (13)*; that packet input is not established by the corollary.*

Proof. Bound each self coefficient in (2) by $\lambda_0 = \gamma/\beta$. This replaces, rather than identifies with, the earlier sharp spectral rate $(\sigma + \varepsilon)/\beta$. The unweighted off-diagonal row sum is at most

$$r_B = \frac{C}{\beta} \left[\frac{\beta q^6}{1 - \beta q^6} + \frac{q^2/\beta}{1 - q^2/\beta} \right].$$

The row sum in the forced weights w_i is at most

$$r_P = \frac{C}{\beta} \left[\frac{\beta}{D - \beta} + \frac{D q^8}{\beta - D q^8} \right].$$

Thus $\lambda_B = \lambda_0 + r_B$ and $\lambda_P = \lambda_0 + r_P$ are both below $\sigma/2$ in the prescribed parameter order. The positive comparison equation splits exactly into initial, packet, and parabolic-error contributions. Its initial part is bounded by $Mbe^{\lambda_B \theta}$, giving the stated B.

For the explicit error put

$$\eta(q) = \sup_{m \ge 1} \frac{e_m}{(\beta D q^6)^m}.$$

This is smaller than every power of q, uniformly when $\kappa_j \le q^{3/2}$ and $S = O(\log(1/q))$. Indeed successive ratios in this supremum are less than $1/2$ for small q, and its first term has Gaussian exponent of order $q^{-7/2}/S$. A direct summation of the four terms of (3) gives

$$\sup_{k \ge 1} \frac{E_k}{(Dq^6)^k} \le C\eta(q) \left[1 + \frac{Dq^6}{1 - Dq^6} + \frac{1}{D - 1} + \frac{\beta q}{1 - \beta q} \right].$$

For example the third sum becomes $D^{-k} \sum_{r<k} \beta^{-r} q^{-6r} e_r \le \eta(q) D^{-k} \sum_{r<k} D^r$. The fourth uses $\sum_{r<k} \beta^{-r} q^{k-r} \le \beta^{-k} \beta q/(1 - \beta q)$. Its weighted convolution is therefore at most $C\eta(q) S e^{\lambda_P S}$, still smaller than every q-power.

The positive-matrix variation of constants and (13) yield the asserted bound. Choose C_2 so $C_2(\sigma - \lambda_B) > 6$; then $B \ll \delta q^6$. Finally $\max_{i<j} w_i = Dq^6$, so small q makes the complete right side less than $\delta/4$, contradicting a first stopping time. The absolute leak envelope has not been called a physical angular source; the exact equation (4) is zero on the all-radial family. In the fixed nonzero seeded family, its error envelope is much smaller even than the prescribed seed b. □

Corollary 1.3 (A static packet bound suffices for inactive density)**.** *For the inactive-density conclusion alone, the packet-history hypothesis* (13) *may be replaced by*

$$\sup_{\theta \le A_j T} \max_{i<j} \frac{f_i(\theta)}{w_i} \le C_* \delta.$$

Fix $C_2 > 6/\sigma$. Choose β, D, and then q, so that

$$C_2(\sigma - \lambda_B) > 6, \qquad C_2 \lambda_P < 1.$$

With $S = C_2 \log(1/q)$, the actual inactive densities still remain below $\delta/4$. Their normalized inward nonradial velocity on the current receiver's fixed collars is

$$O\Big(\delta S q^{8 - C_2 \lambda_P}\Big) + o(q^K) = o(\delta q^6).$$

The static packet bound is an explicit unproved input here. This corollary does not supply a signed transfer or preserve the sharper pre-cap histories needed for phase protection.

Proof. The stated parameter choices are possible since λ_B, λ_P can be made arbitrarily small by taking the fixed β, D large and then q small. The packet contribution to the forced weighted norm is at most $C C_* \delta S e^{\lambda_P S}$. Thus

$$x_i \le B + (Dq^6)^{j-i} \left[C C_* \delta S q^{-C_2 \lambda_P} + o(q^K) \right].$$

The baseline $B = o(\delta q^6)$ and the nearest forced bound $C \delta S q^{6 - C_2 \lambda_P}$ are both below the stopping threshold. The parabolic errors remain smaller than every power of q. For the inward field use the actual older-core multipole estimate:

$$\sum_{i<j} \beta^{i-j} q^{2(j-i)} x_i \le B \frac{q^2}{\beta - q^2} + C C_* \delta S q^{-C_2 \lambda_P} \frac{Dq^8}{\beta - Dq^8} + o(q^K).$$

Since $C_2 \lambda_P < 1$, this is $o(\delta q^6)$. The proof controls magnitudes; it asserts no lower bound on a complex unstable-mode impulse. □