A diffusion cutoff for a planar vorticity cascade: proved estimates and unresolved coupling

Research note

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Abstract

We investigate an infinite, fourfold-symmetric, zero-circulation annular stack as a possible fixed initial datum with more than one vanishing-viscosity limit for the unforced planar Navier–Stokes equations. We prove that diffusion does not erase harmonic moments; construct exactly fourth-moment-neutral annular seeds with a nonzero Euler unstable coordinate; and bound nonlinear fourth-moment creation during one diffusion time, including under the actual foreign drift of an infinite stack. We also prove all-time suppression of an isolated, initially shielded tail of sufficiently small mass relative to viscosity, and a multiplicative comparison for the linear fourth angular mode around the radial heat profile. These results repair a specific failure of a norm-based cutoff argument. They do not establish the requested counterexample: multiplicative viscous stability of the active cascade and suppression under the actual foreign velocity remain unresolved. Conditional statements below are explicitly identified and are not asserted as Navier–Stokes theorems.

1 Problem and conventions

The equation under consideration is

$$\partial_t \omega^\nu + (K * \omega^\nu) \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu, \qquad K(x) = \frac{x^\perp}{2\pi |x|^2}, \qquad x^\perp = (-x_2, x_1), \tag{1}$$

with one initial datum, independent of ν,

$$\omega_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2), \qquad 2 < p < \infty, \qquad \int \omega_0 = 0.$$

The goal is to find two sequences $\nu_n, \widetilde{\nu}_n \downarrow 0$ whose solutions converge to distinct Euler solutions, for example in $C([0,T]; L^1 \cap L^p)$. Every statement in this note concerns the same unforced equation. No viscosity-dependent initial datum is substituted.

Write $z = x_1 + i x_2$, and identify a velocity with $U = u_1 + i u_2$. For a real density f put

$$B_m(f) = \int_{\mathbb{R}^2} \bar{z}^m f(x)\, \mathrm{d}x, \qquad M_k(f) = \int_{\mathbb{R}^2} |x|^k |f(x)|\, \mathrm{d}x.$$

Fourfold symmetry means $f(ix) = f(x)$ in complex notation. The radial projection P_{rad} is the average over all spatial rotations. All unqualified moments are signed; M_k denotes an absolute moment. The usual L^1 well-posedness and positive-time smoothing for (1) are the standard Navier–Stokes input recorded in [1, Theorem 2.2], after rescaling viscosity. Estimates for a coloured scalar are proved directly below.

2 Harmonic moments and the diffusion obstruction

Lemma 2.1. *Suppose f solves*

$$\partial_t f + u \cdot \nabla f = \nu \Delta f, \qquad \operatorname{div} u = 0,$$

and the moments and drift integrals appearing below are finite on compact time intervals. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} B_m(f) = m \int \bar{z}^{m-1} \bar{U} f \, \mathrm{d}x. \tag{2}$$

In particular $B_m(e^{\nu t \Delta} f_0) = B_m(f_0)$ whenever the initial absolute mth moment is finite.

Proof. The polynomial $\bar{z}^m$ is harmonic, and $u \cdot \nabla \bar{z}^m = m \bar{z}^{m-1} \bar{U}$. Testing the equation gives (2). To justify a polynomial test, first multiply it by a cutoff equal to one on B_R and supported in B_{2R}. Its diffusion errors are bounded by tails of absolute moments of orders $m-2$ and m, with the corresponding powers of R^{-1} and R^{-2}; its drift errors are bounded by the assumed integrable drift tails. Let $R \to \infty$ in the integrated identity. For the heat equation polynomial absolute moments remain finite, as follows directly from the Gaussian kernel, and the right-hand side is zero. □

Small L^r norms after diffusion therefore do not imply a small exterior fourth-moment signal. Before heat reaches the next radius, the exterior velocity can remain essentially unchanged.

Lemma 2.2. *Let $f_0 \in L^1$ be supported in B_r, put $m = \|f_0\|_1$, and $d = R - r > 0$. For $|x| \geq R$ and $0 < \nu t \leq d^2$,*

$$|K * e^{\nu t \Delta} f_0(x) - K * f_0(x)| \leq \frac{Cm}{d} \exp\left(-\frac{d^2}{C\nu t} \right).$$

Proof. Let $H_s(x) = (4\pi s)^{-1} e^{-|x|^2/(4s)}$. Since $\Delta K = \nabla^\perp \delta_0$ distributionally,

$$K * H_s - K = \int_0^s \nabla^\perp H_a \, \mathrm{d}a.$$

For $|x - y| \geq d$, the Gaussian derivative is bounded by $Ca^{-3/2} e^{-d^2/(8a)}$. Its integral on $(0, \nu t)$ is bounded by $Cd^{-1} e^{-d^2/(C\nu t)}$. Integrate against $|f_0(y)|$. □

Lemma 2.3 (Shielding of the initial fourth angular mode)**.** *Suppose f_4 is real, bounded, supported in B_r, and has only angular Fourier modes 4 and -4. If $B_4(f_4) = 0$, then $v_4 = K * f_4$ vanishes outside B_r. If $\|f_4\|_\infty \leq CA$, then for $R \geq 2r$,*

$$\int_0^T \frac{|e^{\nu t \Delta} v_4(x)|}{R} \, \mathrm{d}t \leq CA(r/R)^3 T \exp\left(-\frac{cR^2}{\nu T} \right), \qquad |x| \geq R. \tag{3}$$

Proof. The exterior complex-kernel expansion is

$$\overline{K * f_4}(z) = -\frac{i}{2\pi} \sum_{k \geq 0} z^{-k-1} \int y^k f_4(y) \, \mathrm{d}y, \qquad |z| > r.$$

Angular orthogonality permits only $k = 4$, and reality makes that coefficient the conjugate of $B_4(f_4)$, hence zero. The near/far decomposition of Biot–Savart gives $\|v_4\|_\infty \leq CAr$, and its support gives $\|v_4\|_1 \leq CAr^3$. Heat convolution at distance $R - r$ is bounded by

$$CAr^3 (\nu t)^{-1} e^{-(R-r)^2/(8\nu t)}.$$

Using $\int_0^T t^{-1} e^{-a/t} \, \mathrm{d}t \leq (T/a) e^{-a/T}$ proves (3). Heat commutes with K. □

Remark 2.4. A nonlinear annular graph generally also has angular modes $8, 12, \dots$. The lemma shields its fourth angular component only; higher modes and subsequently generated fourth moments require additional estimates.

3 Exactly neutral seeds retain the Euler unstable coordinate

Set $\rho^2 = 3/5$ and consider

$$W = -\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}.$$

Its circulation is zero. Its angular speed at the two boundaries is $\Omega(\rho) = -1/5$, $\Omega(1) = 0$. For two nested domains of these reference areas define

$$Q_i = \frac{1}{2\pi r_i^6} \int_{D_i} \bar{z}^4 \, \mathrm{d}x, \qquad (r_1, r_2) = (\rho, 1), \qquad a = \frac{1}{2}(Q_1 + iQ_2/\rho).$$

Proposition 3.1. *For every sufficiently small complex α, there are nested smooth, fourfold-symmetric domains $D_1(\alpha), D_2(\alpha)$ of areas $\pi\rho^2, \pi$ such that*

$$W_\alpha = -\mathbf{1}_{D_1(\alpha)} + \rho^2 \mathbf{1}_{D_2(\alpha)}, \quad \int W_\alpha = 0, \quad B_4(W_\alpha) = 0, \quad a(W_\alpha) = \alpha.$$

The linearized Euler fourth-mode block has eigenvalues $\pm\sigma + i\tau$, where $\sigma = \rho^5/2 > 0$, $\tau = 3/10$, and a is its unstable coordinate.

Proof. Use the radial graphs

$$r_i(\theta) = r_i h_i(\theta), \qquad h_i(\theta) = \sqrt{1 - 2|z_i|^2} + z_i e^{4i\theta} + \bar{z}_i e^{-4i\theta}.$$

For small z_i they are positive. Since the angular average of h_i^2 is one, their areas are exactly the stated reference areas. Polar integration gives

$$Q_i(z_i) = \frac{1}{12\pi} \int_0^{2\pi} h_i(\theta)^6 e^{-4i\theta} \, \mathrm{d}\theta = z_i + O(|z_i|^2).$$

The real analytic inverse function theorem therefore makes (Q_1, Q_2) local coordinates on this four-real-dimensional family. Prescribe

$$Q_1 = \frac{2\alpha}{1 + i\rho^3}, \qquad Q_2 = \rho^4 Q_1. \tag{4}$$

The exact identity

$$B_4(W_\alpha) = 2\pi(-\rho^6 Q_1 + \rho^2 Q_2)$$

then gives neutrality, and (4) gives $a = \alpha$. Smallness preserves nesting since $\rho < 1$.

For completeness, the linearization is computed directly. If the relative normal displacement on circle r_l is $h_l e^{im\theta}$ and its vorticity coefficient is c_l, its first variation is $c_l r_l h_l \delta(r - r_l) e^{im\theta}$, with $c_1 = -1, c_2 = \rho^2$. Solving $\Delta\psi = \delta\omega$ gives the stream coefficient

$$-\frac{c_l r_l^2 h_l}{2m} \begin{cases} (r/r_l)^m, & r \le r_l, \\ (r_l/r)^m, & r \ge r_l. \end{cases}$$

For $u = \nabla^\perp \psi$, $u_r = -r^{-1}\partial_\theta \psi$. The normal evolution $\dot{h}_i = -im\Omega(r_i)h_i + \delta u_r(r_i)/r_i$ therefore has matrix

$$L_m = \frac{i}{2}\begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

For $m = 4$, its eigenvectors $v_+ = (1, -i\rho)$ and $v_- = (1, i\rho)$ have eigenvalues $\sigma + i\tau$ and $-\sigma + i\tau$. To first order $Q_i = h_i$, and decomposition $Q = av_+ + bv_-$ gives the stated unstable row. □

Corollary 3.2. *Along the classical Euler patch evolution starting at W_α,*

$$\frac{\mathrm{d}}{\mathrm{d}t} B_4(W_\alpha(t))\Big|_{t=0} = 2\sigma\, 2\pi(-\rho^6 - i\rho^3)\alpha + O(|\alpha|^2).$$

Thus heat neutrality does not eliminate instantaneous Euler fourth-mode growth.

Proof. The moment row on $v_\pm$ is $b_\pm = 2\pi(-\rho^6 \mp i\rho^3)$. Neutrality imposes stable coefficient $b = -b_+\alpha/b_- + O(|\alpha|^2)$. The linear moment derivative is $b_+[(\sigma+i\tau)-(-\sigma+i\tau)]\alpha$. The quadratic remainder can be justified without invoking a contour linearization theorem: symmetrize the self-interaction in (2) as in the next section. The resulting kernel is locally bounded; its linear radial-background potential is Lipschitz near the two circles. The radial-graph area integrals have the displayed first variations with quadratic remainders, while the product of two vorticity variations has L^1 norm $O(|\alpha|^2)$. □

4 A rigorous estimate for one diffusion time

Proposition 4.1. *Let a real coloured scalar satisfy*

$$\partial_t f + [\Omega(t)x^\perp + K * f + b(t,x)] \cdot \nabla f = \nu \Delta f, \qquad \operatorname{div} b = 0, \quad |b(t,x)| \le L(t)|x|.$$

Assume $\Omega \in L^1_{\mathrm{loc}}$, $L \ge 0$ measurable, initial support in B_r, $B_4(f_0) = 0$, and

$$m = \|f_0\|_1 \le C_0 A r^2, \qquad \|f_0\|_\infty \le C_0 A.$$

Put $\tau_r = r^2/\nu$, $\mathrm{Re} = Ar^2/\nu$, and $\Lambda = \int_0^{\tau_r} L(t)\,\mathrm{d}t$. If $\mathrm{Re}, \Lambda \le 1$, then

$$\sup_{t \le \tau_r} M_4(f(t)) \le C m r^4, \qquad \sup_{t \le \tau_r} |B_4(f(t))| \le C A r^6 (\mathrm{Re} + \Lambda),$$

where C depends only on C_0.

Proof. The maximum principle gives $\|f(t)\|_1 \le m$ and $\|f(t)\|_\infty \le C_0 A$. Splitting the Biot–Savart integral at an optimizing radius gives

$$\|K * f(t)\|_\infty \le C\sqrt{m\,\|f(t)\|_\infty} \le CAr =: V.$$

Use bounded increasing radial weights

$$\phi_R(x) = \frac{|x|^4}{(1 + |x|^2/R^2)^2}.$$

Direct differentiation shows

$$|\nabla \phi_R| \le 4\phi_R^{3/4}, \quad |x|\,|\nabla \phi_R| \le 4\phi_R, \quad \Delta \phi_R \le 16 \phi_R^{1/2}.$$

Rotation contributes zero. Kato's inequality, first with an additional large compact cutoff, gives for $Y_R = \int \phi_R |f|$,

$$Y_R' \le 4Vm^{1/4}Y_R^{3/4} + 4L(t)Y_R + 16\nu m^{1/2}Y_R^{1/2}.$$

The added cutoff errors vanish: b has at most linear growth, the self velocity is bounded, and f is integrable. With $s = t/\tau_r$ and $y_R = Y_R/(mr^4)$, this becomes

$$y_R' \le C\,\mathrm{Re}\, y_R^{3/4} + 4\tau_r L(\tau_r s) y_R + 16 y_R^{1/2}, \qquad y_R(0) \le 1.$$

Use $y^a \le 1 + y$ for $0 < a < 1$ and Gronwall to obtain $y_R \le C$ on $[0, 1]$. Monotone convergence gives $M_4 \le Cmr^4$, and Hölder gives $M_2 \le Cmr^2$.

In complex notation,

$$\overline{K * f}(z) = -\frac{i}{2\pi} \int \frac{f(y)}{z - y}\,\mathrm{d}y.$$

The original double integral is absolutely convergent, since $|x|^{-1} * |f|$ is bounded and M_3 is finite. Symmetrization yields

$$B_{4,\mathrm{self}}' = -\frac{i}{\pi} \iint \frac{\bar{z}^3 - \bar{y}^3}{z - y} f(z) f(y)\,\mathrm{d}z\,\mathrm{d}y. \tag{5}$$

The quotient has modulus at most $|z|^2 + |z||y| + |y|^2$, so $|B_{4,\mathrm{self}}'| \le CmM_2$. The foreign term is bounded by $4L(t)M_4$, and rotation contributes $-4i\Omega(t)B_4$. Polynomial cutoffs are justified by the proved fourth moment: errors are tails of M_2, M_3, M_4. Removing the rotational phase and integrating gives

$$|B_4(t)| \le Cm^2 r^2 \tau_r + Cmr^4 \Lambda \le CAr^6(\mathrm{Re} + \Lambda).$$

□

The proposition is conditional on a bound for the *actual* foreign field. Diffusion removes material holes and compact supports. A central-hole Taylor expansion for inviscid colours does not establish the global hypothesis $|b(t, x)| \le L(t)|x|$ for viscous colours.

5 A coupled short-window estimate from fourfold symmetry

The global foreign-field hypothesis in Proposition 4.1 can be verified during one diffusion time for the actual coloured stack. The key point is that fourfold symmetry improves the velocity growth estimate at the fixed origin. This improvement does not require a material hole.

Lemma 5.1 (Linear growth at the origin)**.** *Let $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be invariant under the quarter turn $z \mapsto iz$. There is a universal constant C_4 such that*

$$|K * f(x)| \le C_4 \|f\|_\infty |x| \qquad (x \in \mathbb{R}^2).$$

Zero circulation is not needed for this estimate.

Proof. Write $U = K * f$ in complex notation. For $r = |z| > 0$, split

$$U(z) = \frac{i}{2\pi} \int_{|y| \le 2r} \frac{f(y)}{\bar{z} - \bar{y}}\,\mathrm{d}y + \frac{i}{2\pi} \int_{|y| > 2r} \frac{f(y)}{\bar{z} - \bar{y}}\,\mathrm{d}y.$$

The first term is bounded by $C\|f\|_\infty r$, using

$$\int_{|z-y|\le 3r} |z-y|^{-1}\,\mathrm{d}y = 6\pi r.$$

In the second term the absolutely convergent expansion is

$$\frac{1}{\bar z-\bar y} = -\sum_{n\ge 0}\frac{\bar z^n}{\bar y^{n+1}}.$$

Each inverse moment over $\{|y|>2r\}$ vanishes unless $n+1$ is a multiple of four: a quarter turn multiplies its integrand by i^{n+1} while preserving f and the integration region. Thus the surviving powers are precisely $n=3,7,11,\dots$. Their absolute sum is at most

$$Cr^3\int_{|y|>2r}\frac{|f(y)|}{|y|^4}\,\mathrm{d}y \le C\|f\|_\infty r^3\int_{2r}^\infty s^{-3}\,\mathrm{d}s \le C\|f\|_\infty r.$$

The majorant $C|f(y)|/|y|$ justifies termwise integration. The velocity is continuous. Rotational covariance gives $U(0)=0$, which also proves the estimate at the origin. □

Remark 5.2. This is a growth estimate relative to the fixed origin, not a global Lipschitz estimate for the velocity. It therefore does not contradict the usual logarithmic modulus for arbitrary pairs of spatial points. The cancellation through degree two is exactly what removes that logarithmic loss here.

Lemma 5.3 (Smoothing independent of the drift)**.** *Let $S_v(t,s)$ be the solution operator for*

$$\partial_t f + v(t,x)\cdot\nabla f = \nu\Delta f, \qquad \operatorname{div} v = 0,$$

with bounded drift, positive viscosity, and the usual weak solution defined by approximation. Then

$$\|S_v(t,s)f\|_\infty \le \frac{C_N}{\nu(t-s)}\|f\|_1 \qquad (t>s),$$

where C_N is universal and independent of the drift.

Proof. The maximum principle gives positivity and L^1 contraction. The two-dimensional Nash inequality is

$$\|h\|_2^4 \le C\|h\|_1^2\|\nabla h\|_2^2.$$

For completeness, split the Fourier integral at radius Q: the low frequencies are bounded by $CQ^2\|h\|_1^2$, and the high frequencies by $Q^{-2}\|\nabla h\|_2^2$. Optimizing Q gives the displayed inequality. The L^2 energy identity and L^1 contraction consequently give, with $E(t)=\|f(t)\|_2^2$ and $m=\|f(s)\|_1$,

$$E'(t) = -2\nu\|\nabla f(t)\|_2^2 \le -\frac{c\nu}{m^2}E(t)^2.$$

Thus

$$\|S_v(t,s)\|_{L^1\to L^2} \le C[\nu(t-s)]^{-1/2}.$$

The backward adjoint, after time reversal, has divergence-free drift $-v$ and the same diffusion. Applying the same estimate and duality gives the identical $L^2\to L^\infty$ bound. Compose the two bounds at the midpoint of $[s,t]$. Approximation and positivity give the result for signed initial data as well. □

Fix geometric amplitudes and radii

$$A_j = A_1\beta^{j-1}, \qquad R_j = R_1 q^{j-1}, \qquad \beta > 1, \quad 0 < q < 1, \quad \beta q < 1.$$

The indexing convention for R_1 is immaterial. Put

$$r_* := \beta q^2 < 1, \qquad L_q := \log(1/q), \qquad \Theta_{\beta,q} := \frac{1}{\beta - 1} + \frac{r_*}{1 - r_*} + \frac{2L_q r_*}{(1 - r_*)^2}.$$

Proposition 5.4 (Foreign drift over one diffusion time)**.** *Consider finitely or countably many real coloured scalars w_l transported and diffused by the common Navier–Stokes velocity*

$$\partial_t w_l + u^\nu \cdot \nabla w_l = \nu \Delta w_l, \qquad u^\nu = K * \sum_l w_l.$$

Assume each colour is fourfold symmetric and

$$\|w_l(0)\|_1 \le C_0 A_l R_l^2, \qquad \|w_l(0)\|_\infty \le C_0 A_l.$$

For the countable case assume the initial series converges absolutely in $L^1 \cap L^p$, $p > 2$, and use its mild Navier–Stokes solution and the associated linear coloured evolutions. For a fixed j set

$$b_j := \sum_{l \ne j} K * w_l, \qquad \tau_j := R_j^2/\nu, \qquad \mathrm{Re}_j := A_j R_j^2/\nu.$$

Then b_j is divergence free and there is a nonnegative measurable $L_j(t)$ with

$$|b_j(t,x)| \le L_j(t)|x| \quad (t > 0), \qquad \int_0^{\tau_j} L_j(t)\,\mathrm{d}t \le C_* \Theta_{\beta,q}\,\mathrm{Re}_j\,. \tag{6}$$

Here C_* depends only on C_0 and universal kernel and Nash constants, and is independent of viscosity, j, and the number of colours.

Proof. The maximum principles and Lemma 5.3 give

$$\|w_l(t)\|_\infty \le C_1 A_l \min\{1, R_l^2/(\nu t)\}, \qquad C_1 = C_0 \max\{1, C_N\}.$$

Lemma 5.1 therefore proves the spatial bound in (6) with

$$L_j(t) = C_* \sum_{l \ne j} A_l \min\{1, \tau_l/t\}, \qquad C_* = C_4 C_1, \quad \tau_l = R_l^2/\nu.$$

For $t > 0$ the series is finite: only finitely many τ_l exceed t, and the remaining summands are bounded by $A_l R_l^2/(\nu t)$, whose sum converges. The value at $t = 0$ need not be finite and is irrelevant to the integral.

For $l < j$, $\tau_l > \tau_j$, so

$$\sum_{l<j} \int_0^{\tau_j} A_l \min\{1, \tau_l/t\}\,\mathrm{d}t = \tau_j \sum_{l<j} A_l \le \frac{\mathrm{Re}_j}{\beta - 1}.$$

For $l = j + m$, $m \ge 1$, $\tau_l = q^{2m}\tau_j$, and direct integration gives

$$\int_0^{\tau_j} A_l \min\{1, \tau_l/t\}\,\mathrm{d}t = A_l \tau_l [1 + \log(\tau_j/\tau_l)] = \mathrm{Re}_j\, r_*^m (1 + 2mL_q).$$

The geometric identities

$$\sum_{m\ge1} r_*^m = \frac{r_*}{1-r_*}, \qquad \sum_{m\ge1} m r_*^m = \frac{r_*}{(1-r_*)^2}$$

prove the asserted integral bound. Tonelli's theorem applies to these nonnegative estimates, so the argument is uniform also for the infinite stack; no sum of the divergent initial amplitudes $\sum_l A_l$ is taken.

For clarity, the common coloured decomposition causes no circularity. Start with the mild Navier–Stokes velocity of the prescribed total datum; it is bounded because $p > 2$. Solve the linear transport–diffusion equations with that drift. Their $L^1 \cap L^p$ contractions make the colour series converge absolutely, and its sum solves the same scalar equation with the same initial datum as the total vorticity. Linear uniqueness identifies the sum with that vorticity. Rotational covariance and uniqueness preserve each colour's fourfold symmetry. Moreover,

$$\|K * w_l(t)\|_\infty \le C\sqrt{\|w_l(t)\|_1 \, \|w_l(t)\|_\infty} \le C A_l R_l,$$

and $\sum_l A_l R_l < \infty$ since $\beta q < 1$. Thus the velocity series, including every foreign series, converges uniformly and defines the stated divergence-free drift. □

Corollary 5.5 (Coupled creation of a neutral fourth moment)**.** *Under Proposition 5.4, assume in addition that* $\operatorname{supp} w_j(0) \subset B_{R_j}$ *and* $B_4(w_j(0)) = 0$*. If*

$$\mathrm{Re}_j \le \min\{1, (C_*\Theta_{\beta,q})^{-1}\},$$

then

$$\sup_{0\le t\le\tau_j} M_4(w_j(t)) \le C A_j R_j^6, \qquad \sup_{0\le t\le\tau_j} |B_4(w_j(t))| \le C(1 + C_*\Theta_{\beta,q}) A_j R_j^6 \,\mathrm{Re}_j\,.$$

The constants are independent of stack depth. In particular the bound is uniform for $\beta \in [4,5]$ *and* $0 < q \le q_0$ *with sufficiently small fixed* q_0*.*

Proof. Apply Proposition 4.1 with $f = w_j$, $A = A_j$, $r = R_j$, $\Omega = 0$, and the actual foreign drift b_j. Its hypotheses follow from (6); its constants depend only on the initial prefactor C_0. The mass bound gives the first conclusion, and $\mathrm{Re}_j + \Lambda_j \le (1 + C_*\Theta_{\beta,q})\,\mathrm{Re}_j$ gives the second. The displayed expression for $\Theta_{\beta,q}$ remains bounded on the specified parameter range. □

Remark 5.6 (A fixed support prefactor). For annular graphs one may instead have $\operatorname{supp} w_j(0) \subset B_{c_0 R_j}$ with a fixed $c_0 \ge 1$. Provided $c_0 q < 1$, the same calculation up to $c_0^2\tau_j$ replaces $\Theta_{\beta,q}$ by

$$\Theta_{\beta,q,c_0} = \frac{c_0^2}{\beta-1} + \frac{(1 + 2\log c_0) r_*}{1-r_*} + \frac{2L_q r_*}{(1-r_*)^2}.$$

Apply Proposition 4.1 with radius $c_0 R_j$ and require $c_0^2\,\mathrm{Re}_j \le 1$ and $C_*\Theta_{\beta,q,c_0}\,\mathrm{Re}_j \le 1$. Its conclusions on the shorter interval $[0,\tau_j]$ have the same form as Corollary 5.5, with constants depending also on c_0. Thus a fixed geometric support prefactor entails no loss of uniformity in depth.

Remark 5.7 (Scope of the estimate). This proves an actual coupled estimate during the first diffusion time of a sufficiently cold, initially moment-neutral colour. It neither controls that colour for the entire active-source preparation window nor restores the separated-source exterior q^6 gain. Those later-time and spatial impulse estimates are still required for the proposed counterexample.

6 An isolated small-mass tail with no initial fourth moment

This section concerns an isolated Navier–Stokes evolution. A material colour inside a larger solution has an additional foreign drift, so the result below does not by itself establish suppression of a coupled cascade's inactive tail.

We use

$$K(x) = \frac{x^{\perp}}{2\pi|x|^2}, \qquad B_4(f) = \int_{\mathbb{R}^2} \bar{z}^4 f(z)\, dz, \qquad z = x_1 + ix_2.$$

Let P_{rad} denote angular averaging. A function is quarter-turn invariant if $f(ix) = f(x)$ after identifying $\mathbb{R}^2$ with $\mathbb{C}$.

Proposition 6.1 (Isolated-tail suppression)**.** *There are constants $c, C > 0$ with the following property. Suppose $\nu, r > 0$, $p > 2$, and $f_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ is real, quarter-turn invariant, and supported in B_r. Assume*

$$\int f_0 = 0, \qquad B_4(f_0) = 0, \qquad m := \|f_0\|_1 \le c\nu.$$

Let f solve the unforced planar Navier–Stokes vorticity equation

$$\partial_t f + (K * f) \cdot \nabla f = \nu \Delta f, \qquad f(0) = f_0.$$

Then

$$\sup_{t \ge 0} |B_4(f(t))| \le C m r^4 \frac{m}{\nu}. \tag{7}$$

Its nonradial part $h = (I - P_{\mathrm{rad}})f$ satisfies

$$\|h(t)\|_1 \le Cm \left[\left(1 + \frac{\nu t}{r^2}\right)^{-3} + \frac{m}{\nu} \left(1 + \frac{\nu t}{r^2}\right)^{-2} \right], \qquad t \ge 0. \tag{8}$$

The constants do not depend on an initial L^∞ bound or on a decomposition of f_0 into colours.

We specify the weighted linear estimates used in the proof. Put

$$b(y) = (1 + |y|^2)^{1/2}, \qquad \|q\|_M = \left\| b^M q \right\|_2, \qquad M = 10, \qquad \mathcal{L} = \Delta + \tfrac{1}{2} y \cdot \nabla + 1.$$

For $\alpha \in \mathbb{N}^2$ write

$$\mathcal{G}(y) = \frac{1}{4\pi} e^{-|y|^2/4}, \qquad H_\alpha = (-2)^{|\alpha|} \mathcal{G}^{-1} \partial^\alpha \mathcal{G}, \qquad \phi_\alpha = \frac{(-1)^{|\alpha|}}{\alpha!} \partial^\alpha \mathcal{G}.$$

Thus H_α is monic, the two families are dual under integration, and $\mathcal{L}\phi_\alpha = -|\alpha|\phi_\alpha/2$. Define

$$P_k q = \sum_{|\alpha| \le k} \left(\int H_\alpha q \right) \phi_\alpha, \qquad Q_k = I - P_k.$$

In particular, $P_k q = 0$ is equivalent to the vanishing of all ordinary moments of q through degree k. These projections commute with rotations and with the semigroup $S(\tau) = e^{\tau \mathcal{L}}$.

Lemma 6.2 (Weighted semigroup input)**.** *For $k = 1, 3, 5$ one has*

$$\|S(\tau)Q_k q\|_M \le C_k e^{-(k+1)\tau/2} \|q\|_M , \qquad \tau \ge 0. \tag{9}$$

For $k = 3, 5$ and vector fields F with $b^M F \in L^{4/3}$,

$$\|S(\tau)Q_k \operatorname{div} F\|_M \le C_k (1 - e^{-\tau})^{-3/4} e^{-(k+1)\tau/2} \left\| b^M F \right\|_{4/3} , \qquad \tau > 0. \tag{10}$$

The finite projections of $\operatorname{div} F$ *are defined by*

$$\int H_\alpha \operatorname{div} F = -\int \nabla H_\alpha \cdot F.$$

Proof. Estimate (9) is Proposition A.2(b) of Gallay–Wayne [1]: in dimension two its hypothesis is $M > k + 2$, which holds for each stated k. Their Proposition A.5, with input exponent $4/3$, output exponent 2, and one derivative, gives the short-time exponent $1/(4/3) - 1/2 + 1/2 = 3/4$. The identity $S(\tau)\partial_j = e^{-\tau/2}\partial_j S(\tau)$ follows directly from the Gaussian formula for S.

For $0 < \tau \le 1$, the weighted short-time derivative estimate and the bounded finite moment projections prove (10). The required projection coefficients are bounded by weighted Hölder, because ∇H_α has degree at most four and $M = 10$. For $\tau \ge 1$, first apply $S(1/2)$ to $Q_k \operatorname{div} F$, obtaining a weighted L^2 norm bounded by $C_k \left\| b^M F \right\|_{4/3}$. This smoothed function has all moments through degree k zero. Apply (9) for the remaining time $\tau - 1/2$. This proves (10). □

Proof of Proposition 6.1. First rescale

$$x = ry, \qquad t = \frac{r^2}{\nu} s, \qquad f(t, x) = \frac{\nu}{r^2} F(s, y).$$

The rescaled viscosity and support radius are one, and $\|F(0)\|_1 = m/\nu$. We prove the statement with $r = \nu = 1$, writing m for this rescaled mass and requiring m to be sufficiently small. Global L^1 existence, uniqueness, smoothing, and symmetry preservation are those of the planar vorticity theory, for example Theorem 2.2 in [1].

Set

$$G(t) = e^{t\Delta} P_{\mathrm{rad}} f_0, \qquad \rho(t) = f(t) - G(t).$$

Polynomial absolute moments are finite on every compact time interval: the $L^1 \cap L^p$ contractions, $p > 2$, give a bounded velocity, and Kato's inequality with truncated polynomial weights gives this assertion by Gronwall. This preliminary finiteness argument does not use any moment conservation. Below we improve its constants using Nash smoothing. The radial heat solution G is an exact Navier–Stokes solution, because its velocity is tangential. Both f and G have zero circulation. A self-interacting zero-circulation vorticity has constant signed second radial moment:

$$\frac{d}{dt} \int |x|^2 f(x) = 2 \int x \cdot (K * f)(x) f(x) + 4 \int f = 0. \tag{11}$$

Indeed symmetrizing the double integral in its first term replaces x by $(x - y)/2$, and $(x - y) \cdot K(x - y) = 0$. Quarter-turn invariance makes the symmetric second-moment matrix isotropic, and the half-turn contained in this symmetry makes all odd moments zero. Since radial averaging preserves the second radial moment, (11) proves

$$\int x^\alpha \rho(t, x)\, dx = 0, \qquad |\alpha| \le 3, t \ge 0. \tag{12}$$

We record estimates for the first diffusion interval. Vorticity contraction and Nash smoothing give

$$\|f(t)\|_1 \le m, \qquad \|f(t)\|_\infty \le \frac{Cm}{t}, \qquad \|K * f(t)\|_\infty \le \frac{Cm}{\sqrt{t}}, \qquad t > 0.$$

For completeness, the two-dimensional Nash inequality $\|a\|_2^4 \le C\,\|a\|_1^2\,\|\nabla a\|_2^2$, combined with the scalar L^2 identity and L^1 contraction, gives the L^1-to-L^2 bound. Applying the same argument to the adjoint divergence-free drift gives an L^2-to-L^∞ bound by duality. Composition gives the displayed L^∞ estimate; the elementary Biot–Savart split gives its velocity counterpart.

The Kato inequality, tested against b^{2M}, yields

$$\frac{d}{dt}\int b^{2M}|f(t)| \le C_M\left(1+\frac{m}{\sqrt{t}}\right)\int b^{2M}|f(t)|.$$

Here $|\nabla b^{2M}| + |\Delta b^{2M}| \le C_M b^{2M}$. The initial support is in B_1, so Gronwall on $0 \le t \le 1$ gives

$$\sup_{0\le t\le 1}\int b^{2M}|f(t)| \le C_M m, \qquad \|f(1)\|_M + \|G(1)\|_M \le C_M m. \tag{13}$$

The L^2 assertion follows by multiplying the corresponding weighted L^1 bound by $\|f(1)\|_\infty \le Cm$. The same argument applies to G. All polynomial tests here and below can first be truncated. The weighted bounds justify removal of the truncations, and approximation of f_0 justifies the assertions for unbounded initial vorticity.

Every fourth-degree ordinary moment of $\rho(1)$ is of size $O(m^2)$. Initially they all vanish: on quarter-turn invariant functions the degree-four tensor is determined by the radial fourth moment and by the real and imaginary parts of B_4. Radial projection preserves the first quantity and the hypothesis $B_4(f_0) = 0$ removes the latter two. For a homogeneous fourth-degree polynomial P, the viscous term in the evolution of $\int P\rho$ vanishes by (12), since ΔP has degree two. Its self-advection contribution equals

$$\frac{1}{2}\iint [\nabla P(x) - \nabla P(y)]\cdot K(x-y)f(x)f(y)\,dx\,dy.$$

Its absolute value is bounded by $Cm\int |x|^2|f(x)| \le Cm^2$ on $0 \le t \le 1$: the numerator is at most $C_P(|x|^2+|y|^2)|x-y|$, cancelling the kernel singularity. The radial solution has zero advection. Consequently

$$\|P_4\rho(1)\|_M \le Cm^2, \qquad |B_4(f(1))| \le Cm^2. \tag{14}$$

For physical times $t \ge 1$ use

$$\tau = \log t, \qquad y = x/\sqrt{t}, \qquad w(\tau,y) = tf(t,x), \quad H(\tau,y) = tG(t,x), \quad R = w - H.$$

Then H is radial and

$$\|H(\tau)\|_M \le Cme^{-\tau}. \tag{15}$$

Indeed its circulation and first moments vanish, so the $k = 1$ semigroup estimate applies to $G(1)$. Writing $V_a = K * a$, the perturbation equation is

$$R_\tau = \mathcal{L}R - \operatorname{div}\mathcal{F}, \qquad \mathcal{F} = V_H R + V_R H + V_R R. \tag{16}$$

It has $P_3 R = 0$ by (12). The nonlinear divergence in (16) likewise has zero moments through degree three. For its second radial moment this follows by symmetrizing the H–R cross terms

and the R–R term as in (11); its other low moments vanish by circulation and quarter-turn symmetry.

The Biot–Savart estimate $\|V_a\|_4 \le C\,\|a\|_{4/3}$ and weighted Hölder imply

$$\left\|b^M V_a b\right\|_{4/3} \le C\,\|a\|_M\,\|b\|_M\,. \tag{17}$$

In particular, Duhamel and (10) give

$$\begin{aligned}\|R(\tau)\|_M &\le Cme^{-2\tau}\\ &\quad + C\int_0^\tau (1-e^{-(\tau-s)})^{-3/4}e^{-2(\tau-s)}[me^{-s}\,\|R(s)\|_M + \|R(s)\|_M^2]\,\mathrm{d}s.\end{aligned}$$

The quantities

$$\int_0^\tau (1-e^{-(\tau-s)})^{-3/4}e^{-s}\,\mathrm{d}s, \qquad \int_0^\tau (1-e^{-(\tau-s)})^{-3/4}e^{-2s}\,\mathrm{d}s$$

are bounded uniformly in $\tau \ge 0$. A continuity bootstrap, taking m below a fixed sufficiently small constant, therefore proves

$$\|R(\tau)\|_M \le Cme^{-2\tau}, \qquad \tau \ge 0. \tag{18}$$

We now use the initial fourth-moment cancellation. The lower modes vanish, so P_4R consists exactly of the degree-four eigenspace, and (14) gives $\|P_4R(0)\|_M \le Cm^2$. By (15)–(17), $\left\|b^M\mathcal{F}(s)\right\|_{4/3} \le Cm^2e^{-3s}$. The finite-dimensional projected equation yields

$$\|P_4R(\tau)\|_M \le Cm^2e^{-2\tau} + C\int_0^\tau e^{-2(\tau-s)}m^2e^{-3s}\,\mathrm{d}s \le Cm^2e^{-2\tau}.$$

Half-turn symmetry excludes every degree-five mode. Hence $R = P_4R + Q_5R$, and (10) gives

$$\|Q_5R(\tau)\|_M \le Cme^{-3\tau} + Cm^2(1+\tau)e^{-3\tau}.$$

Since $(1+\tau)e^{-\tau}$ is bounded, we conclude that

$$\|R(\tau)\|_M \le C[me^{-3\tau} + m^2e^{-2\tau}]. \tag{19}$$

Weighted L^2 controls L^1, angular projection is bounded, and H is radial. Also physical L^1 is unchanged by the self-similar rescaling. Thus (19) proves (8) for $t \ge 1$; contraction bounds $\|h(t)\|_1 \le 2m$ on $0 \le t \le 1$ complete that estimate.

It remains to bound the harmonic moment at all times. The physical moment satisfies

$$B_4(f(e^\tau)) = e^{2\tau}B_4(w(\tau)).$$

The diffusion contribution vanishes because $\bar z^4$ is harmonic. The scaling contribution cancels when differentiating the displayed physical moment. The radial H–H self-interaction is zero. Weighted Hölder and (17) imply

$$\left|\int \bar y^3\overline{V_a(y)}b(y)\,dy\right| \le C\,\|a\|_M\,\|b\|_M\,.$$

Using (15) and (18), therefore,

$$\left|\frac{d}{d\tau}B_4(f(e^\tau))\right| \le Ce^{2\tau}[\|H\|_M\,\|R\|_M + \|R\|_M^2] \le Cm^2(e^{-\tau}+e^{-2\tau}).$$

This is integrable. Together with (14), it proves $\sup_{t\ge0}|B_4(f(t))| \le Cm^2$ in normalized variables. Finally $B_4(f_{\text{original}}) = \nu r^4 B_4(F)$ and the normalized mass is m_{original}/ν. Undoing the rescaling proves (7) and (8). □

Limitation for the proposed cascade. For a colour satisfying an additional foreign-drift equation

$$\partial_t f + (K * f + b) \cdot \nabla f = \nu \Delta f,$$

the second radial moment acquires $2\int x \cdot b(x) f(x)$ and the invariant condition $P_3 R = 0$ above generally fails. If the foreign field is a solid rotation, it can be removed and the proposition still applies. A cubic harmonic remainder, and density leaking from the outer colours into the central region, require additional estimates. They are not covered by this proposition.

7 A multiplicative linear estimate for the active fourth mode

Fix

$$\rho^2 = \frac{3}{5}, \quad r_1 = \rho, \quad r_2 = 1, \quad c_1 = -1, \quad c_2 = \rho^2, \qquad W = \sum_{i=1}^{2} c_i \mathbf{1}_{B_{r_i}}.$$

For $\kappa > 0$ put $W_\kappa(t) = e^{\kappa t \Delta} W$ and

$$\Omega_\kappa(t, r) = r^{-2} \int_0^r s W_\kappa(t, s)\, \mathrm{d}s.$$

This is the angular velocity of the radial heat solution. Its inviscid values at the reference interfaces are

$$\Omega_0(r_1) = -\frac{1}{5}, \qquad \Omega_0(r_2) = 0.$$

All constants in this note depend only on the displayed fixed profile and the fixed collars chosen below. They do not depend on κ or H.

We use one *complex positive Fourier coefficient*, with angular index $m = 4$. A real perturbation is $f(t, r)e^{4i\theta} + \overline{f(t, r)}e^{-4i\theta}$; its negative coefficient is recovered by conjugation and imposes no additional restriction on the positive coefficient. Write

$$\mu_t(\mathrm{d}r) = f(t, r) r\, \mathrm{d}r, \qquad h_m(r, s) = \left(\frac{\min(r, s)}{\max(r, s)}\right)^m, \qquad \psi_\mu(r) = -\frac{1}{2m} \int_0^\infty h_m(r, s) \mu(\mathrm{d}s).$$

The radial coefficient of the streamfunction is ψ_μ. The equation linearized at the radial heat solution is

$$\partial_t f + im\Omega_\kappa(t, r) f - im \frac{\psi_\mu(r)}{r} \partial_r W_\kappa(t, r) = \kappa \left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}\right) f. \tag{20}$$

Indeed $u_r = -r^{-1}\partial_\theta \psi$; hence the linearized radial advection is $-im\psi_\mu W_\kappa'/r$. The streamfunction formula follows by solving the radial Poisson equation, regular at zero and decaying at infinity. In particular every normalization in (20) refers to the same positive Fourier coefficient.

Choose $d_0 > 0$ so small that the intervals $I_i = (r_i - 2d_0, r_i + 2d_0)$ are disjoint and their closures are contained in $(0, \infty)$. Fix $\chi_i \in C_c^\infty(I_i)$, with $0 \le \chi_i \le 1$, equal to one when $|r - r_i| \le d_0$. Set

$$d(r) = \min\{\mathrm{dist}(r, \{r_1, r_2\}), d_0\}, \qquad q_i(t) = \frac{1}{c_i r_i^2} \int \chi_i(r) \mu_t(\mathrm{d}r),$$

$$F(t) = \|\mu_t\|_{\mathrm{TV}}, \qquad T(t) = \int d(r) |\mu_t|(\mathrm{d}r).$$

The Euler interface matrix is

$$(M_m z)_i = -im\Omega_0(r_i)z_i + \frac{i}{2r_i^2}\sum_j c_j r_j^2 h_m(r_i, r_j) z_j. \tag{21}$$

For $m = 4$, this gives exactly

$$M_4 = \begin{pmatrix} 3i/10 & i\rho^4/2 \\ -i\rho^6/2 & 3i/10 \end{pmatrix}, \qquad \lambda_\pm = \pm\sigma + i\tau, \quad \sigma = \rho^5/2 > 0, \quad \tau = 3/10,$$

with eigenvectors $v_+ = (1, -i\rho)$ and $v_- = (1, i\rho)$.

Proposition 7.1. *There are $a_* > 0$ and $C < \infty$ with the following property. Let $H \geq 1$ and $0 < \kappa H \leq a_*$. Set $\varepsilon = \sqrt{\kappa H}$ and prescribe*

$$\mu_0 = \sum_{i=1}^{2} c_i r_i^2 z_i \delta_{r_i}, \qquad F_0 = \|\mu_0\|_{\mathrm{TV}}.$$

Equation (20) has a unique bounded measure-valued mild solution on $[0, H]$, weakly continuous at zero with initial measure μ_0. It is smooth in the radial variable for positive times. For $0 \leq t \leq H$ it satisfies

$$F(t) \leq CF_0 e^{(\sigma + C\varepsilon)t}, \tag{22}$$

$$T(t) \leq C\varepsilon F_0 e^{(\sigma + C\varepsilon)t}, \tag{23}$$

$$|q(t) - e^{tM_4} z| \leq C\varepsilon t e^{(\sigma + C\varepsilon)t} F_0. \tag{24}$$

The norms on the last line can be any fixed norm on $\mathbb{C}^2$.

Lemma 7.2 (Radial heat layers)**.** *Let $D_i(t, r) = -\partial_r e^{\kappa t\Delta} \mathbf{1}_{B_{r_i}}$. For $0 < t \leq H$ with $\kappa H \leq a_*$ these functions are nonnegative and*

$$\int_0^\infty D_i(t, r)\, \mathrm{d}r = 1 - e^{-r_i^2/(4\kappa t)}, \tag{25}$$

$$\int_0^\infty |r - r_i| D_i(t, r)\, \mathrm{d}r \leq C\sqrt{\kappa t}, \tag{26}$$

$$\int_{|r - r_i| \geq d_0} D_i(t, r)\, \mathrm{d}r \leq Ce^{-c/(\kappa t)}. \tag{27}$$

Moreover

$$|\Omega_\kappa(t, r) - \Omega_0(r_i)| \leq C(|r - r_i| + \sqrt{\kappa t}) \quad (r \in I_i). \tag{28}$$

Proof. Convolution of a disk indicator with a radial decreasing positive kernel is radial decreasing, which proves $D_i \geq 0$. To verify this directly and obtain quantitative bounds, differentiate the disk convolution as a boundary integral. With $a = \kappa t$ and $R = r_i$ this gives

$$D_i(t, r) = \frac{R}{2a} e^{-(r^2 + R^2)/(4a)} I_1\left(\frac{rR}{2a}\right). \tag{29}$$

Here $I_1(s)$ is the modified Bessel function, represented by $(2\pi)^{-1}\int_0^{2\pi} e^{s\cos\theta}\cos\theta\, \mathrm{d}\theta$. For $s \leq 1$ its power series gives $I_1(s) \leq Cs$. For $s \geq 1$, integration by parts gives

$$I_1(s) = \frac{s}{2\pi}\int_0^{2\pi} e^{s\cos\theta}\sin^2\theta\, \mathrm{d}\theta \leq Cse^s(1 + s)^{-3/2}.$$

The last bound follows by $1-\cos\theta \geq c\theta^2$ near zero and the corresponding bound near 2π, with exponentially small contribution elsewhere. Thus for all $s \geq 0$, $I_1(s) \leq Cse^s(1+s)^{-3/2}$. On $R/2 \leq r \leq 2R$, formula (29) is bounded by $Ca^{-1/2}e^{-(r-R)^2/(4a)}$. On the remaining ranges the same formula, and the preceding global bound on I_1, give an exponentially small integral, including after multiplication by $|r-R|$. Near zero the additional factor s in the Bessel bound removes any radial singularity. These elementary Gaussian integrals prove (26) and (27), with slightly reduced exponential constant in the tails. The mass identity is also exact: the heat-convolved disk tends to zero at infinity and has value $1-e^{-R^2/(4a)}$ at the origin.

For (28), the disk translation estimate $\|\mathbf{1}_{B_R}(\cdot-y)-\mathbf{1}_{B_R}\|_1 \leq C_R|y|$ and Gaussian convolution give $\|W_\kappa(t)-W\|_1 \leq C\sqrt{\kappa t}$. For $r \in I_i$, which is bounded away from zero, the displayed formula for Ω_κ bounds $|\Omega_\kappa - \Omega_0|$ by this L^1 error. The function Ω_0 is Lipschitz on the collars, including across its interface values, so the remaining difference is at most $C|r-r_i|$. □

We next justify the measure equation and the estimates used in its proof. Let $P(t,s)$ be the homogeneous propagator obtained by omitting the nonlocal source in (20). In the natural measure $f(r)r\,\mathrm{d}r$ it satisfies

$$\|P(t,s)\eta\|_{\mathrm{TV}} \leq \|\eta\|_{\mathrm{TV}}, \tag{30}$$

$$\int d(r)|P(t,s)\eta|(\,\mathrm{d}r) \leq \int d(r)|\eta|(\,\mathrm{d}r) + C\sqrt{\kappa(t-s)}\|\eta\|_{\mathrm{TV}}. \tag{31}$$

Indeed complex Kato gives

$$\partial_t|f| \leq \kappa\left(\partial_{rr} + r^{-1}\partial_r - m^2r^{-2}\right)|f|.$$

The angular multiplication is imaginary and disappears. There is also a direct kernel argument avoiding any boundary issue at $r=0$. The angular-m heat propagator has a positive radial kernel involving I_m, and $0 \leq I_m(s) \leq I_0(s)$ implies domination by ordinary radial heat. Positivity follows from its power series and the upper bound from the circle-integral representation. Interleave this heat propagator with the unit-modulus multiplications $e^{-im\Omega_\kappa(t,r)\Delta t}$. All these products are dominated by radial heat; their bounded-perturbation limit constructs the homogeneous propagator and preserves domination. The bounded coefficient is continuous in time in L^∞. At time zero, for $r \leq \rho/2$ the heat error in the central constant-vorticity region is $O(e^{-c/(\kappa t)})$, so the integral defining Ω_κ has the same bound there. For $r \geq \rho/2$, the disk L^1 heat error and the factor r^{-2} give $O(\sqrt{\kappa t})$. At positive times L^∞ heat continuity suffices. This proves the claimed continuity. This time-dependent product construction is therefore legitimate and proves (30). The radial function $d(|x|)$ is globally 1-Lipschitz on $\mathbb{R}^2$, so ordinary heat changes its average from a point x by at most $\mathbb{E}|\sqrt{2\kappa}B_{t-s}| \leq C\sqrt{\kappa(t-s)}$, proving (31). For clarity, these facts hold initially for smooth radial coefficients; extension to finite measures follows by approximating measures and using the radial heat kernel. One can also construct P directly by the bounded imaginary perturbation of the angular-m heat propagator. The multiplier Ω_κ is uniformly bounded because $|\Omega_\kappa| \leq \|W\|_\infty/2$.

The nonlocal source as a natural measure is

$$\mathcal{B}_t\mu(\,\mathrm{d}r) = im\psi_\mu(r)W'_\kappa(t,r)\,\mathrm{d}r. \tag{32}$$

It is a bounded map on finite measures, uniformly in $t>0$: $|\psi_\mu| \leq \|\mu\|_{\mathrm{TV}}/(2m)$ and $\int |W'_\kappa|\,\mathrm{d}r \leq \sum_i |c_i|$. Thus the Volterra equation

$$\mu_t = P(t,0)\mu_0 + \int_0^t P(t,s)\mathcal{B}_s\mu_s\,\mathrm{d}s$$

has a unique bounded weakly continuous solution by Picard iteration in finite measures on every finite interval. At positive times each source term has an integrable radial density; its integral is therefore a usual strong measure integral. No strong continuity in total variation at zero is asserted for the singular ring measure. The usual iteration estimate is the exponential series with terms bounded by $(Ct)^n/n!$. The singular initial ring measure is therefore harmless. The kernels are smooth for positive times; on any interval separated from zero, this bounded-source construction and the smooth coefficients give the stated radial regularity. All computations below may alternatively be performed after smoothing the ring initial data and passed to the measure limit with the displayed uniform bounds.

Proof of Proposition 7.1. We first reconstruct the streamfunction from two collar masses. Uniformly in $r > 0$,

$$\left|\psi_\mu(r) + \frac{1}{2m}\sum_i c_i r_i^2 h_m(r, r_i) q_i\right| \le CT, \qquad |\psi_\mu(r)| \le C(|q| + T). \tag{33}$$

To prove this, insert $\chi_1 + \chi_2$ in the integral defining ψ_μ. Outside the regions where these cutoffs equal one, $d(r)$ is bounded below by a fixed positive constant, so the remaining variation is bounded by CT. On the ith collar, $|\partial_s h_m(r, s)| \le m/s$ uniformly in the receiving radius r. The difference from $h_m(r, r_i)$ is consequently bounded by $C|s - r_i| \le Cd(s)$ there, after adjusting the fixed constant where $|s - r_i| \ge d_0$. This proves (33) with no confinement assumption on μ.

Since $W_\kappa' = -\sum_i c_i D_i$, Lemma 7.2 and (33) give

$$\|\mathcal{B}_t\mu\|_{\mathrm{TV}} \le C(|q| + T), \qquad \int d(r)|\mathcal{B}_t\mu|(\,\mathrm{d}r) \le C\varepsilon(|q| + T).$$

Apply (30)–(31) to the mild equation. The initial measure has distance moment zero, so

$$F(t) \le F_0 + C\int_0^t (|q(s)| + T(s))\,\mathrm{d}s, \qquad T(t) \le C\varepsilon\left[F_0 + \int_0^t (|q(s)| + T(s))\,\mathrm{d}s\right]. \tag{34}$$

In the source contribution to the second inequality, the additional heat displacement is at most $C\sqrt{\kappa(t-s)} \le C\varepsilon$ times the source variation.

Testing (20) against χ_i in the natural measure yields

$$\begin{aligned} q_i' = &-\frac{im}{c_i r_i^2}\int \chi_i \Omega_\kappa \mu(\,\mathrm{d}r) \\ &+ \frac{\kappa}{c_i r_i^2}\int (\chi_i'' + r^{-1}\chi_i' - m^2 r^{-2}\chi_i)\mu(\,\mathrm{d}r) \\ &+ \frac{im}{c_i r_i^2}\int \chi_i \psi_\mu W_\kappa'\,\mathrm{d}r. \end{aligned}$$

The first line equals $-im\Omega_0(r_i)q_i$ up to $C(T + \varepsilon F)$, by (28) and the fixed collar geometry. The second is bounded by $C\kappa F \le C\varepsilon F$, since the test function is supported away from zero and $\kappa H \le a_* \le 1$, $H \ge 1$.

For the third line, $|\partial_r h_m(r, s)| \le m/r$ on each receiving collar gives $|\partial_r \psi_\mu| \le CF$. The near-unit mass of D_i, its first distance moment, and the exponentially small D_j mass in the foreign collar imply

$$\int \chi_i \psi_\mu W_\kappa'\,\mathrm{d}r = -c_i\psi_\mu(r_i) + O(\varepsilon F).$$

There is no singular remainder at zero: these tests are supported on fixed positive-radius collars, and Lemma 7.2 already controls the omitted radial gradient mass. Applying (33) at r_i identifies the main term with the coupling in (21). Consequently

$$q' = M_4 q + e, \qquad |e| \le C(T + \varepsilon F). \tag{35}$$

This identity is valid almost everywhere and in the integrated form from time zero; all right-hand sides are locally integrable by the mild bounds. The initial identity $q(0) = z$ follows from weak convergence of the homogeneous heat evolution against χ_i and the $O(t)$ variation of the source integral.

To exploit the exact spectral growth, write $q = \alpha v_+ + \beta v_-$ and put $y = |\alpha| + |\beta|$. This is an equivalent norm on $\mathbb{C}^2$; the upper Dini derivative in (35) satisfies

$$D^+ y \le \sigma y + C(T + \varepsilon F).$$

Define

$$Z(t) = F_0 + \int_0^t (|q(s)| + T(s))\,\mathrm{d}s.$$

Equation (34) gives $F \le CZ$, $T \le C\varepsilon Z$, and therefore, for fixed positive constants C_1, C_2, C_3,

$$D^+ y \le \sigma y + C_1 \varepsilon Z, \qquad Z' \le C_2 y + C_3 \varepsilon Z.$$

Choose $a \ge C_1/\sigma$ and $C_4 \ge \max\{aC_2, C_3\}$. For $Y = y + a\varepsilon Z$, direct substitution gives

$$D^+ Y \le (\sigma + C_4 \varepsilon) Y.$$

Initially $q = z$ and $y(0) \le CF_0$, so $Y(0) \le CF_0$. Gronwall gives $y(t) \le CF_0 e^{(\sigma + C_4\varepsilon)t}$. Integrating the inequality for Z then yields

$$Z(t) \le e^{C_3 \varepsilon t}\left[F_0 + CF_0 \int_0^t e^{(\sigma + (C_4 - C_3)\varepsilon)s}\,\mathrm{d}s\right] \le CF_0 e^{(\sigma + C_4\varepsilon)t},$$

where the fixed positive σ bounds the denominator. This proves (22) and (23). Finally the diagonalization gives $\|e^{tM_4}\| \le Ce^{\sigma t}$. Variation of constants in (35), followed by the Z bound, proves

$$|q(t) - e^{tM_4} z| \le C\varepsilon F_0 \int_0^t e^{\sigma(t-s)} e^{(\sigma + C_4\varepsilon)s}\,\mathrm{d}s \le C\varepsilon t e^{(\sigma + C_4\varepsilon)t} F_0,$$

as claimed. □

Remark 7.3 (Unstable coordinate and scale compatibility). The left unstable coordinate is $a(q) = q_1/2 + iq_2/(2\rho)$. Since $a(M_4 q) = (\sigma + i\tau)a(q)$, (24) gives

$$|a(q(t)) - e^{(\sigma + i\tau)t} a(z)| \le C\varepsilon t e^{(\sigma + C\varepsilon)t} F_0.$$

If $F_0 \le C_0 |a(z)|$, the relative comparison error on $[0, H]$ is at most $CC_0\sqrt{\kappa} H^{3/2} e^{C\sqrt{\kappa} H^{3/2}}$. This includes the zero-initial-fourth-moment stable/unstable combination, whose stable and unstable coefficients have equal modulus.

For geometric cascade scales, the first receiver has $\kappa_{N-1} = \beta q^2 \kappa_N$. On $H = O(\log(1/q))$, this linear relative error tends to zero even when $\kappa_N \gg \beta q^2$, the condition needed to place the next scale in the low-Re regime.

Remark 7.4 (Scope of the proposition). This is a linear statement about a radial heat background. It does not prove the finite-amplitude source-window comparison, the nonlinear stable-complement reset, the control of actual foreign diffusive fields, or a nonunique inviscid limit for one fixed datum. Those statements require additional estimates. In particular, this proposition supplies a multiplicative linear estimate rather than a replacement for the nonlinear contour cascade.

8 What the viscosity windows actually show

Let $\beta > 1$, $R_j = R_1 q^{j-1}$ and $A_j = \beta^{j-1}$, where $0 < q < 1$ and $\beta q^{2/p} < 1$. Choose α_* inside the local chart in Proposition 3.1. Its neutral seeds give the genuine initial datum

$$\omega_0(x) = \sum_{j\ge 1} A_j W_{\alpha_j}(x/R_j), \qquad 0 < |\alpha_j| \le \alpha_*. \tag{36}$$

The series converges absolutely in $L^1 \cap L^p$ because

$$\sum_j A_j R_j^{2/r} < \infty, \qquad r = 1, p.$$

Each summand has zero circulation and zero fourth moment, so the sum has both properties. It is compactly supported and fourfold symmetric. These facts do not prescribe its inviscid accumulation points.

For a cutoff depth N, put

$$\nu_N = \kappa A_N R_N^2, \qquad \kappa = q^{3/2}. \tag{37}$$

Then $\nu_N \downarrow 0$ if $\beta q^2 < 1$. The normalized viscosity at level j is

$$\kappa_j = \frac{\nu_N}{A_j R_j^2} = \kappa(\beta q^2)^{N-j}.$$

At the first inactive child its Reynolds number is

$$\frac{A_{N+1} R_{N+1}^2}{\nu_N} = \beta q^{1/2}.$$

Thus small q is algebraically consistent with weak diffusion at the active level and a diffusion-dominated inner child.

An initially shielded inner aggregate $f_0 = \sum_{j>N} A_j W_{\alpha_j}(x/R_j)$ has support in $B_{CR_{N+1}}$, with $R_{N+1} = R_N q$, and mass $m \le C A_N R_N^2 \beta q^2$. Applied to its *isolated* evolution, the all-time suppression theorem in the preceding section gives

$$\sup_{t\ge 0} \frac{|B_4(f(t))|}{A_N R_N^6} \le C \frac{\beta^2 q^8}{\kappa} = C\beta^2 q^{13/2}. \tag{38}$$

This is smaller than a nominal q^6 transfer by a power of q. It is a proved estimate for an isolated aggregate. The aggregate in the full stack has a foreign velocity, so (38) cannot be used for it without a perturbation argument.

8.1 Why ordinary inviscid convergence is inadequate

There is a straightforward isolated survival bound, but it illustrates the distinction between absolute and relative control. Suppose a zero- circulation Euler velocity U has vorticity Ω with bounded compact support initially, and

$$\Lambda_H = \int_0^H \|\nabla U(s)\|_\infty \, \mathrm{d}s < \infty.$$

Let U^κ solve Navier–Stokes with the same datum and normalized viscosity κ. The relative energy identity, tested against U, gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|U^\kappa - U\|_2^2 + \frac{\kappa}{2}\|\nabla U^\kappa\|_2^2 \le \|\nabla U\|_\infty \|U^\kappa - U\|_2^2 + \frac{\kappa}{2}\|\Omega\|_2^2 .$$

Consequently

$$\sup_{s\le H}\|U^\kappa(s) - U(s)\|_2 \le \sqrt{\kappa H}\,\|\Omega_0\|_2\, e^{\Lambda_H}. \tag{39}$$

The velocities are uniformly bounded by the L^1 and L^∞ vorticity contractions. A Gaussian diffusion representation for the coloured scalar, with bounded drift, therefore gives exponentially small weighted tails outside any ball larger than initial support radius plus H times that velocity bound. A compact cutoff, integration by parts of $\operatorname{curl}(U^\kappa - U)$, and (39) yield

$$\sup_{s\le H}|B_4(\Omega^\kappa(s)) - B_4(\Omega(s))| \le C_H\sqrt{\kappa} + C_H e^{-c_H/\kappa}. \tag{40}$$

For the diffusion representation, positivity bounds $|\Omega^\kappa|$ by the solution with initial density $|\Omega_0|$; its stochastic displacement is bounded by drift displacement plus $\sqrt{2\kappa}|B_s|$. This proves the Gaussian tail bound without a viscous Lipschitz assumption.

The incoming seed at a receiver is of size q^6. Applying (40) at level $N-1$ would require

$$\sqrt{\kappa_{N-1}} \ll q^6, \qquad \text{hence} \qquad \kappa \ll q^{10}/\beta.$$

Damping the next child requires $\kappa \gg \beta q^2$, which is incompatible with that crude requirement. This disproves the adequacy of the absolute estimate, not the possibility of a viscous cascade. A radial solution creates no fourth mode. The estimate required for the cascade must reflect that symmetry and be proportional to the nonradial seed and forcing.

The linear comparison proved above supplies the required proportionality for the linear fourth angular mode of an isolated annulus. Over $H = C\log(1/q)$ its relative error against an unstable coefficient is bounded by

$$C\sqrt{\kappa}\,H^{3/2}\exp(C\sqrt{\kappa}\,H^{3/2}),$$

provided the initial coefficient has a nonzero unstable component comparable to its total variation, as the neutral seed's first variation does. With $\kappa = q^{3/2}$ this tends to zero as $q \to 0$. Thus the linear stability requirement fits the same cutoff window. Its nonlinear and foreign-colour extensions are still needed.

9 Older seeds: a compatible scalar clock, not yet a PDE proof

Let $\sigma = \rho^5/2$, $L = \log(1/q)$, $H = 9L/\sigma$, and let $M > 0$ be a fixed scheduling margin. In an ideal linear growth model suppose a prepared source at depth N caps at S/A_N, and a stage at level m takes at most H/A_m. Give every initial unstable coefficient magnitude $\delta e^{-\sigma S}$.

Lemma 9.1 (Scalar scheduling)**.** *If*

$$S > \max\{S/\beta, H/(\beta-1)\} + 6L/\sigma + M,$$

then every older homogeneous seed, evaluated when its child caps, is at most $\delta e^{-\sigma M} q^6$, uniformly in cutoff depth, in this model. Such an S exists and can be chosen of order $1 + \log(1/q)$.

Proof. For $N > j$, write $n = N - j$. The normalized time at which level $j+1$ caps is bounded above by

$$A_j T_{j+1,N} \le S\beta^{-n} + H \sum_{k=1}^{n-1} \beta^{-k}.$$

The right-hand side is $H/(\beta-1) + \beta^{-(n-1)}[S/\beta - H/(\beta-1)]$, which lies between S/β and $H/(\beta-1)$. Linear growth gives the homogeneous magnitude $\delta e^{-\sigma S + \sigma A_j T_{j+1,N}}$, and the asserted bound follows. Taking S larger than both affine lower bounds proves feasibility. □

All isolated seed delays S/A_j tend to zero. No positive physical waiting time is required. The lemma controls a scalar linear clock only. It neither bounds old seeds under the full coupled equation nor proves that incoming pulses have a nonzero phase.

The unstable coordinate is complex. Its intrinsic rotation and every transfer's complex coefficient must be included. Alternating raw seed phases does not establish alternating macroscopic orientations. A possible robust choice would use an outer-cap phase map of degree one. If a continuous map $F : S^1 \to S^1$ is homotopic through nonvanishing phase maps to $z \mapsto e^{i\theta} z$, it is surjective: a circle map omitting one point is null-homotopic, whereas its degree here is one. The topological assertion is elementary; existence and nonvanishing of the required PDE phase maps are additional hypotheses.

The old Euler inactive-prefix norm $\max_{i<j} x_i/(Dq^6)^{j-i}$, for $Dq^6 < 1$, cannot stay bounded for an infinite stack with any fixed nonzero outer seed. A successful proof has to separate the evolution of those fixed seeds from the perturbation induced by the active cascade. Its old bootstrap cannot simply be reused.

10 The coupling still to be proved

The following are necessary estimates for this specific strategy. They are *not proved in this note.*

1. **Nonlinear multiplicative active stability.** After splitting off radial heat evolution, track a nonradial norm E_j and unstable coordinate a_j with relative viscous errors, schematically
$$\dot a_j = A_j(\sigma + i\vartheta_j)a_j + F_j + \mathcal{R}_j, \qquad |\mathcal{R}_j| \le A_j\varepsilon_j E_j + \varepsilon_j|F_j| + CA_jE_j^2,$$
with stable-complement and accumulated phase estimates uniform in depth. An additive radial-background error independent of E_j, F_j is insufficient. Even a multiplicative linear estimate with an unspecified C_H does not by itself prove this nonlinear assertion on $H \sim \log(1/q)$ windows.

2. **Coupled all-time inner-tail suppression.** Extend isolated suppression to the actual coloured aggregate for the entire receiver window, not only one inner diffusion time. This requires controlling moment generation by the outer cubic field and the leaked outer vorticity. For example, if $b(z) = c_3\bar{z}^3$ and g is radial heat evolution, the exact foreign contribution is

$$B_4' = 4\bar{c}_3 \int |x|^6 g \, \mathrm{d}x.$$

For zero-circulation radial g its signed sixth moment is

$$J_6(t) = J_6(0) + 36\nu t J_4(0) + 288(\nu t)^2 J_2(0),$$

because $\Delta|x|^{2k} = 4k^2|x|^{2k-2}$. Thus the outer field creates a signal even from a shielded radial tail. Its first-order dimensional bound at receiver i is

$$Cx_i \sum_{j>i} \frac{A_j}{A_i} \left[q^{8(j-i)} H + \kappa_i q^{6(j-i)} H^2 + \kappa_i^2 q^{4(j-i)} H^3 \right].$$

Here the cubic coefficient is assumed bounded by $CA_i x_i / R_i^2$. This calculation is exact for the radial heat profile and a prescribed cubic field. It is not a nonlinear bound for the coupled Navier–Stokes solution.

3. **Orientation and observable separation.** Prove that old seeds and diffused tails cannot destroy nonvanishing transfers, and construct phases yielding separated values of one continuous functional of the total trajectory. Opposite cap phases at uncontrolled different cap times do not suffice: one trajectory can have both phases at different times. One needs a common-time observable or a protected first-cap functional with uniform transversality and continuity.

One attempted norm for the first estimate has a specific defect. Smoothness of a patch boundary does not imply analytic angular regularity of the vorticity in radial total variation. For the graph $r = R(1 + \alpha \cos 4\theta)$, $0 < \alpha \ll 1$, put $r = R(1 + \alpha s)$, $-1 < s < 1$. The angular coefficient of its indicator at mode $4n$ is

$$\frac{\sin(n \arccos s)}{\pi n}.$$

Its radial total variation with weight $r \, \mathrm{d}r$ is comparable to $R^2\alpha/n$, uniformly in $n \geq 1$, since $\int_0^\pi |\sin(n\theta)| \sin\theta \, \mathrm{d}\theta$ is bounded above and below by positive constants. Thus exponentially weighted Fourier total variation diverges for every positive analytic radius. Higher graph modes can be quadratic when tested against smooth radial functions, but they are not uniformly quadratic in this total-variation norm. An analytic Fourier-measure bootstrap cannot be applied to these patch seeds as stated. A contour-compatible norm or new smooth-profile construction is required.

No choice of the numerical parameters alone proves these PDE estimates. The isolated and short-window results above close parts of the cutoff argument, but no fixed initial datum with two distinct inviscid accumulation points has been established here.

References

[1] T. Gallay and C. E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier–Stokes and vorticity equations on* $\mathbb{R}^2$, Arch. Rational Mech. Anal. **163** (2002), 209–258.

The checked preprint is arXiv:math/0102197v1, especially Theorem 2.2 and Appendix A, Propositions A.2(b) and A.5.