# Non-unique vanishing-viscosity selection for planar Euler

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**Abstract**

For every finite $p > 2$, we construct one real compactly supported zero-circulation initial vorticity in $L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ whose unforced Navier–Stokes evolutions have two distinct strong vanishing-viscosity accumulation paths. Both limits are renormalized unforced Euler solutions with the same strong initial trace. The construction uses a fixed infinite concentric stack, a smooth unstable fourfold profile, actual positive scalar envelopes, and a finite nonlinear expansion retaining the interacting harmonic histories. Discounted source history is controlled through the complete prospective receiver stop, and exact-cap normalization yields a depth-uniform outgoing reset. Protected first crossings, phase degree and a sparse phase assignment then separate the two limits. The analytical preliminaries, finite-sector spectral scope, source-reference comparison and uniform parameter order are included in the proof.

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# 1 Introduction

The vanishing-viscosity approximation of an incompressible flow combines an existence procedure with a selection question. For each positive viscosity the two-dimensional Navier–Stokes equation has a unique global vorticity evolution from an integrable initial density. As viscosity tends to zero, compactness can produce Euler solutions, and renormalization can upgrade weak convergence to strong convergence. These facts do not by themselves determine whether all viscosities select the same Euler path. The distinction is relevant when the initial vorticity lies in $L^1(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$ for a finite $p>2$.

We study the unforced planar equation

$$\partial_t\omega^\nu + (K*\omega^\nu)\cdot\nabla\omega^\nu = \nu\Delta\omega^\nu, \qquad K(x)=\frac{x^\perp}{2\pi|x|^2}.$$

The initial condition in our construction is fixed once and for all; only $\nu$ varies. Zero circulation is imposed to place its Biot–Savart velocity in the finite-energy class. The strong topology used throughout is the sum norm on $X_p = L^1\cap L^p$, and the Euler solutions have their strong initial trace in this space. The definitions of distributional and renormalized solutions are collected in Section 2.

## 1.1 Main result

**Theorem 1.1** (Non-unique selection from one fixed datum)**.** *For every $2<p<\infty$ there exist a real compactly supported $\omega_0\in L^1(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2}\omega_0=0$, a finite time $T_f>0$, two sequences $\nu_n^+\downarrow 0$ and $\nu_n^-\downarrow 0$, and two distinct renormalized solutions $\omega^+,\omega^-$ of the unforced Euler equation such that*

$$\omega^\pm\in C([0,T_f];X_p), \qquad \omega^\pm(0)=\omega_0, \qquad \lim_{n\to\infty}\|\omega^{\nu_n^\pm}-\omega^\pm\|_{C([0,T_f];X_p)}=0.$$

*Here $\omega^\nu$ is the unique unforced Navier–Stokes solution with viscosity $\nu$ and the same initial condition $\omega_0$ for every $\nu>0$. Distinctness means that the two paths differ on $[0,T_f]$. Consequently this fixed-datum family does not converge as $\nu\downarrow 0$.*

The theorem asserts non-unique accumulation points in a strong vorticity topology. It does not assert that every Euler solution from this datum can be selected by viscosity, or prescribe the total set of accumulation points. The velocity convergence along each of the displayed subsequences is also strong in $C([0,T_f];L^\infty)$, by Lemma 2.6.

## 1.2 Background and analytical setting

The global integrable-vorticity Navier–Stokes theory used below is stated precisely in Proposition 2.4; its existence input is Gallay–Wayne [1, Theorem 2.2]. Smooth Euler evolution and the Biot–Savart formulation are part of the classical planar theory; see Majda–Bertozzi [3]. We give the kernel estimates and the comparison arguments needed by the construction, including the low-frequency cancellation supplied by zero circulation.

The treatment of transported densities follows the renormalization framework of DiPerna–Lions [2]. In the present range $p>2$, the commutator estimates for the limiting total vorticity and for bounded passive colours are proved directly, together with their endpoint traces. For comparison, Ciampa–Crippa–Spirito [4, Theorem 3.2] prove strong convergence to an identified

renormalized inviscid limit under their stated approximation hypotheses. That theorem concerns convergence to a limit already identified along the relevant family, and does not assert uniqueness of selection. The fixed rough datum used here is handled by the self-contained compactness proof in Theorem 2.12, rather than by an unstated application of that approximation theorem.

The auxiliary tools in Section 2 are distinguished from the cascade estimates in the later sections. In particular, compactness and uniqueness of a passive colour for a prescribed Euler velocity are established analytical inputs. They do not imply uniqueness of the nonlinear Euler path.

### 1.3 Construction and proof strategy

The datum is an infinite concentric stack of copies of one smooth radial profile $W$ with zero circulation, perturbed by neutral fourfold seeds:

$$\omega_0(x) = \sum_{j\geq 1} A_j\,[W(x/R_j) + \alpha\Phi_{\vartheta_j}(x/R_j)], \qquad A_j = \beta^{j-1}, \quad R_j = q^{j-1}. \tag{1}$$

The fixed parameters obey $\beta > 1$, $0 < q < 1$ and $\beta q^{2/p} < 1$. The series converges absolutely in $X_p$ and has one common compact support. Each seed has zero circulation and zero fourth complex moment. Its phase is retained as a free circle parameter. The profile, smoothing length, scales, amplitude and eventually all seed phases are fixed independently of viscosity.

The radial profile has a simple unstable pair in angular sectors $\pm 4$. Its linear analysis is modal. Every nonlinear approximation uses only a specified finite set of angular sectors and a finite derivative hierarchy; the actual remainder is controlled by a scalar transport comparison in $X_p$. No spectral claim about an infinite angular direct sum is needed. Finite expansions also produce a canonical smooth ancient Euler orbit and all its finite jets. The same orbit is used at every receiving scale.

For viscosity

$$\nu_N = q^{3/2} A_N R_N^2 = q^{3/2}(\beta q^2)^{N-1},$$

scale $N$ is the deepest selected source. Smaller scales form a diffusing packet. A signed fourth-moment estimate makes its impulse smaller than the active seed. Larger scales remain inactive while the selected source prepares. Their actual core and harmonic estimates are closed using positive scalar envelopes for the complete interacting Navier–Stokes velocity. Thus the packet force is estimated from the actual solution, rather than imposed as an external forcing assumption.

The main small exponents are explicit. We take $6 < K < 643/100$, $\alpha = q^K$ and $\delta = [\log(1/q)]^{-4}$. The selected source reaches its first $\delta$ cap in normalized time $(K/\sigma)\log(1/q)$ to leading order. Its fourth moment transmits an exterior field of order $q^6$ to the next larger colour. A finite nonlinear recursion retains the actual harmonic coefficient histories. Normalization at the receiver's exact cap removes the incoming waveform from its outgoing finite jet, with a uniform intrinsic remainder.

Source history is controlled in three pieces. Before its shrinking cap, a cap-relative exponential envelope controls the discounted past. From that cap to one fixed reference state, one-sided comparison with the canonical orbit controls the short advance, without relating the reference amplitude to a later expansion threshold. After reference time, positive-envelope moments and a separate Gaussian estimate at the parent radius bound the discounted future up to the parent's provisional stop. An upper input bound is obtained before choosing the receiver's stop constant.

Only after reaching a deterministic prospective cap horizon is the nonzero signed transfer coefficient used. This order avoids a circular use of a future coefficient to justify the interval on which it is defined.

The actual stage estimates give uniformly protected first caps through an arbitrarily deep finite descent. Protection comprises an early strict margin and a positive radial slope in a specified late interval around the first crossing. It supplies continuity of the cap time and phase under seed homotopies. At a fixed positive viscosity, scaling all other seeds down to zero leaves a degree-one rotation in the selected seed phase. Homotopy invariance then makes the outer cap phase map surjective.

Finally, phases are assigned at a sparse sequence of depths so that the outer cap phases alternate between two separated values. The summable $X_p$ tail of the datum and continuous dependence at each fixed positive viscosity preserve previously selected outputs. Strong compactness gives a limit along each parity. If the two total Euler paths were equal, the unique bounded passive outer colour transported by their common velocity would give the same first-cap phase. The two selected phases exclude that possibility.

### 1.4 Logical dependencies and organization

Table 1 records the dependencies used in the final proof. Conditional estimates are applied only after their actual hypotheses have been supplied. The passage from one prepared child to its parent is separated from the final induction over the depth.

| Result | Required inputs | Role in the proof |
|---|---|---|
| Kernel, transport and compactness (Section 2) | Fixed $X_p$ datum; bounded Sobolev limiting velocity | Strong limits and recovery of fixed passive colours |
| Profile and isolated activation (Section 3) | Smooth fixed profile; finite modal spectrum | Selected seed, growing coordinate and finite expansions |
| Canonical orbit and signed coupling (Section 4) | Isolated activation and finite jets | One common reference orbit and a nonzero transform |
| Cores, cold tails and inactive prefix (Sections 5–6) | Actual scalar contractions and positive envelopes | Unconditional older control and small deep impulse |
| Source and reference history (Section 7) | Prepared child; actual older control; provisional parent stop | Uniform input norm and reference reset |
| Finite forced recursion (Section 8) | Actual finite harmonic histories and derivative reserve | Exact-cap jet independent of the incoming waveform |
| Actual receiving stage (Section 9) | Prepared-child history and finite recursion | Protected cap and invariant outgoing preparation |
| Uniform induction and phase selection (Section 10) | Selected source; receiver; compactness; passive uniqueness | Theorem 1.1 |

Table 1: Dependency map for the fixed-datum selection theorem.

Section 2 collects the classical analytical inputs and proves the compactness and protected-cap tools in the form used here. Sections 3 and 4 construct the fixed profile and canonical orbit. Sections 5 and 6 control the actual surrounding colours. Section 7 proves the source estimates. Sections 8 and 9 establish the nonlinear transmission step. Section 10 makes all constants independent of depth and proves Theorem 1.1.

### 1.5 Frequently used notation

| Symbol | Meaning |
|---|---|
| $X_p$, $X$ | $L^1 \cap L^p$ with sum norm; $p > 2$ fixed |
| $K$, $Jx$ | Planar Biot–Savart kernel; $Jx = x^\perp$ |
| $\omega^\nu$, $w_j$ | Total vorticity; its actual scalar colour at depth $j$ |
| $A_j$, $R_j$, $\kappa_j$ | Amplitude, radius and normalized viscosity $\nu/(A_j R_j^2)$ |
| $W$, $W_\epsilon$ | The fixed smoothed zero-circulation radial profile |
| $\Phi_\vartheta$, $\alpha$ | Neutral fourfold seed of phase $\vartheta$; its amplitude $q^K$ |
| $\lambda = \sigma + i\tau$ | Unstable eigenvalue of the fixed profile, $\sigma > 0$ |
| $P_{\mathrm{rad}}$, $B_m$ | Angular average; entire moment $\int \bar{z}^m f$ when absolutely integrable |
| $\mathcal{T}_\theta$ | Scalar rotation $f(r, \varphi + \theta/4)$ |
| $T_{\mathrm{adj}}$, $\mathcal{A}$ | Bounded growing adjoint test; $\mathcal{A}(f) = \int T_{\mathrm{adj}} f$ |
| $L$, $\delta$ | $\log(1/q)$ and its shrinking cap $L^{-4}$ |
| $\Omega_{\mathrm{ref}}$, $s_q$ | Fixed canonical orbit; its exact $\delta$-cap time |
| $\nu_N$, $T_f$ | Viscosity selecting depth $N$; final common physical time |

Table 2: Notation used throughout. Local normalized clocks are declared when introduced.

## 2 Preliminaries

### 2.1 Equations, solution classes, and symmetries

All spatial Lebesgue and Sobolev spaces are over $\mathbb{R}^2$. For $2 < p < \infty$ set

$$X_p = L^1 \cap L^p, \qquad \|f\|_{X_p} = \|f\|_1 + \|f\|_p.$$

The notation $X$ means $X_p$ with the exponent fixed in the statement. Vector norms use the Euclidean pointwise norm. The norm on $C([0,T]; X)$ is the supremum of the $X$ norm, and its elements have strongly continuous representatives. The physical vorticity equation is

$$\partial_t \omega^\nu + \operatorname{div}(u^\nu \omega^\nu) = \nu \Delta \omega^\nu, \qquad u^\nu = K * \omega^\nu, \qquad K(x) = \frac{x^\perp}{2\pi|x|^2}, \quad x^\perp = (-x_2, x_1). \tag{2}$$

The velocity is divergence free, so its divergence-form drift agrees with $u^\nu \cdot \nabla \omega^\nu$ whenever the latter expression is defined. Setting $\nu = 0$ gives the Euler vorticity equation.

A weak Euler solution with datum $\omega_0$ on $[0,T]$ is $\omega \in L^\infty([0,T]; X_p)$, with $u = K * \omega$, such that

$$\int_0^T \int \omega(\partial_t \varphi + u \cdot \nabla \varphi) + \int \omega_0 \varphi(0) = 0 \qquad (\varphi \in C_c^\infty([0,T) \times \mathbb{R}^2)).$$

Its weak initial trace is the trace specified by this identity. It is renormalized if the same identity holds with $\omega, \omega_0$ replaced by $b(\omega), b(\omega_0)$ for every $b \in C^1(\mathbb{R})$ with bounded derivative and $b(0) = 0$. Norm conservation is obtained by admissible truncations of powers; it is not part of the definition of a weak solution. All weak and renormalized identities involving a prescribed interior or final time use the time representatives established in their proofs.

Identify $x$ with $z = x_1 + ix_2$. For each integer $m \geq 0$ define

$$M_m(f) = \int |x|^m |f(x)| \, \mathrm{d}x, \qquad B_m(f) = \int \bar{z}^m f(x) \, \mathrm{d}x \quad \text{when } M_m(f) < \infty.$$

Thus $M_0(f) = \|f\|_1$ and $B_0(f)$ is the circulation. An entire moment is used only after its absolute moment has been established. For a fixed smooth compact radial cutoff $\chi$, the cut

moment $\int \chi(|x|)\bar{z}^m f(x)\,\mathrm{d}x$ is defined for every $f \in L^1$. Write $\mathsf{R}_\alpha$ for the counterclockwise planar rotation through angle $\alpha$, and define angular averaging by

$$(P_{\mathrm{rad}} f)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathsf{R}_\alpha x)\,\mathrm{d}\alpha.$$

The notations $P_{\mathrm{rad}}$ and a locally specified $P$ denote this same operator. Fourfold symmetry means $f(\mathsf{R}_{\pi/2}x) = f(x)$ almost everywhere.

The pattern rotation is the scalar action

$$(\mathcal{T}_\theta f)(x) = f(\mathsf{R}_{\theta/4}x), \qquad (\mathcal{T}_\theta f)(r,\varphi) = f(r,\varphi+\theta/4).$$

It corresponds to physical rotation of the pattern through $-\theta/4$. For arbitrary functions $\theta$ is a real parameter; on the fourfold subspace the action is $2\pi$-periodic and hence defines a phase-circle action. Its vector action is the pushforward

$$(\mathcal{T}_\theta V)(x) = \mathsf{R}_{-\theta/4} V(\mathsf{R}_{\theta/4}x).$$

Both actions are isometries in the full-plane Lebesgue and Sobolev norms, commute with the Laplacian, and satisfy

$$K * (\mathcal{T}_\theta f) = \mathcal{T}_\theta (K * f), \qquad \mathcal{T}_\theta(V\cdot\nabla f) = (\mathcal{T}_\theta V)\cdot\nabla(\mathcal{T}_\theta f).$$

These identities follow by change of variables and $K(\mathsf{R}_\alpha x) = \mathsf{R}_\alpha K(x)$. For every finite absolute moment,

$$B_m(\mathcal{T}_\theta f) = e^{im\theta/4} B_m(f), \qquad B_4(\mathcal{T}_\theta f) = e^{i\theta} B_4(f).$$

In particular, a fourfold function has $B_m(f) = 0$ whenever $m$ is not divisible by four and the moment is defined.

## 2.2 Heat and Sobolev estimates

**Lemma 2.1** (Heat estimates)**.** *Let $H_\tau(x) = (4\pi\tau)^{-1} e^{-|x|^2/(4\tau)}$ and $S_\tau f = H_\tau * f$ for $\tau > 0$, with $S_0$ the identity. For exponents $1 \le r \le s \le \infty$ and each integer $a \ge 0$,*

$$\|D^a S_\tau f\|_s \le C_{a,r,s}\,\tau^{-a/2-(1/r-1/s)}\|f\|_r.$$

*In particular $S_\tau$ is a contraction in every $L^r$, and*

$$\|\nabla S_\tau f\|_X \le C\tau^{-1/2}\|f\|_X.$$

*If $f, \Delta f, \dots, \Delta^{N+1} f \in L^r$, then*

$$\left\| S_\tau f - \sum_{l=0}^{N} \frac{\tau^l}{l!}\Delta^l f \right\|_r \le \frac{\tau^{N+1}}{(N+1)!}\|\Delta^{N+1} f\|_r.$$

*For smooth compact profiles this also holds after taking any fixed number of derivatives or replacing $L^r$ by $H^k$. If $f$ is supported in $F$ and $\operatorname{dist}(E,F) \ge d > 0$, then*

$$\|\mathbf{1}_E D^a S_\tau f\|_r \le C_a \tau^{-a/2} e^{-d^2/(8\tau)}\|f\|_r \qquad (1 \le r \le \infty).$$

*At $\tau = 0$ the separated-support value is zero directly.*

*Proof.* Scaling the Gaussian and applying Young's convolution inequality give the first estimates. Repeated integration of $\partial_\tau S_\tau f = S_\tau \Delta f$ gives the Taylor remainder

$$\frac{\tau^{N+1}}{N!}\int_0^1 (1-s)^N S_{s\tau}\Delta^{N+1} f \,\mathrm{d}s;$$

contraction gives its bound. In the off-diagonal estimate only $|x-y| \geq d$ contributes. The $L^1$ norm of the corresponding Gaussian derivative tail is at most $C_a \tau^{-a/2} e^{-d^2/(8\tau)}$, after absorbing its polynomial factor into half the exponential. Young's inequality finishes the proof. □

**Lemma 2.2** (Sobolev energy and Nash estimates)**.** *For every integer $k \geq 3$, $H^k \subset W^{1,\infty}$ and*

$$\|fg\|_{H^k} \leq C_k \|f\|_{H^k}\|g\|_{H^k}.$$

*The Biot–Savart derivative satisfies $\|\nabla K * g\|_{H^j} \leq C\|g\|_{H^j}$ for every integer $j \geq 0$. For smooth bounded $v$ and $g \in H^k$,*

$$\sum_{|a|\leq k} \|D^a(v\cdot\nabla g) - v\cdot\nabla D^a g\|_2 \leq C_k(\|\nabla v\|_\infty \|g\|_{H^k} + \|\nabla v\|_{H^{k-1}}\|\nabla g\|_\infty).$$

*If $v$ is divergence free and $\partial_t g + v\cdot\nabla g = \kappa\Delta g + F$, $\kappa \geq 0$, its $H^k$ energy estimate follows by pairing each differentiated equation with the corresponding derivative of $g$; the undifferentiated transport has zero pairing and diffusion has nonpositive pairing. Finally, for $g \in L^1 \cap H^1$,*

$$\|g\|_2^4 \leq C\|g\|_1^2\|\nabla g\|_2^2.$$

*Proof.* Fourier inversion and Cauchy–Schwarz give the stated embeddings; the integrability of $(1+|\xi|^2)^{-k}$ for $k > 1$ also gives the Sobolev product bound by Fourier convolution. The multiplier of $\nabla K*$ is bounded, so Plancherel gives its $H^j$ estimate. Leibniz's rule gives the transport commutator. Its endpoint terms put one derivative in $L^\infty$ and the other factor in $L^2$; the intermediate terms are bounded by Sobolev interpolation between these endpoints in dimension two. Integration by parts gives the energy assertion, with expanding cutoffs justified by bounded $v$ and integrability of $|D^a g|^2$. For Nash's inequality split the Fourier integral at radius $\rho$:

$$\|g\|_2^2 \leq C\rho^2\|g\|_1^2 + C\rho^{-2}\|\nabla g\|_2^2.$$

Optimizing $\rho$ and squaring proves the last assertion; zero norms are handled by approximation. □

### 2.3 Classical Euler continuation

**Proposition 2.3** (Finite-Sobolev planar Euler evolution)**.** *Let $k \geq 3$ be an integer, and let $\omega_0$ be real, compactly supported, in $H^k$, and have zero circulation. Then there is a unique global Euler solution in the finite-Sobolev class*

$$\omega \in C([0,\infty); H^k) \cap C^1([0,\infty); H^{k-1}), \qquad u = K*\omega \in C([0,\infty); H^{k+1}) \cap C^1([0,\infty); H^k).$$

*Its vorticity is transported by the volume-preserving classical flow, so its $L^r$ norms, $1 \leq r \leq \infty$, and its circulation are preserved. Its velocity energy is conserved. If $\operatorname{supp}\omega_0 \subset B_{R_0}$, then*

$$\|u(t)\|_\infty \leq V, \qquad \operatorname{supp}\omega(t) \subset B_{R_0+Vt}, \qquad V = C(\|\omega_0\|_1 + \|\omega_0\|_\infty).$$

*All Sobolev norms in the statement are bounded on each finite time interval. Uniqueness here is in the indicated finite-Sobolev class.*

*Proof.* Compact support gives $\omega_0 \in L^1$, and zero circulation gives $u_0 \in L^2$ by Lemma 2.6. The order-zero derivative multiplier in Lemma 2.2 then gives $u_0 \in H^{k+1}$. The classical finite-Sobolev local Euler construction is obtained by regularization and the energy commutator estimates; see [3] for this standard local construction. For $G(t) = \|u(t)\|_{H^{k+1}}$, its velocity energy estimate is

$$\frac{\mathrm{d}^+}{\mathrm{d}t} G \le C_k \|\nabla u\|_\infty G.$$

Indeed, the divergence-free transport and pressure have zero energy pairing, and the commutator estimate of Lemma 2.2 at order $k+1$ bounds the remaining terms by $C_k \|\nabla u\|_\infty G^2$. In particular $G' \le C_k G^2$ gives a local existence time controlled by $G(0)$, and the same construction can restart while $G$ stays bounded. The difference of two such solutions satisfies the $L^2$ energy inequality with coefficient $\|\nabla u\|_\infty$, giving uniqueness.

The local solution has a $C^1$ spatial flow. Its Jacobian is one, and the scalar vorticity is constant along trajectories. Thus $m = \|\omega_0\|_1$ and $M = \|\omega_0\|_\infty$ bound the same norms throughout its existence interval. The near–far split of Biot–Savart gives $\|u\|_\infty \le C(m + M)$ and hence the displayed support bound.

We give the continuation estimate explicitly. The off-origin kernel of $\nabla K *$ has size $C|y|^{-2}$ and zero angular mean; a possible local multiple of $\omega$ contributes at most $CM$. For $0 < \rho < 1$, cancellation in the ball of radius $\rho$, the intermediate annuli, and the exterior region give

$$\|\nabla u\|_\infty \le C\big([\omega]_{C^{1/2}} \rho^{1/2} + M(1 + \log(1/\rho)) + m\big).$$

Sobolev embedding bounds $[\omega]_{C^{1/2}}$ by $C_k G$. Take $\rho = (2 + G)^{-2}$ to obtain

$$\|\nabla u\|_\infty \le C_{k,m,M}\big(1 + \log(2 + G)\big).$$

Thus $Y = \log(2 + G)$ satisfies $Y' \le C_{k,m,M}(1 + Y)$. Its integrating-factor bound prevents finite-time growth of $G$ to infinity, so the local solution extends globally. The vorticity and velocity equations give the asserted time-derivative regularity. Their strong Sobolev time continuity is part of the local construction and is preserved under restart. □

## 2.4 The classical viscous solution and its dependence on data

**Proposition 2.4** (Planar Navier–Stokes inputs)**.** *For $\nu > 0$ and real $\omega_0 \in L^1$, equation (2) has a unique global mild solution in*

$$C([0,\infty); L^1) \cap C((0,\infty); L^\infty).$$

*It preserves circulation and is smooth at positive times. For $1 \le r \le \infty$,*

$$\|\omega^\nu(t)\|_r \le C_r (\nu t)^{-(1-1/r)} \|\omega_0\|_1 \qquad (t > 0).$$

*Whenever $\omega_0 \in L^r$, the solution also satisfies the $L^r$ contraction $\|\omega^\nu(t)\|_r \le \|\omega_0\|_r$. If $\omega_0 \in X_p$, then $\omega^\nu \in C([0,\infty); X_p)$, and on each finite interval the solution map is locally Lipschitz in $X_p$ for this fixed $\nu$. For $\omega_0 \in X_p$, if additionally $u_0 = K * \omega_0 \in L^2$, its velocity satisfies*

$$\|u^\nu(t)\|_2^2 + 2\nu \int_0^t \|\nabla u^\nu(s)\|_2^2 \,\mathrm{d}s \le \|u_0\|_2^2.$$

*For $\omega_0 \in X_p$ and its bounded drift $u^\nu$, a prescribed colour $f_0 \in L^1 \cap L^r$, $1 < r < \infty$, has a unique mild path $f \in C_t(L^1 \cap L^r)$ solving*

$$\partial_t f + \operatorname{div}(u^\nu f) = \nu \Delta f, \qquad f(0) = f_0.$$

*It preserves its integral, its $L^1$ and $L^r$ norms contract, and positivity and order are preserved, and an initial $L^\infty$ bound contracts as well. At fixed viscosity these colour paths depend continuously on their initial colours and on the total initial datum in the indicated norms.*

*Proof.* The global $L^1$ existence, uniqueness, positive-time $L^\infty$ continuity, circulation, and smoothing estimate at viscosity one are Theorem 2.2 of Gallay–Wayne [1, Theorem 2.2]. To obtain viscosity $\nu$, let $\Omega$ solve that equation with datum $\omega_0/\nu$ and set

$$\omega^\nu(t,x) = \nu \Omega(\nu t, x), \qquad u^\nu(t,x) = \nu (K * \Omega)(\nu t, x).$$

Substitution proves (2) and its asserted estimates. The mild equation is

$$\omega^\nu(t) = S_{\nu t}\omega_0 - \int_0^t \nabla \cdot S_{\nu(t-s)}\big[(K * \omega^\nu(s))\omega^\nu(s)\big]\,\mathrm{d}s.$$

Positive-time smoothness follows by heat bootstrap on intervals bounded away from zero.

For smooth approximations, multiply the scalar equation by the derivative of a convex regularization of $|s|^r$ and use expanding cutoffs. Divergence freedom removes transport, and diffusion is nonpositive. For $1 < r < \infty$ this yields

$$\|\omega^\nu(t)\|_r^r + \frac{4\nu(r-1)}{r} \int_0^t \|\nabla |\omega^\nu(s)|^{r/2}\|_2^2 \,\mathrm{d}s \le \|\omega_0\|_r^r.$$

Absolute-value regularizations give the $L^1$ contraction; the maximum principle, or passage $r \to \infty$, gives the $L^\infty$ contraction. These estimates pass through initial-data approximation. The same arguments give contraction and order preservation for the linear colour equation.

For $X_p$ data, Lemmas 2.6 and 2.1 make the mild equation a contraction on a bounded $C_t X_p$ ball for a sufficiently short interval. The scalar contractions prevent loss of this bound, giving continuation and strong $X_p$ continuity from zero. If two paths lie in an $X_p$ ball of radius $M$, their difference $d(t)$ obeys, after a restart at any time $a$,

$$d(t) \le d(a) + \frac{C_p M}{\sqrt{\nu}} \int_a^t (t-s)^{-1/2} d(s)\,\mathrm{d}s.$$

Choose a length $\ell > 0$ with $2C_pM\sqrt{\ell/\nu} \le 1/2$. On each such interval the supremum of $d$ is at most $2d(a)$. Iteration over $[0,T]$ proves

$$\sup_{t \le T} d(t) \le C_{\nu,T,M}\, d(0).$$

This is a fixed-viscosity estimate.

The linear colour mild equation has the same integrable heat-gradient kernel. If $Y = L^1 \cap L^r$, two colour paths satisfy on a short restarted interval

$$d_f(t) \le d_f(a) + \frac{C}{\sqrt{\nu}} \int_a^t (t-s)^{-1/2} \big(V d_f(s) + M_f \|u_1(s) - u_2(s)\|_\infty\big)\,\mathrm{d}s,$$

where $V$ bounds the drifts and $M_f$ bounds the second colour in $Y$. Iteration and the total-path estimate prove the asserted colour continuity and well-posedness. The convex scalar calculation justifies all its contractions and comparison statements by approximation.

Under the additional finite-energy assumption, mollify $u_0$ and its vorticity together. Smooth finite-energy solutions satisfy the velocity energy identity by divergence freedom and integration by parts. Their uniform $L^2$ bound and dissipation pass to the mild solution by weak lower semicontinuity, using the already established initial-data continuity. This proves the stated energy inequality. □

For a countable decomposition $\omega_0 = \sum_l f_{l,0}$ with $\sum_l \|f_{l,0}\|_{X_p} < \infty$, define every colour by the linear equation with the actual common velocity $u^\nu$. Contraction gives absolute convergence of $\sum_l f_l$ in $C([0,T]; X_p)$. Its sum solves that same linear equation with datum $\omega_0$, so linear uniqueness gives $\sum_l f_l = \omega^\nu$. These are the actual colours used in the interaction estimates. Positive envelopes are defined by the same linear equation with nonnegative initial data; comparison gives $|f_l| \le P_l$ when $P_l(0) = |f_{l,0}|$.

### 2.5 Scalar comparison

**Lemma 2.5** (Scalar and positive-majorant comparison)**.** *Let $v$ be smooth and divergence free, with at most linear growth: $|v(t,x)| \le C(t)(1+|x|)$, where $C$ is integrable on the time interval. A smooth solution of*

$$\partial_t z + \operatorname{div}(vz) = \kappa \Delta z + F, \qquad \kappa \ge 0,$$

*satisfies, whenever $z \in L^\infty_t L^r$ and $F \in L^1_t L^r$,*

$$\|z(t)\|_r \le \|z(a)\|_r + \int_a^t \|F(s)\|_r \, \mathrm{d}s, \qquad 1 \le r \le \infty.$$

*These estimates and positivity extend to the standard mild viscous solutions by approximation. If $P$ solves the same scalar equation with datum $|z(a)|$ and source $|F|$, then $|z| \le P$. Consequently, for $X = X_p$, an error equation with source $-(K * z) \cdot \nabla W_* + R$ satisfies*

$$\|z(t)\|_X \le \|z(a)\|_X + \int_a^t \left( C_p \|\nabla W_*(s)\|_X \|z(s)\|_X + \|R(s)\|_X \right) \mathrm{d}s.$$

*For a finite nonnegative system, suppose*

$$x_i(t) \le b_i e^{\lambda t} + \int_0^t e^{\lambda(t-s)} \left( \sum_j k_{ij}(s) x_j(s) + F_i(s) \right) \mathrm{d}s,$$

*with $b_i, F_i, k_{ij} \ge 0$, locally integrable coefficients, and continuous $x_i \ge 0$. If $Y' = (\lambda I + K)Y + F$, $Y(0) = b$, then $x \le Y$. For fixed weights $\rho_i > 0$, put $r(s) = \max_i \sum_j k_{ij}(s)\rho_j/\rho_i$. Then*

$$\max_i \frac{x_i(t)}{\rho_i} \le e^{\lambda t + \int_0^t r} \max_i \frac{b_i}{\rho_i} + \int_0^t e^{\lambda(t-s) + \int_s^t r} \max_i \frac{F_i(s)}{\rho_i} \, \mathrm{d}s.$$

*Proof.* Convex regularization gives the local inequality $\partial_t |z| + \operatorname{div}(v|z|) \le \kappa \Delta |z| + |F|$. On a cutoff annulus of radius $R \ge 1$, the drift flux for $|z|^r$ is at most $C_r C(t) \int_{|x|>R} |z(t,x)|^r$. Its time integral tends to zero by dominated convergence. Thus expanding cutoffs give the global $L^1$ estimate and positive envelope comparison even with this linear growth. Regularized powers and

Hölder's inequality give the finite-$r$ estimates. For $r = \infty$, use the maximum principle with a barrier $\epsilon A(t)(1+|x|^2)$, choosing the positive factor $A$ to dominate the integrable drift growth and diffusion, and then send $\epsilon \downarrow 0$. Sum the estimates at $r = 1, p$ and use Lemma 2.6 for the error-equation conclusion. For the nonnegative system, iterate its Volterra inequality. The remainder after $n$ iterations tends to zero on bounded time intervals by the factorial bound for an iterated integrable kernel. The resulting positive series is the solution $Y$. Its weighted maximum obeys the scalar inequality with coefficient $\lambda + r(s)$; the integrating-factor formula gives the last bound. □

For completeness, a divergence-free drift with the same at-most-linear growth gives its standard viscous scalar evolution the $L^1 \to L^\infty$ bound $C/(\kappa(t-a))$, $\kappa > 0$. The $L^2$ energy identity and Nash's inequality give $L^1 \to L^2$ norm $C/\sqrt{\kappa(t-a)}$. The backward adjoint has the same estimate because its drift is also divergence free. Duality gives $L^2 \to L^\infty$; composition at the midpoint proves the claim. Initial-data approximation justifies it for the mild colour paths used here.

### 2.6 Physical and local colour normalization

Let $\omega_j^\nu(t_{\text{phys}}, x)$ denote a physical colour transported and diffused by the common total velocity $u^\nu$. At vorticity amplitude $A_j > 0$ and length $R_j > 0$, define

$$s_j = A_j t_{\text{phys}}, \qquad y = x/R_j, \qquad w_j(s_j, y) = A_j^{-1}\omega_j^\nu(s_j/A_j, R_j y),$$

$$U_j(s_j, y) = (A_j R_j)^{-1} u^\nu(s_j/A_j, R_j y), \qquad \kappa_j = \frac{\nu}{A_j R_j^2}.$$

The normalized colour satisfies

$$\partial_{s_j} w_j + \operatorname{div}_y(U_j w_j) = \kappa_j \Delta_y w_j.$$

The self-induced part of $U_j$ is $K * w_j$; its other parts are all foreign colours expressed in this same receiving frame. If $w_l^{(l)}$ is colour $l$ in its own frame, its expression in frame $j$ is

$$w_l^{(j)}(s_j, y) = \frac{A_l}{A_j} w_l^{(l)}\left(\frac{A_l}{A_j} s_j, \frac{R_j}{R_l} y\right).$$

Thus its own clock is $s_l = (A_l/A_j)s_j$. Translated clocks $s_j = A_j(t_{\text{phys}} - t_*)$ have the same diffusivity. At corresponding times the norm, velocity, and moment scalings are

$$\|\omega_j^\nu\|_r = A_j R_j^{2/r}\|w_j\|_r, \qquad \|K * \omega_j^\nu\|_\infty = A_j R_j \|K * w_j\|_\infty, \qquad B_m(\omega_j^\nu) = A_j R_j^{m+2} B_m(w_j).$$

The last equality is asserted only when the absolute moment is finite. Statements in a local frame use these normalized quantities; field bounds in physical variables include their displayed amplitude and length factors.

**Lemma 2.6** (The velocity estimates used below)**.** *Let $p > 2$, $\alpha = 1 - 2/p$, and $w \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Then*

$$\|K * w\|_\infty + \|K * w\|_p \le C_p(\|w\|_1 + \|w\|_p), \qquad [K * w]_{C^\alpha} \le C_p\|w\|_p, \qquad \|\nabla K * w\|_p \le C_p\|w\|_p.$$

*If $w$ has compact support and zero integral, then $K * w \in L^2$.*

*Proof.* For the supremum norm split $K$ at radius one: its near part is in $L^{p'}$ since $p' < 2$, and its far part is bounded. For the $L^p$ norm the near part is in $L^1$ and the far part is in $L^p$; apply Young's inequality. For a velocity difference at points a distance $d$ apart, the two radius-$2d$ disks contribute at most $C_p\|w\|_p d^\alpha$. On their complement use $|K(x-z) - K(y-z)| \le Cd|x-z|^{-2}$ and Hölder's inequality; the same power $d^\alpha$ follows.

We recall a direct proof of the derivative bound to specify the analytic input. Each component $T$ of $\nabla K*$ has a Fourier multiplier bounded in absolute value, and hence is bounded on $L^2$ by Plancherel. Its off-origin kernel satisfies $|t(x)| \le C|x|^{-2}$, $|\nabla t(x)| \le C|x|^{-3}$ and has zero angular average; a possible multiple of the identity is harmless. For $f \in L^1 \cap L^2$ decompose at height $\lambda$ into $g + \sum b_Q$ using maximal dyadic squares with average of $|f|$ larger than $\lambda$. The squares are disjoint, and

$$\|g\|_\infty \le 4\lambda, \quad \|g\|_1 \le \|f\|_1, \quad \sum |Q| \le \|f\|_1/\lambda, \quad \int b_Q = 0, \quad \sum \|b_Q\|_1 \le 2\|f\|_1.$$

The $L^2$ bound controls the level set of $Tg$ by $C\|f\|_1/\lambda$. Cancellation and the kernel derivative estimate give

$$\int_{\mathbb{R}^2\setminus 5Q} |Tb_Q(x)|\,\mathrm{d}x \le \int_Q |b_Q(y)| \int_{\mathbb{R}^2\setminus 5Q} |t(x-y) - t(x-c_Q)|\,\mathrm{d}x\,\mathrm{d}y \le C\|b_Q\|_1.$$

The last inner integral is bounded by $C|y-c_Q| \int_{C\ell(Q)}^\infty r^{-3} r\,\mathrm{d}r$. This proves weak type $(1,1)$, first for finite sums and then by their $L^2$ convergence. Splitting $f$ above and below a height proportional to $\lambda$ consequently gives

$$|\{|Tf| > \lambda\}| \le C\left[\lambda^{-1}\int_{|f|>c\lambda} |f| + \lambda^{-2}\int_{|f|\le c\lambda} |f|^2\right].$$

Integration against $s\lambda^{s-1}\,\mathrm{d}\lambda$, with $1 < s < 2$, and Fubini bound the $L^s$ norm: the two integrals produce finite factors $(s-1)^{-1}$ and $(2-s)^{-1}$. The adjoint has the same bounds, so duality with $s = p'$ proves the required $L^p$ estimate. Density identifies the result with the distributional derivative of $K * w$.

Finally, compact support gives a finite first absolute moment. At low frequencies zero circulation implies $|\widehat{w}(\xi)| \le |\xi| \int |x||w(x)|\,\mathrm{d}x$. At high frequencies use $w \in L^2$, which follows by interpolation between $L^1$ and $L^p$. The Fourier multiplier $i\xi^\perp/|\xi|^2$ and Plancherel prove $K * w \in L^2$. □

## 2.7 A fixed passive colour is determined by the total Euler path

The next result allows signed smooth colours as well as indicator colours. It therefore does not require a patch reference profile.

**Proposition 2.7** (Recovery of fixed passive colours)**.** *Let $u$ be bounded and divergence free on $[0,T]\times\mathbb{R}^2$, with $u \in L^1([0,T]; W^{1,p}_{\mathrm{loc}})$ for some $p > 1$. For each compactly supported $f_0 \in L^1 \cap L^\infty$ there is at most one distributional solution of*

$$\partial_t f + \operatorname{div}(uf) = 0, \qquad f(0) = f_0,$$

*in $L^\infty([0,T]; L^1 \cap L^\infty)$ with its weak initial trace. Every such solution is renormalized and has a representative in $C([0,T]; L^1 \cap L^2)$; its $L^1$ and $L^2$ norms are conserved.*

*Suppose that the bounded measurable drifts $u_n$ are divergence free, $u_n \to u$ uniformly on $[0,T]\times\mathbb{R}^2$, $\sup_n \|u_n\|_\infty \le V$, and $\nu_n \to 0$, with $0 \le \nu_n \le 1$. Let $f_n \in C([0,T];L^1\cap L^2)\cap L^\infty([0,T];L^\infty)$ solve*

$$\partial_t f_n + \operatorname{div}(u_n f_n) = \nu_n \Delta f_n, \qquad f_n(0) = f_0,$$

*in the weak sense, with the $L^1$, $L^2$, and $L^\infty$ contraction bounds at every time. Then the transport limit $f$ is unique and*

$$f_n \longrightarrow f \quad \text{in } C([0,T];L^1\cap L^2).$$

*This includes transported colours with $\nu_n = 0$ under the same continuity, weak-equation, and contraction hypotheses.*

*Proof.* We first prove renormalization, including the prescribed trace. The transport equation and the uniform $L^2$ bound give a weakly continuous $L^2$ representative: compact smooth pairings are absolutely continuous in time, and density extends their continuity to all $L^2$ tests. Its values at every time retain the $L^1$ and $L^\infty$ bounds, by approaching that time through bounded slices, taking weak limits on finite balls, and then enlarging the balls.

**Renormalization at the prescribed trace.** Extend the solution to $-1 < t < 0$ by $\widetilde{f}(t) = f_0$ and the drift by $\widetilde{u}(t) = 0$. The weak initial-trace identity gives

$$\partial_t \widetilde{f} + \operatorname{div}(\widetilde{u}\widetilde{f}) = 0 \quad \text{on } (-1,T)\times\mathbb{R}^2.$$

Thus there is no distributional jump at zero, and the extended drift has the same boundedness, divergence, and local Sobolev properties. For a spatial mollifier, write $\widetilde{f}_\varepsilon = \widetilde{f} * \eta_\varepsilon$. Its equation is

$$\partial_t \widetilde{f}_\varepsilon + \widetilde{u}\cdot\nabla\widetilde{f}_\varepsilon = r_\varepsilon, \qquad r_\varepsilon(t,x) = \int \nabla\eta_\varepsilon(y)\cdot[\widetilde{u}(t,x)-\widetilde{u}(t,x-y)]\widetilde{f}(t,x-y)\,\mathrm{d}y.$$

On a compact spatial set the difference-quotient estimate bounds its $L^1$ norm by

$$C\|\nabla\widetilde{u}(t)\|_p\|\widetilde{f}(t)\|_{p'}$$

on a slightly larger set. To see this, express the velocity difference as the integral of its gradient along the segment and apply Hölder's inequality. Approximate the drift by divergence-free spatial convolutions in $L^1_t W^{1,p}_{\mathrm{loc}}$. For a smooth approximating drift, divergence freedom allows subtraction of $\widetilde{f}(t,x)$ inside the commutator integral. Translation continuity of $\widetilde{f}$ then gives convergence to zero in local $L^1_{t,x}$. The difference-quotient bound controls the drift-approximation error. Consequently $r_\varepsilon \to 0$ locally in $L^1$.

For $\beta \in C^1$ with bounded derivative, the chain rule for the mollified equation and the commutator convergence give

$$\partial_t\beta(\widetilde{f}) + \operatorname{div}(\widetilde{u}\beta(\widetilde{f})) = 0.$$

This identity has the prescribed renormalized trace $\beta(f_0)$ at zero: its compact smooth pairings have continuous representatives, and on negative times they equal their pairings with $\beta(f_0)$. There are also renormalized traces at every prescribed time. Indeed, for each fixed $\varepsilon$ the mollified equation is absolutely continuous in time in local spatial norms. Integrate its renormalized equation between any fixed $s,t \in [0,T]$. At both endpoints spatial approximate identities give $f_\varepsilon(s) \to f(s)$ and $f_\varepsilon(t) \to f(t)$ strongly in $L^2$ on finite balls; the Lipschitz function $\beta$ passes those endpoint terms in local $L^1$. The commutator error tends to zero. This proves the renormalized identity between every such pair of times, including the endpoints.

For $\beta(0) = 0$, expanding spatial cutoffs yield

$$\int \beta(f(t)) = \int \beta(f_0) \qquad (0 \le t \le T),$$

because the flux error is at most $C\|u\|_\infty R^{-1}\|\beta'\|_\infty \sup_t \|f(t)\|_1$ times the length of the time interval. Choose a bounded-derivative extension of $s^2$ from the bounded range of $f$ to obtain conservation of $\|f(t)\|_2$. Using $\sqrt{s^2+\epsilon^2}-\epsilon$ and sending $\epsilon \downarrow 0$ gives conservation of $\|f(t)\|_1$. Apply the same square argument to the difference of two solutions with zero weak initial trace to prove uniqueness. Weak $L^2$ continuity and conservation of the square norm give strong $L^2$ continuity at every time.

Renormalization by absolute value also gives finite propagation. If $\operatorname{supp} f_0 \subset B_{R_0}$ and $\|u\|_\infty \le V$, test the equation for $|f|$ against a smooth nondecreasing function $\psi(|x| - R_0 - Vt)$ which vanishes for nonpositive arguments. Its material derivative is nonpositive. Expanding spatial cutoffs justify this bounded exterior test using $|f| \in L^\infty_t L^1$. The initial pairing is zero, so the nonnegative pairing remains zero. Letting the transition width of $\psi$ tend to zero proves

$$\operatorname{supp} f(t) \subset B_{R_0+Vt} \qquad (0 \le t \le T).$$

Local $L^1$ continuity follows from strong $L^2$ continuity, and this common support ball gives continuity in global $L^1$.

For the approximating sequence, the weak equation gives for compact smooth $\varphi$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi f_n\right| \le \|f_0\|_1(V\|\nabla\varphi\|_\infty + \|\Delta\varphi\|_\infty).$$

The contraction bounds, a countable dense set of $L^2$ tests, and this equicontinuity give weak $L^2$ convergence uniformly in time along a subsequence. The limit has the $L^1$ and $L^\infty$ bounds by local lower semicontinuity. Uniform convergence of the drifts and $\nu_n \to 0$ pass the weak equation to $\partial_t f + \operatorname{div}(uf) = 0$ with initial trace $f_0$. Uniqueness identifies every subsequential limit.

Suppose uniform strong $L^2$ convergence fails, and select bad times $t_n \to t$. Uniform weak convergence gives $f_n(t_n) \rightharpoonup f(t)$ in $L^2$, while contraction and renormalization imply

$$\limsup_n \|f_n(t_n)\|_2 \le \|f_0\|_2 = \|f(t)\|_2.$$

Uniform convexity of $L^2$ gives strong convergence to $f(t)$, and strong time continuity of $f$ contradicts the choice of the times. Thus convergence is uniform strongly in $L^2$.

For the $L^1$ upgrade set $m = \|f_0\|_1$ and choose a fixed ball $B_R$ containing the support of $f(t)$ for all $t \le T$. Conservation of the limiting $L^1$ norm and contraction of the approximating norms give

$$\int_{B_R^c} |f_n(t)| \le m - \int_{B_R} |f_n(t)| \le \int_{B_R} |f_n(t) - f(t)|.$$

Consequently

$$\sup_{t\le T} \|f_n(t) - f(t)\|_1 \le 2|B_R|^{1/2} \sup_{t\le T} \|f_n(t) - f(t)\|_2 \longrightarrow 0.$$

The assumed strong continuity of the approximating paths makes these uniform estimates convergence in the asserted $C_t$ spaces. □

For $p > 2$, Lemma 2.6 gives the useful bound

$$\|K * h\|_\infty \le C_p(\|h\|_1 + \|h\|_p) : \tag{3}$$

strong convergence of total vorticity paths in $C_t(L^1 \cap L^p)$ therefore implies uniform convergence of their velocities. Whenever the limiting velocities have the Sobolev regularity in Proposition 2.7, that proposition applies to every fixed bounded compactly supported initial colour. Equal total Euler paths then have equal recovered fixed colours.

Fix finitely many bounded compactly supported initial colours $f_{1,0}, \dots, f_{k,0}$, complex coefficients $c_l$, and complex tests $\psi_l \in C_b^2(\mathbb{R}^2)$, where the function and its derivatives through order two are bounded. These choices define the one fixed functional

$$a_u(t) = \sum_{l=1}^{k} c_l \int \psi_l(x) f_l[u](t,x)\,\mathrm{d}x. \tag{4}$$

Expanding spatial cutoffs in the weak equation justify these bounded noncompact tests: the new drift and diffusion errors are bounded by $C\|f_0\|_1(V/R + \nu/R + \nu/R^2)$ and vanish. In particular

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int \psi_l f_l\right| \le \|f_{l,0}\|_1(V\|\nabla\psi_l\|_\infty + \nu\|\Delta\psi_l\|_\infty).$$

Thus the observable is uniformly Lipschitz for bounded drifts and $0 \le \nu \le 1$, and strong $C_tL^1$ convergence makes its pairings converge uniformly. Proposition 2.7 therefore makes it continuous under the strong total-path convergence specified above. For degree shooting we use tests with the covariance

$$\psi_l(\mathsf{R}_{\theta/4}x) = e^{-i\theta}\psi_l(x),$$

where $\mathsf{R}$ is the planar rotation matrix. This includes a fixed bounded adjoint-coordinate test and the compact moment tests $\psi_l(x) = \Phi_l(x) = \chi_l(|x|)\bar{z}^4$ with fixed smooth radial compact cutoffs. These compact tests cover one smooth outer colour or several patch colours. To recover uncut fourth moments in an Euler limit, choose their cutoffs equal to one on the common support ball supplied by finite propagation.

## 2.8 Protected first caps and phase shooting

We use the first crossing of one fixed observable. Its continuity is controlled by a strict early margin and a positive radial slope in a fixed neighbourhood of that crossing.

**Lemma 2.8** (Continuity of a protected first-cap phase)**.** *Fix $T, \delta, c, e, h, L > 0$, with $2e < T$. Let $a : [0,T] \to \mathbb{C}$ be $L$-Lipschitz, and define*

$$\tau(a) = \inf\{t \in [0,T] : |a(t)| \ge \delta\}, \qquad \mathcal{P}(a) = a(\tau(a))/\delta,$$

*where the infimum of the empty set is $+\infty$ and $\mathcal{P}$ is defined when the cap exists. Suppose*

$$\begin{aligned} |a(0)| &\le \delta - h, \qquad e \le \tau(a) \le T - e, \\ |a(t)| &\le \delta - h \quad (0 \le t \le \tau(a) - e), \\ |a(t)| - |a(s)| &\ge c(t-s) \quad (\tau(a) - e \le s < t \le \tau(a) + e). \end{aligned}$$

*For any continuous $b : [0,T] \to \mathbb{C}$ with $\varepsilon = \|a - b\|_\infty < \min(h/2, ce/4)$, its first cap exists and*

$$|\tau(a) - \tau(b)| \le 2\varepsilon/c, \qquad |\mathcal{P}(a) - \mathcal{P}(b)| \le \delta^{-1}(1 + 2L/c)\varepsilon.$$

*Families satisfying the displayed conditions with common constants retain those conditions under uniform convergence. Hence $\mathcal{P}$ is continuous on this protected class. The lower-slope assertion is confined to the displayed interval around the first crossing.*

*Proof.* Continuity and the strict initial margin give $|a(\tau(a))| = \delta$. If $\varepsilon = 0$ the conclusions are immediate. Suppose $\varepsilon > 0$. Before $\tau(a) - e$ the strict margin excludes a cap of $b$. On $[\tau(a) - e, \tau(a)]$, the radial lower-slope inequality gives

$$|a(t)| \le \delta - c(\tau(a) - t).$$

Thus $b$ cannot cap before $\tau(a) - 2\varepsilon/c$. At $\tau(a) + 2\varepsilon/c$ its modulus is at least $\delta + \varepsilon$, so it has capped by that time. Both comparison times lie inside the protected interval and inside $[0, T]$. This proves the time estimate. Both first-cap values have modulus $\delta$, and

$$|a(\tau(a)) - b(\tau(b))| \le L|\tau(a) - \tau(b)| + \|a - b\|_\infty$$

proves the phase estimate.

For a uniformly convergent protected sequence, extract a convergent subsequence of cap times. The common Lipschitz bound, early margin, and integrated radial lower-slope inequality pass to the limit; for interval endpoints first pass on strict interior intervals and then use continuity. The limiting time has modulus $\delta$, every earlier time has strictly smaller modulus, and it is the first cap. The limiting path satisfies the same protection conditions. This applies to every subsequence, giving the closure and continuity assertions. □

**Corollary 2.9** (Separation of total inviscid paths)**.** *Consider one fixed total initial datum and fixed bounded compactly supported initial colours. On both subsequences use the same initial colours $f_{l,0}$, coefficients $c_l$, tests $\psi_l$, and cap level $\delta$, so that (4) defines one common functional. Suppose two Navier–Stokes subsequences converge strongly in $C_t(L^1 \cap L^p)$, $p > 2$, their limits satisfy the velocity regularity of Proposition 2.7, and the approximating observables have first caps protected by common constants on $[0, T]$. If the two limiting first-cap phases of this common functional differ, the total Euler paths differ. The cap times may differ.*

*Proof.* Proposition 2.7 and Lemma 2.8 pass the fixed colours, observable paths, and first-cap phases to the limit. Equal total paths would have equal velocities and recovered colours, hence equal observable paths and first-cap phases. This contradicts the assumed phase separation. □

**Proposition 2.10** (Conditional degree shooting)**.** *Fix $\nu > 0$ and an active index $N \ge 2$. Suppose a continuous coloured initial-data family is indexed by $(r, \theta) \in [0, 1] \times \mathbb{S}^1$: the active seed at $N$ has phase $\theta$, and all other seed amplitudes are multiplied by $r$. Assume the total and passive-colour solution paths at this viscosity depend continuously on these parameters. Use fixed coefficients, tests $\psi_l \in C_b^2$ satisfying $\psi_l(\mathsf{R}_{\theta/4}x) = e^{-i\theta}\psi_l(x)$, and cap level in (4), and assume the observable paths have first caps protected by common constants from Lemma 2.8.*

*At $r = 0$, assume the full coloured initial datum at phase $\theta$ is the spatial rotation through angle $-\theta/4$ of the datum at phase zero. This covariance holds when only the active seed rotates and all background profiles and other initial colours are radial. Then the output first-cap phase map at $r = 1$ is surjective onto $\mathbb{S}^1$.*

*Proof.* The first-cap phase map is continuous by Lemma 2.8. Rotational covariance of the full equation and the fixed tests give

$$a_{0,\theta}(t) = e^{i\theta}a_{0,0}(t), \qquad \mathcal{P}(a_{0,\theta}) = e^{i\theta}\mathcal{P}(a_{0,0}).$$

Its degree is one. To see directly that degree is invariant along the homotopy, lift its continuous argument on $[0,1]\times[0,2\pi]$. The endpoint difference is $2\pi d(r)$ for an integer $d(r)$; continuity makes this integer constant. A circle map omitting a point has a periodic continuous argument and degree zero. The map at $r=1$ therefore cannot omit a point. □

Fixed-positive-viscosity continuity follows from the mild equations. In $X=L^1\cap L^p$, $p>2$, (3) makes $(K*w)w$ locally Lipschitz on bounded $X$ balls. If two total paths are bounded there by $M$, their difference satisfies

$$d(t)\le d(0)+\frac{C_pM}{\sqrt{\nu}}\int_0^t(t-s)^{-1/2}d(s)\,\mathrm{d}s.$$

Take short intervals on which the integral coefficient is at most $1/2$ and iterate to obtain continuity for each fixed $\nu$. The passive mild equation has the same heat-gradient kernel; its additional source is bounded by the uniform drift difference times the bounded colour norm. It gives continuity of colours in $C_t(L^1\cap L^2)$ when their initial data vary in $L^1\cap L^2$. For indicator colours, continuity in symmetric-difference measure of the initial domains supplies precisely this initial-data continuity. Uniform cap protection across the homotopy remains a substantive coupled PDE hypothesis.

**Proposition 2.11** (Conditional sparse-phase construction)**.** *Fix $p>2$ and $\beta q^{2/p}<1$. Let a family of infinitely seeded, zero-circulation compact initial data be indexed by a phase at every depth. Assume its defining series has a summable bound in $X=L^1\cap L^p$ uniform in all phases, that changing only the phase at depth $N$ changes the datum by at most $\eta_N\to 0$ uniformly in the other phases, and that depths $N\ge 2$ leave the fixed outer initial colours unchanged. Fix the coefficients, tests, and cap level of one outer observable (4) throughout the family.*

*Suppose $\nu_N\downarrow 0$, and at every depth $N\ge 2$, for every choice of the other phases, Proposition 2.10 applies at $\nu_N$ with common cap-protection constants on $[0,T]$. Suppose also that for each datum in the family its Navier–Stokes paths are relatively compact in $C([0,T];X)$ as $\nu\downarrow 0$, and every limit satisfies the velocity hypothesis of Proposition 2.7. Then one single datum in this family has two distinct vanishing-viscosity accumulation paths on $[0,T]$.*

*Proof.* Start with default phases at all depths. At step $k$, choose a new depth $N_k>N_{k-1}$, $N_k\ge 2$, with $\nu_{N_k}<\nu_{N_{k-1}}/2$ when $k>1$, and

$$\eta_{N_k}<2^{-k}\min(1,r_1,\dots,r_{k-1}),$$

where at $k=1$ the minimum is 1. Shoot its phase to give first-cap output $(-1)^k$ at $\nu_{N_k}$, and call the resulting datum $g_k$. Fixed-positive-viscosity continuity gives $r_k>0$ such that an admissible datum within $r_k$ of $g_k$ changes that output by less than $1/4$. All later admissible phase choices have protected caps by the common hypothesis.

The summable changes give a limit $g_\infty\in X$. Each coordinate is changed at most once, and uniform absolute convergence of the defining series identifies the limit with the resulting phase sequence in the assumed family. For every $k$,

$$\|g_\infty-g_k\|_X\le\sum_{l>k}2^{-l}r_k<r_k.$$

Thus its selected-viscosity output lies within $1/4$ of $(-1)^k$. Relative compactness supplies a limiting path along each parity. Colour convergence and protected-cap continuity put their first-cap phases in the disjoint closed $1/4$-neighbourhoods of $+1$ and $-1$. Corollary 2.9 makes the

total paths different. Circulation passes by strong $L^1$ convergence. Compact support follows from membership of $g_\infty$ in the assumed compact-data phase family; a common support ball, as in the scaled stack construction, also gives it directly. □

The proposition is conditional on uniform cap protection and phase shooting for the actual coupled system. Every intermediate datum retains the complete infinite background, and compactness alone supplies no cap or phase-shooting estimate.

### 2.9 Compactness for one fixed compact zero-circulation datum

We prove the strong compactness used in phase separation for one fixed initial datum. The Navier–Stokes inputs are its global solution, strongly continuous vorticity paths, the $L^1$ and $L^p$ contraction bounds, and the finite-energy inequality. They follow from smooth approximation of the datum and do not require uniqueness of its Euler evolution.

**Theorem 2.12** (Strong vanishing-viscosity precompactness)**.** *Let $2 < p < \infty$ and let one fixed real initial vorticity satisfy*

$$\omega_0 \in L^1 \cap L^p, \qquad \operatorname{supp}\omega_0 \subset B_{R_0}, \qquad \int \omega_0 = 0.$$

*For every sequence $\nu_n \downarrow 0$, the unforced planar Navier–Stokes vorticities with this datum have a subsequence converging strongly in $C([0,T]; L^1 \cap L^p)$ for each fixed $T$. Every limit is a renormalized Euler solution with initial vorticity $\omega_0$. Its velocity is bounded and belongs to $L^\infty([0,T]; W^{1,p})$. At every time its vorticity preserves its $L^1$ and $L^p$ norms and circulation, and*

$$\operatorname{supp}\omega(t) \subset B_{R_0+Vt}, \qquad V = C_p(\|\omega_0\|_1 + \|\omega_0\|_p).$$

*In particular the compactness and Sobolev-velocity hypotheses for fixed-colour first-cap separation hold for this class.*

*Proof.* Set $m = \|\omega_0\|_1$, $P = \|\omega_0\|_p$, and $E = \|K * \omega_0\|_2$, finite by Lemma 2.6. Discarding finitely many indices ensures $0 < \nu_n \le 1$. Vorticity contraction and the kinetic-energy inequality give

$$\|\omega_n(t)\|_1 \le m, \quad \|\omega_n(t)\|_p \le P, \quad \|u_n(t)\|_2 \le E, \quad \|u_n(t)\|_\infty \le V, \qquad V = C_p(m+P). \tag{5}$$

Lemma 2.6 also bounds $[u_n(t)]_{C^\alpha}$, $\|u_n(t)\|_p$, and $\|\nabla u_n(t)\|_p$, where $\alpha = 1 - 2/p > 0$.

*Local velocity compactness.* Project the velocity equation onto divergence-free fields. The projection has a bounded matrix Fourier multiplier and is bounded on every $H^s$. Since $\|u_n \otimes u_n\|_2 \le VE$, the equation implies

$$\|\partial_t u_n\|_{H^{-2}} \le CE(V+1).$$

For a spatial mollifier $\eta_h$, $0 < h \le 1$, its $H^2$ norm is at most $Ch^{-3}$ in two dimensions. Spatial Hölder continuity and $H^{-2}$–$H^2$ duality therefore give

$$\|u_n(t) - u_n(s)\|_\infty \le Ch^\alpha + Ch^{-3}|t-s|.$$

Take $h = |t-s|^{1/(3+\alpha)}$ for $|t-s| \le 1$. Arzelà–Ascoli on compact cylinders and diagonal extraction give

$$u_n \longrightarrow u \quad \text{locally uniformly on } [0,T] \times \mathbb{R}^2.$$

The limiting field is bounded and divergence free. Weak passage of the global velocity and gradient bounds yields $u \in L_t^\infty W^{1,p}$.

*Weak vorticity compactness.* For compact smooth $\varphi$, the scalar equation gives

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi\omega_n\right| \le m(V\|\nabla\varphi\|_\infty + \|\Delta\varphi\|_\infty).$$

A countable dense set of compact smooth tests in $L^{p'}$, the uniform $L^p$ bound, and Arzelà–Ascoli produce a weakly continuous $L^p$ path $\omega$ with

$$\sup_{t\le T}\left|\int \varphi(\omega_n(t)-\omega(t))\right| \longrightarrow 0 \qquad \text{for every } \varphi \in L^{p'}. \tag{6}$$

It has $\omega(0)=\omega_0$. At every time its $L^1$ norm is at most $m$, by lower semicontinuity on finite balls and then enlargement. Local uniform convergence of $u_n$ and weak convergence of $\omega_n$ pass the drift product in compact smooth tests. The diffusion term is bounded by $\nu_n P\|\Delta\varphi\|_{p'}$ and vanishes. Thus

$$\partial_t\omega + \operatorname{div}(u\omega) = 0, \qquad \omega(0)=\omega_0.$$

We will identify $u = K*\omega$ after obtaining strong convergence.

*Renormalization and endpoint traces.* Spatial mollification gives

$$\partial_t\omega_\varepsilon + u\cdot\nabla\omega_\varepsilon = r_\varepsilon,$$

where divergence freedom allows the expression

$$r_\varepsilon(t,x) = \int \nabla\eta_\varepsilon(y)\cdot[u(t,x)-u(t,x-y)][\omega(t,x-y)-\omega(t,x)]\,\mathrm{d}y.$$

Sobolev difference quotients and Hölder's inequality yield

$$\|r_\varepsilon(t)\|_{p/2} \le C\|\nabla u(t)\|_p \sup_{|y|\le\varepsilon}\|\omega(t,\cdot-y)-\omega(t)\|_p.$$

Translation continuity and dominated convergence imply $r_\varepsilon \to 0$ in $L_t^1 L_x^{p/2}$, hence locally in space-time $L^1$ because $p/2>1$. For each $C^1$ function $\beta$ with bounded derivative the chain rule for the mollified equation therefore passes to

$$\partial_t\beta(\omega) + \operatorname{div}\big(u\beta(\omega)\big) = 0.$$

This equation has the renormalized trace $\beta(\omega(t))$ at every prescribed time $t$, including zero. Indeed, the spatially mollified weakly continuous path is absolutely continuous in local spatial norms. Integrate its renormalized equation between any fixed $s,t$. At each endpoint the spatial approximate-identity property gives $\omega_\varepsilon(s)\to\omega(s)$ and $\omega_\varepsilon(t)\to\omega(t)$ strongly in $L^p$. The Lipschitz function $\beta$ passes these endpoint terms in local $L^1$, and the commutator error vanishes. This proves the renormalized identity with the indicated traces between every pair of times.

For $\beta(0)=0$, use cutoffs equal to one on $B_R$ and vanishing outside $B_{2R}$. Their flux error is at most $CTVm\|\beta'\|_\infty/R$. Sending $R\to\infty$ gives

$$\int\beta(\omega(t)) = \int\beta(\omega_0) \qquad (0\le t\le T).$$

To recover the $L^p$ norm use the convex tangent truncations

$$\beta_M(s) = \begin{cases} |s|^p, & |s| \le M, \\ pM^{p-1}|s| - (p-1)M^p, & |s| > M. \end{cases}$$

They are $C^1$ with bounded derivative, nonnegative, and increasing to $|s|^p$: on $|s| > M$ their derivative in $M$ is $p(p-1)M^{p-2}(|s| - M) \ge 0$. Monotone convergence proves

$$\|\omega(t)\|_p = \|\omega_0\|_p \qquad (0 \le t \le T). \tag{7}$$

The functions $\sqrt{s^2+\epsilon^2} - \epsilon$ followed by $\epsilon \downarrow 0$ give conservation of $\|\omega(t)\|_1$. Taking $\beta(s) = s$ gives conservation of circulation.

The renormalized equation for $|\omega|$ gives finite propagation exactly by an exterior test $\psi(|x| - R_0 - Vt)$ with nonnegative derivative and zero initial pairing. Its material derivative is nonpositive, and its bounded exterior test is justified by expanding cutoffs and the global $L^1$ bound. Sending the transition width to zero yields $\operatorname{supp}\omega(t) \subset B_{R_0+Vt}$ at every time.

*Uniform strong convergence.* For $p \ge 2$, Clarkson's inequality reads

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \le \frac{\|f\|_p^p + \|g\|_p^p}{2}.$$

Weak time continuity and the constant norm (7) imply $\omega \in C_t L^p$ by this inequality and lower semicontinuity. If uniform strong $L^p$ convergence failed, choose bad times $t_n \to t$. Equation (6) gives $\omega_n(t_n) \rightharpoonup \omega(t)$, while contraction gives

$$\limsup_n \|\omega_n(t_n)\|_p \le P = \|\omega(t)\|_p.$$

The same inequality gives strong convergence to $\omega(t)$; continuity of $\omega$ contradicts the bad times. Thus convergence is strong in $C_t L^p$.

Choose $R > R_0 + VT$. The conserved limiting $L^1$ norm and the approximating contraction imply the tail estimate

$$\int_{B_R^c} |\omega_n(t)| \le m - \int_{B_R} |\omega_n(t)| \le \int_{B_R} |\omega_n(t) - \omega(t)|. \tag{8}$$

Consequently

$$\sup_{t \le T} \|\omega_n(t) - \omega(t)\|_1 \le 2|B_R|^{1-1/p} \sup_{t \le T} \|\omega_n(t) - \omega(t)\|_p \longrightarrow 0.$$

The common support ball also makes the limiting path continuous in $L^1$. This proves strong convergence in $C_t(L^1 \cap L^p)$.

Finally Lemma 2.6 gives uniform global convergence $u_n = K * \omega_n \to K * \omega$. The previously obtained local limit is therefore $u = K * \omega$. The transport equation is the Euler vorticity equation, with all the stated regularity and conservation properties. For one subsequence on all finite windows, perform the extractions on $T = 1, 2, \ldots$ and take a diagonal subsequence. □

**Corollary 2.13.** *For the compact zero-circulation initial-data class of Theorem 2.12, the relative compactness as $\nu \downarrow 0$ and Sobolev-velocity assumptions in Proposition 2.11 hold automatically. The remaining substantive hypotheses are uniform protected caps and phase shooting for the actual coupled system.*

# 3 The fixed profile, datum and isolated activation

**Lemma 3.1** (Admissibility of a fixed infinite smooth stack)**.** *Let $W$ be a fixed real smooth compactly supported radial density with zero integral, and let $\Phi_\theta$ be a bounded smooth family with common compact support, zero integral, and $B_4(\Phi_\theta) = 0$. Suppose $\Phi_\theta$ is fourfold symmetric. Fix $\alpha > 0$, $A_j = A_1\beta^{j-1}$, $R_j = R_1 q^{j-1}$, and $A_1, R_1 > 0$, $\beta > 1$, $0 < q < 1$, and $\beta q^{2/p} < 1$, where $2 < p < \infty$. For any phases $\theta_j$, the single datum*

$$\omega_0(x) = \sum_{j\geq 1} A_j[W(x/R_j) + \alpha\Phi_{\theta_j}(x/R_j)]$$

*converges absolutely in $L^1\cap L^p$, has compact support and zero circulation, and satisfies $B_4(\omega_0) = 0$. Changing only phase $\theta_N$ changes its $L^1 \cap L^p$ norm by at most $C\alpha A_N(R_N^2 + R_N^{2/p})$, which tends to zero.*

*Proof.* For either summand its $L^r$ norm is at most $CA_jR_j^{2/r}$, uniformly in phase, for $r = 1, p$. The geometric ratio is less than one for both norms. All supports are in a common ball. Integral and phase claims follow by absolute convergence. The absolute fourth moment of each summand is bounded by $CA_jR_j^6$; this series converges since $\beta q^6 < 1$. Radial symmetry gives $B_4(W) = 0$, so termwise integration proves the whole-moment assertion. □

## 3.1 An isolated nonlinear active stage for a fixed smooth profile

This section proves the nonlinear activation estimate for one isolated annulus with a fixed smooth radial profile. The smoothing length is independent of viscosity and of the small parameter below.

**Theorem 3.2** (Isolated smooth activation)**.** *There exist a real, radial, compactly supported $C^\infty$ vorticity $W_\epsilon$, with zero circulation, a number $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$ with $\sigma_\epsilon > 0$, and a nonzero complex number $b_\epsilon$, with the following property. For every $2 < p < \infty$ and $K_* > 0$ there is $q_0 > 0$ such that the conclusions below hold for $0 < q < q_0$. Put*

$$L_q = \log(1/q), \quad \alpha_q = q^{K_*}, \quad \kappa_q = q^{3/2}, \quad \delta_q = L_q^{-4}, \quad H_q = \frac{1}{\sigma_\epsilon}\log\frac{\delta_q}{\alpha_q}.$$

*There is a $2\pi$-periodic family of smooth real seeds $\Phi_\vartheta$, supported in two fixed annular collars, such that*

$$\int \Phi_\vartheta = 0, \qquad B_4(\Phi_\vartheta) = 0, \qquad \Phi_\vartheta(ix) = \Phi_\vartheta(x).$$

*Let $\omega^{q,\vartheta}$ be the unforced Navier–Stokes solution with viscosity $\kappa_q$ and initial vorticity $W_\epsilon + \alpha_q\Phi_\vartheta$. Then, uniformly in $\vartheta \in \mathbb{R}$,*

$$\sup_{0\leq t\leq H_q} \|\omega^{q,\vartheta}(t) - e^{\kappa_q t\Delta}W_\epsilon\|_{L^1\cap L^p} \leq C\delta_q, \tag{9}$$

$$\int_0^{H_q} \|K * (\omega^{q,\vartheta}(t) - e^{\kappa_q t\Delta}W_\epsilon)\|_\infty \,\mathrm{d}t \leq C\delta_q, \tag{10}$$

$$B_4(\omega^{q,\vartheta}(H_q)) = \delta_q e^{i(\vartheta+\tau_\epsilon H_q)}b_\epsilon + o(\delta_q). \tag{11}$$

*Here $\|f\|_{L^1\cap L^p} = \|f\|_1 + \|f\|_p$; all fourth absolute moments in (11) are finite. In particular, (11) is a statement about the whole moment, rather than only a compactly tested moment. The constants depend on the fixed profile, $p$ and $K_*$, but not on $q$ or $\vartheta$.*

The logarithmic cap satisfies $H_q\sqrt{\delta_q} \to 0$. In the fixed-data construction, $q$ and hence this positive cap are subsequently fixed.

### 3.1.1 The fixed profile and its positive angular spectrum

Fix $\rho^2 = 3/5$ and choose a nonnegative, radial, radially decreasing $\chi \in C_c^\infty(B_1)$ with integral one. Define

$$\chi_\epsilon(x) = \epsilon^{-2}\chi(x/\epsilon), \qquad W_\epsilon = \chi_\epsilon * (-\mathbf{1}_{B_\rho} + \rho^2 \mathbf{1}_{B_1}).$$

The convolution is smooth, radial, supported in $B_{1+\epsilon}$ and has zero integral. It is constant on $B_{\rho-\epsilon}$. Choose $\epsilon_* < (1-\rho)/4$ and subsequently $0 < \epsilon < \epsilon_*$. The limits below refer first to the mollified family as $\epsilon \downarrow 0$. After the spectral and moment limits here and the transfer-coefficient limits in Lemma 4.8 and the interior-field calculation following it have been checked, choose one sufficiently small $\epsilon$ satisfying all their nonvanishing and spectral conclusions, and retain that choice for every later finite expansion and derivative order. Its radial derivative has the form

$$W_\epsilon'(r) = D_{\rho,\epsilon}(r) - \rho^2 D_{1,\epsilon}(r), \qquad D_{a,\epsilon} \geq 0, \quad \operatorname{supp} D_{a,\epsilon} \subset [a-\epsilon, a+\epsilon], \quad \int_0^\infty D_{a,\epsilon}(r)\,\mathrm{d}r = 1. \tag{12}$$

To verify the sign, express the radial decreasing mollifier by its superlevel disks. The area of intersection of two disks is nonincreasing in their center separation: along each line parallel to their displacement, the overlap of the two interval sections is nonincreasing. Layer-cake integration gives radial monotonicity of the disk convolution. Its value is one at the origin and zero at infinity, giving the displayed derivative mass. The support assertion follows because the convolution is exactly constant at points whose distance to the disk boundary exceeds $\epsilon$.

Set

$$\Omega_\epsilon(r) = r^{-2}\int_0^r sW_\epsilon(s)\,\mathrm{d}s.$$

On the two derivative collars, respectively,

$$\Omega_\epsilon(r) = \Omega_0(r_i) + O(\epsilon), \qquad (r_1, r_2) = (\rho, 1), \quad (\Omega_0(r_1), \Omega_0(r_2)) = (-1/5, 0). \tag{13}$$

Indeed disk translation changes its $L^1$ norm by at most a constant times the translation length. Thus $\|W_\epsilon - W_0\|_1 \leq C\epsilon$, where $W_0 = -\mathbf{1}_{B_\rho} + \rho^2\mathbf{1}_{B_1}$. The radial circulation formula, on radii bounded away from zero, gives $|\Omega_\epsilon - \Omega_0| \leq C\epsilon$. The piecewise smooth $\Omega_0$ is Lipschitz there. Direct integration gives the interface values.

For a positive angular mode $m \geq 1$ use the convention $f(r)e^{im\theta}$. Put

$$h_m(r,s) = \left(\frac{\min(r,s)}{\max(r,s)}\right)^m, \qquad \psi_f(r) = -\frac{1}{2m}\int_0^\infty h_m(r,s)f(s)s\,\mathrm{d}s.$$

The function $\psi_f e^{im\theta}$ solves $\Delta\psi = fe^{im\theta}$, is regular at the origin and decays at infinity. This follows either by solving the radial second-order equation on each side of $r = s$, or by checking the derivative jump $1/s$ of the displayed Green kernel. Since $u_r = -im\psi_f/r$, the frozen Euler linearization at $W_\epsilon$ is

$$\mathcal{L}_m f = -im\Omega_\epsilon f + im\psi_f W_\epsilon'/r. \tag{14}$$

For $m = 0$ the linearization is zero: two radial vorticities have tangential velocities and do not advect one another. Negative sectors are the complex conjugates of the positive sectors. From this point onward all angular sectors belong to $4\mathbb{Z}$. No spectral assertion is made for other symmetry classes.

**Lemma 3.3** (The fixed smooth unstable pair)**.** *There are $\epsilon_0 > 0$ and $\eta > 0$ such that, for every $0 < \epsilon < \epsilon_0$, $0 < \eta < \sigma_\epsilon/4$ and the following modal conclusions hold. For each $m \in 4\mathbb{N}$ and each*

*integer $k \geq 0$, realize (14) as a bounded operator on $H^k(I)$, using the Green potential of the zero extension from a fixed interval $I \Subset (0,\infty)$ containing both derivative collars strictly in its interior. The only spectral value of any of these individual operators with real part greater than $\eta$ is the algebraically simple $m = 4$ eigenvalue $\lambda_\epsilon = \sigma_\epsilon + i\tau_\epsilon$. Its radial eigenfunction $f_\epsilon$ is smooth and supported in the two derivative collars. For negative sectors use the conjugate operators; their only spectral value with real part greater than $\eta$ is $\overline{\lambda_\epsilon}$ in sector $m = -4$. The radial generator is zero. Moreover*

$$\lambda_\epsilon \longrightarrow \rho^5/2 + 3i/10 \quad \text{as } \epsilon \downarrow 0.$$

*For each fixed finite set $S \subset 4\mathbb{Z}$ and integer $k \geq 0$, let $X_{k,S} = \bigoplus_{m\in S} H^k(I)$ and let $\mathcal{L}_S$ be its bounded direct-sum generator. Its propagator has the decomposition*

$$e^{t\mathcal{L}_S} = e^{\lambda_\epsilon t}P_+ + e^{\overline{\lambda_\epsilon} t}P_- + \mathcal{R}_{k,S}(t), \qquad \|\mathcal{R}_{k,S}(t)\|_{X_{k,S}\to X_{k,S}} \leq C_{k,S}e^{2\eta t}, \quad t \geq 0. \tag{15}$$

*The growing terms are included only if their sectors are present. The constants may depend on $\epsilon, k, S$. Smooth inputs supported in the fixed derivative collars remain supported in the union of those two collars. No assertion is made about the spectrum or propagator of an infinite angular direct sum.*

*Proof.* Fix $0 < \eta < \rho^5/8$ before the angular cutoff is chosen. First consider an eigenvalue $\lambda$ in an individual sector with $\Re\lambda > 0$. The eigenvalue equation forces $f = 0$ off the derivative collars and gives

$$f(r)r\,\mathrm{d}r = g(r)W_\epsilon'(r)\,\mathrm{d}r, \qquad g(r) = \frac{im\psi_f(r)}{\lambda + im\Omega_\epsilon(r)}.$$

There is no division by $W_\epsilon'$ here. In the Hilbert space $L^2(|W_\epsilon'|\,\mathrm{d}r)$ the equation for $g$ is

$$\lambda g = -im\Omega_\epsilon g - \frac{i}{2}\int h_m(r,s)g(s)W_\epsilon'(s)\,\mathrm{d}s.$$

Each same-collar block is skew-adjoint, since the sign of $W_\epsilon'$ is constant there and $h_m(r,s)$ is real and symmetric. Angular multiplication is also skew-adjoint. The two opposite-collar blocks have operator norms at most their Hilbert–Schmidt norms, bounded by

$$Ca_*^m, \qquad a_* = (\rho + \epsilon_*)/(1 - \epsilon_*) < 1,$$

because their two weighted masses are 1 and $\rho^2$ and their kernels are bounded by $a_*^m$. Consequently

$$\Re\lambda \leq Ca_*^m. \tag{16}$$

This fixes a finite angular cutoff in any half-plane $\Re\lambda \geq \eta > 0$, independently of $\epsilon$.

For one of the finitely many positive sectors, eliminate $f$ to obtain the equation on the two collars

$$\psi(r) = -\frac{i}{2}\int h_m(r,s)\frac{W_\epsilon'(s)\psi(s)}{\lambda + im\Omega_\epsilon(s)}\,\mathrm{d}s =: \mathcal{T}_\epsilon(\lambda)\psi(r). \tag{17}$$

Work first in the sup-norm space of continuous functions on the two collars. Let $P$ evaluate at $r_1, r_2$ and let $E$ extend a vector to constants on the two collars, so $PE = I$. The receiving-radius derivative of the kernel is bounded there for fixed $m$. Hence

$$\|(I - EP)\mathcal{T}_\epsilon(\lambda)\| \leq C\epsilon$$

uniformly on compact subsets of $\Re\lambda \geq \eta$. For $\psi = Ep + w$, $Pw = 0$, solve the $w$ equation by an analytic Neumann series. The remaining two-dimensional equation has a determinant converging uniformly on these compact sets to the determinant of

$$p_i = \frac{i}{2} \sum_j \frac{c_j h_m(r_i, r_j)}{\lambda + im\Omega_0(r_j)} p_j, \qquad (c_1, c_2) = (-1, \rho^2).$$

The sign follows from $W_0' = -\sum c_j \delta_{r_j}$. Using interface variables $f(r) r\,\mathrm{d}r = \sum c_i r_i^2 q_i \delta_{r_i}$, this equation is equivalent to the matrix eigenvalue problem

$$M_m q = \lambda q, \qquad M_m = \frac{i}{2} \begin{pmatrix} 2m/5 - 1 & \rho^m \\ -\rho^{m+2} & 3/5 \end{pmatrix}.$$

For $m = 4$ its roots are $\pm\rho^5/2 + 3i/10$, both simple. For $m = 8, 12, \ldots$ they are imaginary, since the real-matrix discriminant is

$$\left(\frac{2(m-4)}{5}\right)^2 - 4\rho^{2m+2} > 0.$$

At $m = 8$ the first term is $64/25$ and the second is $4(3/5)^9 < 1$; thereafter the first increases and the second decreases. Large $|\lambda|$ gives no omitted roots: once $|\lambda| > C_m$, the operator in (17) has norm less than one, uniformly in $\epsilon$, because the denominator then has size comparable with $|\lambda|$. Uniform determinant convergence and Rouché's theorem therefore give one simple root near the growing $m = 4$ root and no other roots in $\Re\lambda \geq \eta$. Combined with (16), this proves the claimed positive spectrum for all sufficiently small $\epsilon$. By the growing-root limit, decrease $\epsilon_0$ so that $4\eta < \sigma_\epsilon$ throughout $0 < \epsilon < \epsilon_0$. The chosen $\eta$ is unchanged in this step.

We give the operator-domain and multiplicity details needed for the propagator claim. Choose a closed interval $I \Subset (0, \infty)$ containing both collars strictly in its interior. On $H^k(I)$ define $\psi_f$ by the Green integral against the zero extension of $f$ from $I$. This extension need not be globally $H^k$: multiplication of the potential by $W_\epsilon'$ restricts output strictly inside $I$. The radial elliptic equation, or direct differentiation of its Green kernel, gives two local derivatives. Thus the coupling in (14) is compact on $H^k(I)$ and the angular multiplier is bounded there. Its spectrum is imaginary: off the multiplier's value range the inverse is multiplication by $(\lambda + im\Omega_\epsilon)^{-1}$. The compact-operator Fredholm alternative consequently reduces all spectrum in $\Re\lambda > 0$ to eigenfunctions. The equation bootstraps such eigenfunctions to smoothness and forces their support into the collars, so the roots already found are exactly the positive spectrum on every $H^k(I)$.

To check algebraic simplicity independently of a sup-norm determinant, start on $X_0 = L^2(I)$. Write $Af = -im\Omega_\epsilon f$, $Cf = \psi_f|_{\text{collars}}$ and $B\psi = imW_\epsilon'\psi/r$. Then $\mathcal{L}_m = A + BC$ and $\mathcal{T}_\epsilon = C(\lambda - A)^{-1}B$. Off the imaginary multiplier range, the resolvent identity is

$$(\lambda - \mathcal{L}_m)^{-1} = (\lambda - A)^{-1} + (\lambda - A)^{-1}B(I - \mathcal{T}_\epsilon)^{-1}C(\lambda - A)^{-1}. \tag{18}$$

Here $C : X_0 \to C(\text{collars})$ is bounded by its elliptic gain, and $B : C(\text{collars}) \to X_0$ is bounded. The invertible complementary Schur factor and the simple two-by-two determinant zero imply a rank-one simple pole in (18). The residue is nonzero, since an eigenfunction exists. On $X_k = H^k(I)$ the resolvent equation and its radial elliptic bootstrap give, locally uniformly for $\lambda$ off the imaginary multiplier range,

$$\|v\|_{H^k(I)} \leq C_k(\|f\|_{H^k(I)} + \|v\|_{L^2(I)}), \qquad (\lambda - \mathcal{L}_m)v = f.$$

The $X_k$ resolvent is the restriction of the $X_0$ resolvent; the simple pole has no higher-order singularity by this estimate and has the same nonzero smooth one-dimensional range. This proves algebraic simplicity on every space used below. The $m = -4$ conclusion follows by conjugation; $m = 0$ is the identity propagator.

After removing the growing Riesz projections, the remaining bounded generator has spectral bound at most $\eta$. Its exponential is bounded by $Ce^{2\eta t}$. For clarity this last fact does not assume a spectral growth theorem for unbounded generators. For a bounded operator, integrate $e^{tz}(z-\mathcal{L})^{-1}$ on a finite rectangular contour enclosing its remaining spectrum and having right edge $\Re z = 2\eta$. Expansion of the resolvent on a large circle and termwise integration gives the exponential series; contour deformation gives the rectangle formula. The resolvent is bounded on this fixed contour, proving the estimate. The finitely many sectors and derivative orders give (15). Coefficients initially smooth and supported in the collars remain so under this propagator: multiplication preserves that support and the coupling has smooth output flat at the collar endpoints. This proves the lemma. □

### 3.1.2 An exactly neutral smooth seed

Normalize the growing radial eigenfunction $f_\epsilon$ so that its first collar mass tends to $c_1 r_1^2 = -\rho^2$. The finite-dimensional reduction then implies

$$f_\epsilon(r) r \, \mathrm{d}r \;\rightharpoonup\; -\rho^2 \delta_\rho - i\rho^3 \delta_1.$$

Indeed its interface vector tends to $(1, -i\rho)$, the growing eigenvector of $M_4$. Define

$$b_\epsilon = 2\pi \int_0^\infty r^5 f_\epsilon(r) \, \mathrm{d}r. \tag{19}$$

The weak convergence on the two compact collars gives

$$b_\epsilon \longrightarrow 2\pi(-\rho^6 - i\rho^3) \neq 0.$$

Consequently $b_\epsilon \neq 0$ for every sufficiently small $\epsilon$. This condition is included in the final fixed choice of the smoothing length. Since $\mathcal{L}_4$ is $i$ times a real operator, $\overline{f_\epsilon}$ is an eigenfunction of $\mathcal{L}_4$ with eigenvalue $-\overline{\lambda_\epsilon} = -\sigma_\epsilon + i\tau_\epsilon$. Define the real seed

$$\Phi_\vartheta(r,\theta) = e^{i\vartheta} \left( f_\epsilon(r) - \frac{b_\epsilon}{\overline{b_\epsilon}} \overline{f_\epsilon(r)} \right) e^{4i\theta} + \text{complex conjugate}. \tag{20}$$

Its circulation is zero and its only initial sectors are $\pm 4$. For the convention $B_4 = \int \bar{z}^4 \omega$, angular integration gives $B_4(f(r)e^{4i\theta}) = 2\pi \int r^5 f(r) \, \mathrm{d}r$. Therefore $B_4(\Phi_\vartheta) = 0$ exactly, while the growing coefficient is $e^{i\vartheta}$. The stable coefficient has modulus one.

### 3.1.3 The finite seed–viscosity expansion

For the rest of the proof abbreviate the fixed profile to $W$, and write $\sigma = \sigma_\epsilon$, $\tau = \tau_\epsilon$, $W_\kappa(t) = e^{\kappa t \Delta} W$ and $U_\kappa = K * W_\kappa$. The latter is an exact radial heat solution. Put $X = L^1 \cap L^p$ with the sum norm. Splitting the Biot–Savart convolution at radius one gives

$$\|K * g\|_\infty \leq C_p \|g\|_X. \tag{21}$$

On the near part use Hölder and $|x|^{-1} \in L^{p'}(B_1)$, which holds because $p' < 2$. On the far part use its bounded kernel and the $L^1$ norm. This estimate requires no zero-circulation assumption.

The frozen full linear operator is $\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W$. Its sectors are those in (14). Choose integers $M, N \geq 1$, to be fixed below, and construct

$$f_{\mathrm{app}} = \sum_{j=1}^{M} \sum_{\ell=0}^{N} \alpha^j \kappa^\ell f_{j,\ell}, \qquad \omega_{\mathrm{app}} = W_\kappa + f_{\mathrm{app}}. \tag{22}$$

Set $f_{1,0}(0) = \Phi_\vartheta$ and all other initial coefficients zero. With $W_r(t) = t^r \Delta^r W / r!$, solve the triangular recursion

$$\begin{aligned}(\partial_t - \mathcal{L}) f_{j,\ell} = \mathbf{1}_{\ell \geq 1} \Delta f_{j,\ell-1} - \sum_{r=1}^{\ell} \left[ (K * W_r) \cdot \nabla f_{j,\ell-r} + (K * f_{j,\ell-r}) \cdot \nabla W_r \right] \\ - \sum_{\substack{j_1+j_2=j,\ j_1,j_2 \geq 1 \\ \ell_1+\ell_2=\ell}} (K * f_{j_1,\ell_1}) \cdot \nabla f_{j_2,\ell_2}.\end{aligned} \tag{23}$$

Every right-hand side uses earlier coefficients in the order $j + \ell$. They are smooth and supported in the fixed collars: Laplacians preserve the support of smooth flat functions; $\Delta^r W$, $r \geq 1$, has that support; and every advective factor differentiates such a supported function. The linear source is supported where $W'$ is supported. Each coefficient has zero circulation, because each forcing is a Laplacian or divergence of an integrable vector field. Its frequencies belong to $4\mathbb{Z}$ and satisfy $|m| \leq 4j$.

**Lemma 3.4** (Coefficient bounds and seeded residual)**.** *For every finite derivative order $k$, the finitely many coefficients in (23) satisfy*

$$\|f_{j,0}(t)\|_{H^k} \leq C_{j,k} e^{j\sigma t}, \tag{24}$$

$$\|f_{j,\ell}(t)\|_{H^k} \leq C_{j,\ell,k} (1+t)^D e^{j\sigma t}, \qquad \ell > 0. \tag{25}$$

*There is a finite $D$, enlarged as needed, such that if $a(t) = \alpha e^{\sigma t} \leq \delta_0 \leq 1$ on $[0, H]$ and $\kappa(1 + H)^D \leq c$, the residual*

$$R_{\mathrm{app}} = \partial_t \omega_{\mathrm{app}} + (K * \omega_{\mathrm{app}}) \cdot \nabla \omega_{\mathrm{app}} - \kappa \Delta \omega_{\mathrm{app}}$$

*satisfies*

$$\|R_{\mathrm{app}}(t)\|_X \leq C \left[ a(t)^{M+1} + \kappa^{N+1} (1+t)^D a(t) \right]. \tag{26}$$

*All constants are uniform in $\vartheta$.*

*Proof.* The first coefficient is the exact growing-plus-decaying solution from (20). For $j \geq 2$, $\ell = 0$, induction and Sobolev multiplication bound its forcing by $Ce^{j\sigma t}$. Convolution against the growing exponent $\sigma$, or the remaining exponent $2\eta < \sigma$, is bounded by a constant times $e^{j\sigma t}$, because $j\sigma - \sigma \geq \sigma$. This proves (24) with no polynomial factor. For positive $\ell$ the same induction applies, with finite polynomial losses from $W_r(t)$ and possible resonance with the first growing mode. This proves (25). Derivative loss causes no infinite iteration: the coefficient set is finite and the profile is smooth in every order. At each induction step apply the preceding coefficient estimates in a sufficiently higher fixed order and the propagator lemma in the requested order. Compact support and zero circulation also control $K * f$ and its needed derivatives, including low frequencies. More explicitly $|\widehat{f}(\xi)| \leq C|\xi|$ near zero, from its zero mass and finite first absolute moment, while the high-frequency multiplier has order minus one.

Expand the radial heat base only in its couplings to $f_{\mathrm{app}}$. The integral Taylor formula and heat contraction in $L^1, L^p$ give

$$G_N(t) := W_\kappa(t) - \sum_{r=0}^{N} \kappa^r W_r(t), \qquad \|G_N(t)\|_X + \|\nabla G_N(t)\|_X \le C\kappa^{N+1} t^{N+1}.$$

Its residual couplings obey

$$\|(K * G_N) \cdot \nabla f_{\mathrm{app}} + (K * f_{\mathrm{app}}) \cdot \nabla G_N\|_X \le C\kappa^{N+1}(1+t)^D a(t)$$

by (21) and the coefficient bounds. These are the only noncompact factors. Every other coefficient is supported in one fixed bounded set, where its high Sobolev norms control the required $L^1, L^p$ product norms. The recursion cancels exactly all monomials with $1 \le j \le M$, $0 \le \ell \le N$. Terms of seed order $j > M$ are bounded by $Ca^{M+1}$: the pure-seed bounds contain no time polynomial, and each positive-viscosity polynomial is absorbed after enlarging $D$ and using $\kappa(1+H)^D \le c$. Terms of viscosity order $\ell > N$, including the diffusion of $f_{j,N}$, have the second bound in (26); $a^j \le a$ for $a \le 1$ absorbs their higher seed powers. Crucially the exact radial heat base is part of (22), so no residual term without a seed factor occurs. Finally rotations of (20) change only unit complex phases; all bounds are uniform in $\vartheta$. □

### 3.1.4 Nonlinear closure in the actual vorticity norm

Let $\omega^\kappa$ be the NS solution from $W + \alpha\Phi_\vartheta$ and put $\zeta = \omega^\kappa - \omega_{\mathrm{app}}$. Its exact equation, with zero initial data, is

$$\partial_t \zeta + (K * \omega^\kappa) \cdot \nabla \zeta = \kappa\Delta\zeta - (K * \zeta) \cdot \nabla \omega_{\mathrm{app}} - R_{\mathrm{app}}. \tag{27}$$

Kato's inequality gives its $L^1$ estimate, and multiplication by $|\zeta|^{p-2}\zeta$ gives its $L^p$ estimate; transport disappears by divergence freedom and diffusion is nonpositive in both inequalities. Using (21) and adding gives

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\|\zeta\|_X \le C_p\|\nabla\omega_{\mathrm{app}}\|_X\|\zeta\|_X + \|R_{\mathrm{app}}\|_X. \tag{28}$$

These inequalities follow first with smooth convex regularizations of the norm and compact cutoffs, then by their limits. All solutions here are smooth at positive time, have finite $X$ norm, and are continuous in that norm at time zero, so the integrated inequality starts at zero. No derivative bound on the actual NS vorticity enters its coefficient.

We choose parameters without circular dependence. Fix

$$C_1 = C_p(\|\nabla W\|_X + 1), \qquad r_1 = (C_1/\sigma - 1)_+,$$

before choosing expansion orders. Choose $M$ with $(M+1)\sigma > C_1$ and $N$ with

$$\frac{3}{2}(N+1) > K_* r_1. \tag{29}$$

Construct the finite coefficients, then fix a cap threshold $\delta_0$ using the pure-seed coefficients alone and reduce the positive-viscosity threshold so that $\|\nabla f_{\mathrm{app}}\|_X \le 1$ whenever $a \le \delta_0$ and $\kappa(1+H)^D$ is sufficiently small. This is possible by the finite coefficient bounds: make the pure-seed gradient at most $1/2$ by the cap choice, and the finitely many positive-viscosity contributions at most $1/2$ by the later viscosity threshold. Thus a later increase of $N$ need not reduce a previously fixed pure-seed cap. The exact heat gradient obeys $\|\nabla W_\kappa\|_X \le \|\nabla W\|_X$, so $C_1$ is the already fixed

exponent in (28). The coefficient constants affect only the later cap threshold; they do not change the previous choice of $M, N$.

Convolving (26) against $e^{C_1(t-s)}$ gives the all-time estimate

$$\|\zeta(t)\|_X \le Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)_+t}. \tag{30}$$

The first integral has denominator $(M+1)\sigma - C_1 > 0$. For the second use a bound by $e^{\max(C_1,\sigma)t}$ and an additional polynomial factor if the two exponents coincide. At $H = \sigma^{-1}\log(\delta/\alpha)$ this yields

$$\|\zeta(H)\|_X \le C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \tag{31}$$

For the theorem's parameters, (29) implies that the second bound is $o(\delta_q)$: a strict positive power of $q$ beats every fixed power of $\log(1/q)$. Also $\kappa_q(1+H_q)^D \to 0$, $\delta_q \to 0$, and $\alpha_q/\delta_q \to 0$. The coefficient estimates and (30) therefore prove (9), including uniformity through the entire interval. They also give the actual nonradial density bound

$$\|(I - P_{\mathrm{rad}})\omega^\kappa(t)\|_1 \le Ca(t) + Ca(t)^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)_+t}. \tag{32}$$

Indeed radial averaging contracts $L^1$, and the heat base is radial. This estimate concerns the actual density; it is not inferred from a contour coordinate.

Similarly (21), the coefficient bounds and (30) imply

$$\int_0^H \|K * (\omega^\kappa - W_\kappa)\|_\infty \,\mathrm{d}t \le C\delta + C\delta^{M+1} + C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1}. \tag{33}$$

To integrate the approximate terms use $\int_0^H a(t)^j \,\mathrm{d}t \le \delta^j/(j\sigma)$. The error terms grow at a fixed positive exponential rate; their nondecreasing polynomial factor can be evaluated at $H$, so there is no additional factor $H$. This proves (10).

### 3.1.5 Whole-moment control and the cap

We verify the tails needed to remove the moment cutoff. Write $u^\kappa = U_\kappa + v^\kappa$, with $U_\kappa$ tangential, and $B_H = \int_0^H \|v^\kappa\|_\infty \,\mathrm{d}t$. By (33), $B_{H_q} = O(\delta_q)$. Let $d(x) = \sqrt{1+|x|^2}$. Then $|\nabla d| \le 1$, $\Delta d \le 2$, and $U_\kappa \cdot \nabla d = 0$. Kato's inequality with $e^{\lambda d}$ gives, for every $\lambda > 0$,

$$\int e^{\lambda d(x)}|\omega^\kappa(t,x)|\,\mathrm{d}x \le m\exp\{\lambda d_0 + \lambda B_H + 2\kappa\lambda t + \kappa\lambda^2 t\}, \tag{34}$$

where $m = \|\omega^\kappa(0)\|_1$ and $d_0 = \sqrt{1+(1+\epsilon)^2}$. No initial exponential tail is presupposed beyond compact support. To justify the unbounded test use

$$F_L(d) = \frac{e^{\lambda d}}{1 + e^{\lambda d}/L}.$$

It has $F_L' \le \lambda F_L$ and $F_L'' \le \lambda^2 F_L$; therefore $\Delta F_L(d) \le (\lambda^2 + 2\lambda)F_L(d)$. For fixed $L$ add a compact cutoff and let its radius tend to infinity. Finite mass and bounded drift make the cutoff errors vanish. Gronwall then gives (34) uniformly in $L$, and monotone convergence removes $L$. The drift is bounded here by (21) and the scalar $L^1, L^p$ contractions.

Choose a fixed $R > 1 + \epsilon$ with $d_R - d_0 > 2c_* > 0$. For small $q$, $B_{H_q} < c_*/2$ and $\kappa_q H_q \to 0$. Exponential Markov and optimization of (34) give

$$\int_{|x|>s} |\omega^\kappa(t,x)|\,\mathrm{d}x \le m\exp\left\{-\frac{(d_s - d_0 - B_H - 2\kappa t)^2}{4\kappa t}\right\}, \qquad s \ge R, \quad 0 < t \le H.$$

At $t = 0$ the exterior tail vanishes by the initial support. Integrating this mass bound in $s$ gives

$$\sup_{0\le t\le H_q}\int_{|x|>R}|x|^4|\omega^{q,\vartheta}(t,x)|\,\mathrm{d}x \le Ce^{-c/(\kappa_q H_q)}. \tag{35}$$

For example the weighted tail equals $R^4$ times the mass tail plus $4\int_R^\infty s^3$ times the lower-radius mass tail. The displayed Gaussian estimate bounds both terms after reducing $c$. This also proves finiteness of the required absolute moments.

Let $\xi$ be a smooth radial cutoff equal to one on $B_R$ and supported in $B_{R+1}$. Every formal nonradial coefficient is supported in $B_R$; the exact heat base pairs to zero with both $\bar z^4$ and $\xi\bar z^4$, since it is radial. At the cap the first coefficient has growing moment

$$\alpha e^{\lambda_\epsilon H}e^{i\vartheta}b_\epsilon = \delta e^{i(\vartheta+\tau_\epsilon H)}b_\epsilon$$

and stable moment of size $C\alpha^2/\delta$. The remaining pure-seed coefficients have compactly tested moment $O(\delta^2)$; positive-viscosity coefficients have moment $O(\kappa(1+H)^D\delta)$. The actual error pairs against $\xi\bar z^4$ with bound $\|\xi\bar z^4\|_\infty\|\zeta(H)\|_1$. Using (31) and (35) proves

$$\begin{aligned}|B_4(\omega^\kappa(H)) - \delta e^{i(\vartheta+\tau_\epsilon H)}b_\epsilon| \le{}& C\delta^2 + C\alpha^2/\delta + C\kappa(1+H)^D\delta + C\delta^{M+1}\\ &+ C\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_1} + Ce^{-c/(\kappa H)}.\end{aligned}$$

For $\delta = \delta_q$, every bound divided by $\delta_q$ tends to zero. This proves (11) and completes the theorem.

*Remark* 3.5 (Scaling and scope). Let $A, R > 0$ and set

$$\omega_{A,R}(t,x) = A\omega^{q,\vartheta}(At, x/R), \qquad W^\nu_{A,R}(t,x) = A(e^{\kappa_q At\Delta}W_\epsilon)(x/R), \qquad \nu = \kappa_q AR^2.$$

Homogeneity of Biot–Savart gives $K * \omega_{A,R}(t,x) = AR(K*\omega^{q,\vartheta})(At, x/R)$, so the physical cap time is $H_q/A$. For $1 \le r < \infty$, change of variables gives

$$\|\omega_{A,R}(t) - W^\nu_{A,R}(t)\|_r = AR^{2/r}\|\omega^{q,\vartheta}(At) - e^{\kappa_q At\Delta}W_\epsilon\|_r.$$

Thus the separate density bounds are $CAR^2\delta_q$ in $L^1$ and $CAR^{2/p}\delta_q$ in $L^p$, and their sum is

$$\sup_{0\le t\le H_q/A}\|\omega_{A,R}(t) - W^\nu_{A,R}(t)\|_{L^1\cap L^p} \le CA(R^2 + R^{2/p})\delta_q.$$

The cap moment and integrated perturbation velocity satisfy

$$B_4(\omega_{A,R}(H_q/A)) = AR^6\delta_q e^{i(\vartheta+\tau_\epsilon H_q)}b_\epsilon + o(AR^6\delta_q),$$

$$\int_0^{H_q/A}\|K*(\omega_{A,R}(t) - W^\nu_{A,R}(t))\|_\infty\,\mathrm{d}t = R\int_0^{H_q}\|K*(\omega^{q,\vartheta}(s) - e^{\kappa_q s\Delta}W_\epsilon)\|_\infty\,\mathrm{d}s \le CR\delta_q.$$

All constants are independent of $A, R$. These are isolated-colour estimates. Coupled use also requires bounds for the foreign-colour terms in the coefficient and error equations.

# 4 The canonical ancient orbit and signed coupling

## 4.1 The canonical ancient profile of the isolated smooth activation

We retain the fixed profile, eigenvalue and seeds constructed in Section 3.1. Abbreviate them to $W = W_\epsilon$, $\lambda = \sigma + i\tau = \lambda_\epsilon$, $b = b_\epsilon \neq 0$, and $\Phi_\vartheta$. In this section the cap $\delta > 0$ is fixed and sufficiently small. It does not decrease with the vanishing parameter. An additional Sobolev estimate permits identification of a unique isolated ancient profile. This identification requires no convergence of an infinite formal series. Here $\delta$ specifies the ancient growing amplitude and the linear preparation level. It does not assert that $s = 0$ is the exact first hitting time of $|\mathcal{A}(\Omega_\delta)| = \delta$; at that time $\mathcal{A}(\Omega_\delta(0)) = \delta + O(\delta^2)$.

**Theorem 4.1** (The isolated ancient limit)**.** *Fix an integer $k \geq 3$ and $2 < p < \infty$. There exists $\delta_* > 0$ such that, for $0 < \delta < \delta_*$, there is a real quarter-turn invariant Euler solution $\Omega_\delta(s)$ on $-\infty < s < \infty$, with zero circulation, having the following properties. For every fixed $K_* > 0$, put*

$$\alpha_q = q^{K_*}, \qquad \kappa_q = q^{3/2}, \qquad H_q = \sigma^{-1}\log(\delta/\alpha_q), \qquad \vartheta_q = -\tau H_q.$$

*Let $\omega^q$ be the isolated Navier–Stokes solution with viscosity $\kappa_q$ and initial vorticity $W + \alpha_q\Phi_{\vartheta_q}$. For every $J < \infty$ and every $0 < \varepsilon < k$,*

$$\omega^q(H_q + \cdot) \longrightarrow \Omega_\delta \quad \text{in } C([-J,0]; L^1 \cap L^p) \text{ and in } C([-J,0]; H^{k-\varepsilon}) \quad \text{as } q \downarrow 0. \tag{36}$$

*The parameter $\delta_*$ and the limiting profile can be chosen independently of $K_*$; the threshold for small $q$ may depend on $K_*$. On $s \leq 0$, $\Omega_\delta$ is supported in one fixed ball and has a uniform $H^k$ bound. Its whole fourth harmonic moment satisfies*

$$B_4(\Omega_\delta(s)) = \delta b e^{\lambda s} + O(\delta^2 e^{2\sigma s}) \qquad (s \leq 0). \tag{37}$$

*For $s \geq 0$ it satisfies*

$$|B_4(\Omega_\delta(s))| \leq C_\delta(1+s)^4. \tag{38}$$

*These assertions concern the isolated annulus.*

### 4.1.1 A high Sobolev error estimate without derivative loss

Fix once and for all a radius $R > 1 + \epsilon$ with a positive gap in $d(x) = \sqrt{1+|x|^2}$, for example $R = 3+\epsilon$. This radius and the corresponding compact-support norm constants are fixed before the energy exponent or expansion orders. The later cap smallness condition will make the integrated perturbation drift smaller than this fixed gap. For this proof let

$$\|g\|_{Z_k} = \|g\|_1 + \|g\|_{H^k}.$$

The finite expansion from (22) and (23) can be constructed in every finite derivative order. In particular, the residual estimate holds in $Z_k$. A useful version separates its pure-seed constant from constants depending on the viscosity order:

$$\|R_{\mathrm{app}}(t)\|_{Z_k} \leq [C_M + C_{M,N}\kappa(1+t)^D]a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^D a(t), \qquad a(t) = \alpha e^{\sigma t} \leq 1. \tag{39}$$

Here $C_M$ is independent of $N$. To verify the additional norm and this separation, the pure-seed products are estimated by their compactly supported coefficient Sobolev bounds. Every

other omitted seed-order product contains at least one positive power of $\kappa$, and its finite time polynomial is incorporated in $D$. The noncompact heat remainder products are

$$(K * G_N) \cdot \nabla f_{\mathrm{app}} + (K * f_{\mathrm{app}}) \cdot \nabla G_N.$$

The heat Taylor formula gives

$$\|K * G_N\|_{W^{k,\infty}} + \|\nabla G_N\|_1 + \|\nabla G_N\|_{H^k} \le C_{k,N} \kappa^{N+1} t^{N+1}.$$

For the first term use heat Taylor on the smooth initial velocity $K * W$ and contraction of each bounded derivative. For the other terms use heat contraction on $\nabla \Delta^{N+1} W$ in $L^1$ and $H^k$. The coefficient bounds give $\|K * f_{\mathrm{app}}\|_{W^{k,\infty}} + \|\nabla f_{\mathrm{app}}\|_1 + \|\nabla f_{\mathrm{app}}\|_{H^k} \le Ca(t)$ after the positive-viscosity threshold is made small. Leibniz's rule proves the claimed seeded remainder in both norms.

Let $\zeta = \omega^\kappa - \omega_{\mathrm{app}}$ and $u_{\mathrm{app}} = K * \omega_{\mathrm{app}}$. Rewriting (27) gives

$$\partial_t \zeta + u_{\mathrm{app}} \cdot \nabla \zeta + (K * \zeta) \cdot \nabla \zeta = \kappa \Delta \zeta - (K * \zeta) \cdot \nabla \omega_{\mathrm{app}} - R_{\mathrm{app}}. \tag{40}$$

We record the estimates that avoid a quasilinear derivative loss. Sobolev embedding and the near–far kernel split give, for $k \ge 3$,

$$\|K * g\|_\infty \le C_k(\|g\|_1 + \|g\|_{H^k}), \tag{41}$$

$$\|\nabla K * g\|_{H^j} \le C_j \|g\|_{H^j}, \qquad j \ge 0. \tag{42}$$

The second is Plancherel's theorem for a bounded order-zero Fourier multiplier. In particular $\|\nabla K * g\|_\infty \le C_k \|g\|_{H^k}$.

For each integer derivative order up to $k$, the undifferentiated transport term has zero energy pairing. The commutator with $(K * \zeta) \cdot \nabla$ is bounded by

$$C_k \left( \|\nabla K * \zeta\|_\infty \|\zeta\|_{H^k} + \|\nabla \zeta\|_\infty \|\nabla K * \zeta\|_{H^{k-1}} \right) \le C_k \|\zeta\|_{H^k}^2.$$

This commutator inequality follows from Leibniz's rule: the endpoint terms put the first derivative in $L^\infty$ and the last derivative in $L^2$, and the intermediate terms follow by Sobolev interpolation in dimension two. The embeddings for both displayed endpoint terms hold because $k > 2$. No derivative beyond order $k$ of $\zeta$ occurs. Similarly,

$$\|(K * \zeta) \cdot \nabla \omega_{\mathrm{app}}\|_{H^k} \le C \left( \|\nabla \omega_{\mathrm{app}}\|_{H^k} + \sum_{j=1}^{k} \|\nabla^j \omega_{\mathrm{app}}\|_\infty \right) \|\zeta\|_{Z_k}.$$

For the term where no derivative lands on $K * \zeta$, use (41); for every term with a positive number of such derivatives use (42) in $L^2$ and put the corresponding derivative of the smooth approximation in $L^\infty$. The $H^k$ transport estimate with $u_{\mathrm{app}}$ uses only its fixed $W^{k,\infty}$ bounds. Its coefficients are bounded, independently of $\alpha, \kappa, H$, by fixed norms of $W$ plus one, once the approximate perturbation is made small in the finitely many required norms.

Together with Kato's $L^1$ inequality this gives, for $Y = \|\zeta\|_{Z_k}$,

$$\frac{\mathrm{d}^+ Y}{\mathrm{d}t} \le C_k Y + C_k Y^2 + \|R_{\mathrm{app}}\|_{Z_k}, \qquad Y(0) = 0. \tag{43}$$

Diffusion has a nonpositive energy contribution. The constant $C_k$ can be fixed before $M, N$: the pure-seed approximation is made small by choosing its cap after $M$, and each positive-viscosity coefficient is made small by choosing its viscosity threshold after $N$.

Set $E_k = 2C_k$, increasing it if necessary to dominate $C_p(\|\nabla W\|_{L^1\cap L^p}+1)$ and the compact-support $L^1\cap L^p$ stability constant used below. Choose one finite $M$ with

$$(M+1)\sigma > E_k. \tag{44}$$

After its pure-seed coefficients are known, choose $\delta_* > 0$ small enough for all their required norms and their pure-seed remainder. For each subsequently prescribed $K_*$ choose $N$ so that, with $r_k = (E_k/\sigma - 1)_+$,

$$\tfrac{3}{2}(N+1) > K_* r_k. \tag{45}$$

The $N$-dependent coefficients are controlled by decreasing the threshold for $q$, not by changing $\delta_*$. In particular arrange $C_{M,N}\kappa(1+H)^D \le 1$ in (39). On a bootstrap interval $Y \le 1$, (43) now gives

$$Y(t) \le C_M a(t)^{M+1} + C_{M,N}\kappa^{N+1}(1+t)^{D+1}a(t)e^{(E_k-\sigma)_+ t}. \tag{46}$$

At the cap, the second term is

$$C_{M,N}\kappa^{N+1}(1+H)^{D+1}\delta(\delta/\alpha)^{r_k} = o(1)$$

in the prescribed polynomial window. The first is smaller than $1/2$ after reducing $\delta_*$. Continuity contradicts a first bootstrap exit, so the estimate holds through the full cap interval. It also yields a uniform $H^k$ bound for the actual vorticity. This bound is obtained from the actual error equation, rather than assumed from the formal approximation.

### 4.1.2 Finite homological coefficients and compactness

For a real angle $\theta$ define the rotation

$$(\mathcal{T}_\theta f)(r,\varphi) = f(r,\varphi+\theta/4), \qquad \mathcal{L}_{\mathrm{rot}} = \mathcal{L} - (\tau/4)\partial_\varphi.$$

It acts isometrically on every radial or full-plane Sobolev norm, commutes with the radial linearization, and respects Biot–Savart advection. The pure-seed coefficient $f_{j,0}$ with seed phase zero has a limit

$$e^{-j\sigma t}\mathcal{T}_{-\tau t}f_{j,0}(t) \longrightarrow F_j \quad \text{in every finite } H^m \text{ as } t\to\infty. \tag{47}$$

For $j=1$ the decaying part is $O(e^{-2\sigma t})$ after this normalization, so

$$F_1 = f_\epsilon(r)e^{4i\varphi} + \text{complex conjugate}.$$

For $j \ge 2$, put $g_j = e^{-j\sigma t}\mathcal{T}_{-\tau t}f_{j,0}$. Its equation is

$$\partial_t g_j = (\mathcal{L}_{\mathrm{rot}} - j\sigma)g_j - \sum_{j_1+j_2=j}(K * g_{j_1})\cdot\nabla g_{j_2}.$$

On its finite set of sectors, the homogeneous propagator decays at least as $Ce^{-(j-1)\sigma t}$ by (15). Induction, the coefficient bounds, and dominated convergence in Duhamel's integral prove (47) and the finite homological relations

$$F_j = -(j\sigma - \mathcal{L}_{\mathrm{rot}})^{-1}\sum_{j_1+j_2=j}(K*F_{j_1})\cdot\nabla F_{j_2}, \qquad 2\le j\le M. \tag{48}$$

The inverse is the integral of the exponentially decaying propagator. All these functions are smooth and supported in the fixed collars. Only the finitely many sectors in $4\mathbb{Z}$ with $|m|\le 4M$ are used. No all-mode Sobolev spectral-growth assertion is needed.

Define the fixed polynomial ancient profile

$$P_M(s) = W + \sum_{j=1}^{M} \delta^j e^{j\sigma s} \mathcal{T}_{\tau s} F_j, \qquad s \le 0. \tag{49}$$

For the phase $\vartheta_q = -\tau H_q$, (47) shows that every pure-seed term of the cap-shifted approximation converges to the corresponding term in (49). The positive-viscosity terms tend to zero because $\kappa_q(1+H_q)^D \to 0$, while $e^{\kappa_q(H_q+s)\Delta} W \to W$. For each fixed $s \le 0$, (46) gives the asymptotic error $C_M(\delta e^{\sigma s})^{M+1}$, since its viscosity remainder tends to zero by (45).

We justify actual compactness separately. The kernel bound and (46), integrated as in (33), give

$$\int_0^{H_q} \|K * (\omega^q(t) - W_{\kappa_q}(t))\|_\infty \, \mathrm{d}t \le C_M \delta + o(1).$$

Choose $\delta_*$ so this is less than the positive radial gap fixed before the expansion orders. The exponential Kato proof of (35), with this fixed cap and the already fixed radius $R$, gives

$$\sup_{0 \le t \le H_q} \int_{|x|>R} |x|^4 |\omega^q(t,x)| \, \mathrm{d}x \le C e^{-c/(\kappa_q H_q)}. \tag{50}$$

Initial velocities have uniformly bounded finite energy, since the initial vorticities are compactly supported and have zero circulation. The NS velocity energy inequality and the uniform vorticity $H^k$ bound therefore bound the velocities in $H^{k+1}$. The equation bounds $\partial_t \omega^q$ uniformly in $H^{k-2}$ on the cap interval.

On each shifted interval $[-J, 0]$, Rellich compactness and interpolation give local compactness in $C(H^{k-\varepsilon})$. The tail bound and the uniform $H^k$, hence $L^\infty$, bound give global $L^2$ tightness. Interpolation applied to a smooth exterior cutoff then gives global $H^{k-\varepsilon}$ tightness; the cutoff is chosen outside the same fixed $B_R$. Thus compactness holds globally in that norm. The mass tail gives global $L^1$ tightness, and local convergence plus uniform $L^\infty$ bounds gives compactness in $C(L^1 \cap L^p)$ for every finite $p$. A diagonal subsequence over $J$ gives an ancient Euler solution: the transport term passes by the stated strong convergence, and $\kappa_q \Delta \omega^q \to 0$ on every compact shifted interval. The solution is supported in $B_R$ on $s \le 0$, since the right side of (50) tends to zero.

By lower semicontinuity of the $H^k$ norm and the strong $L^1$ convergence, every such limit satisfies, for all $s \le 0$,

$$\|\Omega(s) - P_M(s)\|_{Z_k} \le C_M(\delta e^{\sigma s})^{M+1}. \tag{51}$$

One may first take rational times; continuity in the weaker compactness norm and weak $H^k$ continuity extend the bound to every time.

### 4.1.3 Uniqueness of the ancient subsequential limit

We use a norm in which no extra derivative of a comparison orbit is required. If $\Omega_1, \Omega_2$ are two ancient subsequential limits, their compact supports and uniform $H^k$ bounds imply a uniform bound for $\|\nabla \Omega_j\|_{L^1 \cap L^p}$. Indeed compact support controls the $L^1$ norm by the $L^2$ derivative norm, and $H^k$, $k \ge 3$, controls every finite $L^p$ derivative norm. Their perturbations from $W$ are small in these norms after reducing $\delta_*$, so the stability coefficient is bounded by the fixed constant dominated by $E_k$ above.

For $D = \Omega_1 - \Omega_2$ the exact Euler equation is

$$\partial_s D + (K * \Omega_1) \cdot \nabla D = -(K * D) \cdot \nabla \Omega_2.$$

The same Kato and $L^p$ argument as (28) gives, for $s_0 \le s_1 \le 0$,

$$\|D(s_1)\|_{L^1\cap L^p} \le e^{E_k(s_1-s_0)}\|D(s_0)\|_{L^1\cap L^p}.$$

Both limits have the same polynomial (49). Using (51) at $s_0 = -T$ yields

$$\|D(0)\|_{L^1\cap L^p} \le C_M\delta^{M+1}e^{-[(M+1)\sigma-E_k]T} \longrightarrow 0.$$

The same comparison with any fixed endpoint $s_1 \le 0$ proves equality throughout the ancient interval. The strict inequality (44) uses one finite $M$ only. Thus every subsequence has the same limit, proving the full convergence in (36). Repeating the argument for any different $K_*$ gives the same limit; only its permissible viscosity expansion order and small-$q$ threshold change.

For clarity, a direct $H^k$ difference stability estimate would require an $H^{k+1}$ bound on the comparison vorticity because it differentiates $(K*D)\cdot\nabla\Omega_2$ $k$ times. The $L^1\cap L^p$ comparison above deliberately avoids this extra hypothesis. The high Sobolev estimate is used for compactness and for the available first derivatives.

#### 4.1.4 Forward extension and the whole-moment asymptotics

The ancient construction initially gives compactly supported $H^k$ vorticity at zero and finite-energy velocity in $H^{k+1}$, with $k \ge 3$, and preserves zero circulation by strong $L^1$ convergence. Lemma 2.3 therefore extends this same solution uniquely to every $s \ge 0$, with $H^k$ vorticity and $H^{k+1}$ velocity on each fixed interval. This first extension uses only the finite order $k$; the all-order regularity of the fixed orbit is proved separately below. Its conserved $L^1$ and $L^p$ norms and (21) give a time-independent velocity bound $V$.

The bounded velocity transports its support into $B_{R+Vs}$. Consequently

$$\int |x|^4|\Omega_\delta(s,x)|\,\mathrm{d}x \le \|\Omega_\delta(0)\|_1(R+Vs)^4 \qquad (s \ge 0),$$

which proves (38). On $s \le 0$, its support is in the fixed $B_R$. Pairing (51) with $\bar z^4$ is therefore bounded by its $L^1$ error times $R^4$. The first homological coefficient has moment $b$, rotation multiplies it by $e^{i\tau s}$, and every higher term is $O(\delta^2 e^{2\sigma s})$. This proves (37) and completes the theorem.

**Corollary 4.2** (Every fixed source window, including after the cap)**.** *For the isolated family in Theorem 4.1, every fixed $J < \infty$ satisfies*

$$\omega^q(H_q+\cdot) \longrightarrow \Omega_\delta \quad \text{in } C([-J,J];L^1\cap L^p),$$

$$\sup_{|s|\le J}|B_4(\omega^q(H_q+s)) - B_4(\Omega_\delta(s))| \longrightarrow 0.$$

*Write $B_q(s) = B_4(\omega^q(H_q+s))$ and $B_\delta(s) = B_4(\Omega_\delta(s))$. For fixed $0 \le S,U < \infty$ and $a \in \mathbb{R}$, its source moment therefore satisfies*

$$\int_{-S}^{U} e^{-as}|B_q(s) - B_\delta(s)|\,\mathrm{d}s \longrightarrow 0.$$

*Proof.* The negative window follows from (36) and (50). On $0 \le s \le J$, let $\zeta_q(s) = \omega^q(H_q+s) - \Omega_\delta(s)$. Its exact equation is

$$\partial_s\zeta_q + (K*\omega^q(H_q+s))\cdot\nabla\zeta_q = \kappa_q\Delta\zeta_q - (K*\zeta_q)\cdot\nabla\Omega_\delta + \kappa_q\Delta\Omega_\delta.$$

The global Euler extension has bounded $H^k$ norm and compact support on this fixed interval. Since $k \geq 3$, its second derivatives belong to $H^1$, which embeds in every finite $L^p$ in dimension two; compact support also bounds their $L^1$ norms. Thus $\|\nabla\Omega_\delta\|_{L^1\cap L^p} + \|\Delta\Omega_\delta\|_{L^1\cap L^p} \leq C_J$. Kato's inequality, the $L^p$ inequality and (21) give

$$\sup_{0\leq s\leq J} \|\zeta_q(s)\|_{L^1\cap L^p} \leq e^{C_J J}(\|\zeta_q(0)\|_{L^1\cap L^p} + C_J\kappa_q) \longrightarrow 0.$$

No actual NS derivative bound is used in this comparison.

We remove the moment cutoff separately. Scalar norm contraction gives one uniform post-cap velocity bound $V$. Choose a fixed radius $R' > R + VJ + d$, also enclosing the Euler support on $[0, J]$, with $d > 0$. The positive post-cap propagator has a forward diffusion with $|X_s - y| \leq VJ + \sqrt{2\kappa_q}|B_s|$. For $|y| \leq R$, its terminal fourth moment outside $B_{R'}$ is at most $C_J e^{-c_J/\kappa_q}$ by Gaussian endpoint moments. For unrestricted $y$ its full fourth moment is at most $C_J(1 + |y|^4)$. The positive domination of the signed solution by the propagation of $|\omega^q(H_q)|$ therefore gives

$$\sup_{0\leq s\leq J} \int_{|x|>R'} |x|^4|\omega^q(H_q + s, x)|\,\mathrm{d}x \leq C_J e^{-c_J/\kappa_q} + C_J \int_{|y|>R} (1 + |y|^4)|\omega^q(H_q, y)|\,\mathrm{d}y \longrightarrow 0$$

by (50). A compact radial cutoff of $\bar{z}^4$, the just-proved $L^1$ convergence, and this tail prove uniform whole-moment convergence. On a fixed weighted time window the weight is bounded, so its final assertion follows by integration. □

**Corollary 4.3** (A nonzero bilateral discounted moment)**.** *Define*

$$\mathcal{F}_\delta(z) = \int_{-\infty}^{\infty} e^{-zs} B_4(\Omega_\delta(s))\,\mathrm{d}s.$$

*It is analytic and is not identically zero on* $0 < \Re z < \sigma$*. More precisely it extends meromorphically to* $0 < \Re z < 2\sigma$ *as*

$$\mathcal{F}_\delta(z) = \frac{\delta b}{\lambda - z} + \mathcal{G}_\delta(z), \tag{52}$$

*where* $\mathcal{G}_\delta$ *is analytic on that larger strip. For every fixed real* $c$*, the exceptional set*

$$\{\beta > 1 : \mathcal{F}_\delta((\sigma - ic)/\beta) = 0\}$$

*is discrete in* $(1, \infty)$*. In particular one can select arbitrarily large* $\beta$ *outside it, including for* $c = 1/2$*.*

*Proof.* The negative-time estimate (37) and the positive-time polynomial bound give absolute, locally uniform convergence on $0 < \Re z < \sigma$, including after any fixed number of $z$ derivatives. Subtract the leading negative-time exponential. Its integral is

$$\int_{-\infty}^{0} \delta b e^{(\lambda - z)s}\,\mathrm{d}s = \frac{\delta b}{\lambda - z}.$$

The remaining negative-time integral is analytic for $\Re z < 2\sigma$, and the positive-time integral is analytic for $\Re z > 0$. This proves (52). Its pole at $z = \lambda$ has nonzero coefficient, so the transform cannot be identically zero on the smaller strip. Its zeros there are isolated. The curve $\beta \mapsto (\sigma - ic)/\beta$ is nonconstant, lies in that strip for $\beta > 1$, and has no interior accumulation of its zeros. Any terminal interval of large $\beta$ contains values outside the zero set, proving the last assertion. Accumulation as $\beta \to \infty$ is possible at the boundary $z = 0$ and is not excluded or needed. □

**Corollary 4.4** (Changing the canonical cap)**.** *If $0 < \delta' < \delta < \delta_*$ and $s_0 = \sigma^{-1}\log(\delta'/\delta)$, then*

$$\Omega_{\delta'}(s) = \mathcal{T}_{-\tau s_0}\Omega_\delta(s+s_0) \qquad (-\infty < s < \infty), \qquad \mathcal{F}_{\delta'}(z) = e^{(z-i\tau)s_0}\mathcal{F}_\delta(z).$$

*Thus the zero set of the transform is independent of the canonical cap. At $z = (\sigma - ic)/\beta$ its modulus scales by $(\delta'/\delta)^{1/\beta}$.*

*Proof.* The time-translated and rotated orbit has the polynomial (49) with $\delta$ replaced by $\delta'$, since $\delta e^{\sigma s_0} = \delta'$. It has the same order-$M+1$ remainder. The ancient uniqueness argument therefore identifies it with $\Omega_{\delta'}$ on negative times, and forward Euler uniqueness extends the identity to every time. Rotation $\mathcal{T}_\theta$ multiplies $B_4$ by $e^{i\theta}$. Substituting $v = s + s_0$ into the absolutely convergent bilateral transform gives the stated exponential factor, which never vanishes. Its modulus at the displayed point is $e^{\sigma s_0/\beta}$. □

**Corollary 4.5** (All finite regularity orders and finite jets of one orbit)**.** *The already fixed orbit $\Omega_\delta$ belongs to every finite spatial Sobolev order at every time. For each integer $\ell \geq k$,*

$$\sup_{s\leq 0}\|\Omega_\delta(s)\|_{L^1\cap H^\ell} < \infty,$$

*and its $H^\ell$ norm is bounded on each fixed positive-time interval. In particular its vorticity is spatially $C^\infty$. For each fixed $J \geq 1$ and integer $r \geq 0$, the homological coefficients $F_1, \dots, F_J$ satisfy, for some $s_{J,r} < 0$ and $C_{J,r} < \infty$,*

$$\left\|\Omega_\delta(s) - W - \sum_{j=1}^{J}\delta^j e^{j\sigma s}\mathcal{T}_{\tau s}F_j\right\|_{L^1\cap H^r} \leq C_{J,r}(\delta e^{\sigma s})^{J+1}, \qquad s \leq s_{J,r}. \tag{53}$$

*The coefficients are given by (48) for each finite target degree, and are independent of the larger cutoff used to construct them. All intersection norms here are sum norms.*

*Proof.* Fix an integer $\ell \geq k$. Repeat the finite-coefficient and actual-error construction above in order $\ell$, with amplitude $\delta_\ell$ smaller than $\delta$ and both the original and the order-$\ell$ permissible thresholds. The profile and smoothing length remain fixed. Choose any fixed $K_* > 0$. The exact Navier–Stokes family used in this construction, with cap shift $H_q = \sigma^{-1}\log(\delta_\ell/\alpha_q)$ and seed phase $-\tau H_q$, is precisely the isolated family already identified by Theorem 4.1 at amplitude $\delta_\ell$. Its full $L^1 \cap L^p$ limit is unique. The order-$\ell$ construction therefore gives the same $\Omega_{\delta_\ell}$ and a uniform $H^\ell$ bound on negative times. Lemma 2.3 yields its $H^\ell$ bounds on every fixed positive-time interval.

Let $s_\ell = \sigma^{-1}\log(\delta_\ell/\delta) < 0$. Corollary 4.4 and forward uniqueness give

$$\Omega_{\delta_\ell}(v) = \mathcal{T}_{-\tau s_\ell}\Omega_\delta(v + s_\ell) \qquad (-\infty < v < \infty).$$

For $s \leq s_\ell$ use the negative-time bound on $\Omega_{\delta_\ell}(s - s_\ell)$. For $s_\ell \leq s \leq 0$ use its positive-time bound on $0 \leq v \leq -s_\ell$. Rotation is an isometry, so these two bounds give the asserted uniform order-$\ell$ bound for the same $\Omega_\delta$ on $s \leq 0$. Translating fixed positive intervals gives its remaining bounds. Their constants may depend on $\ell, \delta$. Since $\ell$ is arbitrary, Sobolev embedding proves spatial smoothness. No intersection of infinitely many small-amplitude thresholds is used.

For the finite jet, fix $J, r$ and choose $\ell \geq \max\{k, r+1, 3\}$. Choose a finite expansion degree $M \geq J$ large enough that $(M+1)\sigma > E_\ell$, and choose a sufficiently small $\delta_\ell < \delta$ for this finite construction. The same isolated family and uniqueness again identify its limit with $\Omega_{\delta_\ell}$. The

pure-seed triangular recursion gives the same $F_1, \dots, F_J$ independently of $M$; passing its finite expansion and actual-error estimate to the limit gives

$$\left\| \Omega_{\delta_\ell}(v) - W - \sum_{j=1}^{J} \delta_\ell^j e^{j\sigma v} \mathcal{T}_{\tau v} F_j \right\|_{L^1 \cap H^r} \le C_{J,r} (\delta_\ell e^{\sigma v})^{J+1}, \qquad v \le 0.$$

Indeed the error has degree $M+1$, and the finitely many discarded degrees $J+1, \dots, M$ have the displayed degree-$J+1$ bound. Translate by $s_\ell$ and rotate by $\tau s_\ell$. The amplitude identity $\delta_\ell e^{\sigma(s-s_\ell)} = \delta e^{\sigma s}$ and addition of the rotation phases give (53) for $s \le s_\ell$. The constant and onset time may depend on the finite target orders. This argument proves each finite jet and does not assume convergence of an infinite formal series. □

*Remark* 4.6 (Scope). The profile and source transform in this section belong to the isolated annulus. Their use in a coupled receiver requires the foreign-colour and weighted source-history estimates in addition to this isolated limit.

## 4.2 The smooth receiver's protected coordinate and transfer coefficient

Throughout this section $W = W_\epsilon$ is the one fixed profile of Section 3.1, $\lambda = \sigma + i\tau$ its growing fourth-sector eigenvalue, and $f_+ = f_\epsilon$ its normalized radial eigenfunction. All angular conventions are unchanged.

**Lemma 4.7** (A global growing-coordinate test)**.** *There is a smooth bounded complex test*

$$T_{\mathrm{adj}}(x) = a(r) e^{-4i\theta}$$

*whose derivatives of every fixed order are bounded, such that*

$$\mathcal{A}(g) = \int_{\mathbb{R}^2} T_{\mathrm{adj}}(x) g(x)\, \mathrm{d}x, \qquad \mathcal{A}(f_+(r)e^{4i\theta}) = 1, \qquad \mathcal{A}(\overline{f_+(r)} e^{4i\theta}) = 0,$$

*and $\mathcal{A}(\mathcal{L}g) = \lambda \mathcal{A}(g)$ for smooth integrable perturbations for which the pairings are defined. Near the origin $T_{\mathrm{adj}}$ is a constant multiple of $\bar{z}^4$; outside a fixed ball it is a constant multiple of $z^{-4}$. In particular, $\Delta T_{\mathrm{adj}}$ is compactly supported away from the origin.*

*Proof.* On a fixed radial interval containing the derivative collars, the rank-one Riesz projection in the positive fourth sector has form $P_+ g = f_+ \ell(g)$ with $\ell(f_+) = 1$ and $\ell(\mathcal{L}_4 g) = \lambda \ell(g)$. Represent this continuous $L^2$ functional by a bilinear density $2\pi a(r) r\, \mathrm{d}r$; complex conjugation is included in the representing density rather than in the pairing. Substitution of (14) gives

$$(\lambda + 4i\Omega_\epsilon(s)) a(s) = -\frac{i}{2} \int_0^\infty a(r) W_\epsilon'(r) h_4(r,s)\, \mathrm{d}r. \tag{54}$$

On the collars, the radial Green equation and this nonvanishing denominator bootstrap $a$ to smoothness. Extend it to every $s > 0$ by the right side of (54) divided by its denominator. This agrees with the original density where $W'$ is supported, so it still satisfies the equation everywhere. Integration of the Green kernel and Fubini then verify the global eigenfunctional identity.

For $s$ smaller than every derivative collar, the numerator is a constant times $s^4$ and $\Omega_\epsilon$ is the constant core angular velocity $-1/5$. Thus $a(s) = Cs^4$ there and $T_{\mathrm{adj}} = C\bar{z}^4$. Beyond the profile support the numerator is a constant times $s^{-4}$ and $\Omega_\epsilon = 0$ by zero circulation. Thus $a(s) = C's^{-4}$ and $T_{\mathrm{adj}} = C'z^{-4}$. These two extensions are harmonic; on the remaining fixed annulus the test is smooth. This proves all the regularity and Laplacian assertions. Normalize by $2\pi \int a f_+ r\, \mathrm{d}r = 1$. Since the stable eigenvalue is $-\bar{\lambda} \ne \lambda$, the eigenfunctional identity annihilates its eigenfunction. The angular test kills every sector other than $+4$. □

**Lemma 4.8** (Exact inverse-quintic forcing coefficient)**.** *Let a source $w$ be real, integrable, supported in $B_{R_s}$, quarter-turn symmetric and of zero circulation. For each integer $m \geq 0$ used below, write*

$$B_m(w) = \int_{\mathbb{R}^2} \bar{y}^m w(y)\,\mathrm{d}y, \qquad \int_{\mathbb{R}^2} |y|^m |w(y)|\,\mathrm{d}y < \infty.$$

*These absolute moments are finite for this compactly supported source. At points in the exterior region $|z| > R_s$ its complex velocity is*

$$U(z) = \frac{i}{2\pi} \sum_{\ell=1}^{\infty} \frac{B_{4\ell}(w)}{\bar{z}^{4\ell+1}}. \tag{55}$$

*For every finite $L \geq 1$, its remainder after degree $4L$ satisfies*

$$\left| U(z) - \frac{i}{2\pi} \sum_{\ell=1}^{L} \frac{B_{4\ell}(w)}{\bar{z}^{4\ell+1}} \right| \leq \frac{1}{2\pi |z|^{4L+5} (1 - (R_s/|z|)^4)} \int |y|^{4L+4} |w(y)|\,\mathrm{d}y. \tag{56}$$

*The first term forces the receiver's positive fourth vorticity sector by $B_4 F_\epsilon(r) e^{4i\theta}$, where*

$$F_\epsilon(r) = -\frac{i W_\epsilon'(r)}{4\pi r^5}.$$

*Its growing projection is $\gamma_\epsilon B_4$, with $\gamma_\epsilon = \mathcal{A}(F_\epsilon e^{4i\theta})$. For the mollified family before the smoothing length is fixed,*

$$\gamma_\epsilon \longrightarrow \gamma_0 = \frac{i\rho^{-6} - \rho^{-1}}{8\pi} \neq 0 \quad (\epsilon \downarrow 0). \tag{57}$$

*Proof.* In complex notation Biot–Savart is $U(z) = \frac{i}{2\pi} \int (\bar{z} - \bar{y})^{-1} w(y)\,\mathrm{d}y$. For $|z| > R_s$, the geometric series converges uniformly in $y$ on the source support and absolutely after integration. Quarter-turn symmetry gives $B_m = (-i)^m B_m$, eliminating every degree not divisible by four; zero circulation eliminates $B_0$. This proves (55). Summing the absolute values of the remaining degrees and using $|y| \leq R_s$ proves (56). The exterior expansion is not used in an inner hole of an annular support. For its first term,

$$u_r = \Re(U e^{-i\theta}) = \frac{i B_4}{4\pi r^5} e^{4i\theta} + \text{complex conjugate}.$$

The linearized forcing is $-u_r W'$, giving $F_\epsilon$.

We check nonvanishing without assuming an adjoint convergence theorem. The inhomogeneous resolvent Schur reduction in the proof of Lemma 3.3 applies to this forcing. Its natural radial measure has limit

$$F_\epsilon(r) r\,\mathrm{d}r \rightharpoonup \sum_{i=1}^{2} \frac{i c_i}{4\pi r_i^4} \delta_{r_i}.$$

Thus in interface variables its forcing vector tends to $h = (i/(4\pi\rho^6), i/(4\pi))$. The growing dual of $M_4$, normalized on $(1, -i\rho)$, is $\ell_0(q) = (q_1 + iq_2/\rho)/2$. Its pairing with $h$ is $\gamma_0$ in (57). More explicitly, on a small fixed contour around the growing root the inhomogeneous Schur equations converge uniformly to $(z - M_4)q = h$. Integrating their resolvent in $z$ gives convergence of the rank-one projected forcing coefficient. The eigenfunction normalization agrees with that in Section 3.1. This proves the limit and nonvanishing for all sufficiently small $\epsilon$. This condition is included in the final fixed choice of the smoothing length. □

The direction of this interaction matters. A smaller source acts on the larger receiver by the inverse-quintic exterior field above. A larger annulus acts in its interior through a cubic harmonic field. They are different coefficients and cannot be interchanged.

For comparison, an exterior annulus's interior harmonic field $U(z) = c_3\bar{z}^3$ has positive fourth radial coefficient $\bar{c}_3 r^3/2$. Its frozen-base forcing is $-\bar{c}_3 r^3 W'/2$, with growing projection $\gamma_\epsilon^{\mathrm{out}}\bar{c}_3$. The same resolvent limit gives

$$\gamma_\epsilon^{\mathrm{out}} \longrightarrow (\rho^2 + i/\rho)/4 \neq 0.$$

Indeed its interface forcing vector is $(\rho^2/2, 1/2)$, paired against the same growing dual. Thus older cubic fields can genuinely seed the growing coordinate; their size must be bounded, rather than discarded by symmetry. All the required spectral and nonvanishing conditions now hold for every sufficiently small smoothing length. Fix once one $\epsilon$ for which Lemma 3.3 holds and $b_\epsilon, \gamma_\epsilon, \gamma_\epsilon^{\mathrm{out}}$ are nonzero, and retain it in every later finite expansion and Sobolev order.

**Lemma 4.9** (Duhamel coefficient for a child source)**.** *Let $A_c = \beta A_r$, $R_c = qR_r$, and use receiver time $s = A_r t$. Suppose, in a frame removing all common solid rotation, that the source's normalized fourth moment is $B(u)$, with child time $u = \beta s - s_*$. The receiver core has angular velocity $-A_r/5$; this rotates the child's moment by $e^{4is/5}$ in the receiver frame. The ideal linear growing receiver coefficient obeys*

$$a'(s) = \lambda a(s) + \beta q^6 \gamma_\epsilon e^{4is/5} B(\beta s - s_*).$$

*For the response with no independent growing coefficient at $s = -\infty$,*

$$e^{-\lambda s} a(s) = q^6 \gamma_\epsilon e^{-\mu_\beta s_*} \int_{-\infty}^{\beta s - s_*} e^{-\mu_\beta u} B(u)\,\mathrm{d}u, \qquad \mu_\beta = \frac{\sigma + i(\tau - 4/5)}{\beta}. \tag{58}$$

*This integral exists whenever $B(u) = O(e^{\sigma u})$ at $-\infty$ and has polynomial growth at $+\infty$, provided $\beta > 1$.*

*Proof.* The dimensional fourth moment is $A_c R_c^6 B$. Dividing the child velocity by $A_r R_r$ leaves the factor $\beta q^6$, so Lemma 4.8 gives the forcing magnitude. Rotation through angle $\Omega t$ multiplies $B_4$ by $e^{-4i\Omega t}$. The parent constant core has vorticity $-2A_r/5$, hence angular velocity $-A_r/5$, which gives the factor $e^{4is/5}$. Multiplication by the integrating factor $e^{-\lambda s}$ and the substitution $u = \beta s - s_*$ give (58). Its negative tail has exponential exponent $\sigma - \sigma/\beta > 0$; its positive tail has discount $\sigma/\beta > 0$. □

**Lemma 4.10** (A nonzero bilateral source transform from an ancient orbit)**.** *Suppose a global Euler source trajectory has a fourth moment $B(u)$ with*

$$B(u) = cb_\epsilon e^{\lambda u} + O(e^{2\sigma u}) \quad (u \to -\infty), \qquad c \neq 0,$$

*and $|B(u)| \le C(1+u)^4$ for $u \ge 0$. Define*

$$\mathcal{F}(z) = \int_{\mathbb{R}} e^{-zu} B(u)\,\mathrm{d}u, \qquad 0 < \Re z < \sigma.$$

*Then $\mathcal{F}$ is analytic and not identically zero. For every finite interval $[B_*, B_* + 1] \subset (1, \infty)$ there is a choice of $\beta$ in that interval such that $\mathcal{F}(\mu_\beta) \neq 0$.*

*Proof.* The two explicit bounds give absolute locally uniform convergence of the integral and of all complex derivatives, proving analyticity. Split at zero and subtract the leading negative-time asymptotic. It gives

$$\mathcal{F}(z) = \frac{cb_\epsilon}{\lambda - z} + \mathcal{G}(z), \qquad 0 < \Re z < \sigma,$$

where $\mathcal{G}$ is analytic in $0 < \Re z < 2\sigma$: the remainder on the negative half-line has rate $2\sigma$, and the positive half-line has polynomial growth. Thus its meromorphic extension has a nonremovable simple pole at $z = \lambda$. In particular it cannot vanish identically on the original strip. The analytic nonconstant curve $\beta \mapsto \mu_\beta$ lies inside that strip for real $\beta > 1$. If the transform vanished along any real interval, its zeros would have an interior accumulation point and the identity theorem would force identical vanishing. Its zeros along the displayed compact interval are therefore discrete, and a nonzero choice exists. □

The canonical ancient Euler orbit of Theorem 4.1 has precisely this negative-time estimate. It comes from the high-Sobolev construction and not merely from the isolated $X$-norm cap theorem. Its positive-time moment bound is elementary once a compactly supported bounded-vorticity orbit exists: $L^1, L^p$ conservation bounds its velocity by (21), its support radius grows at most linearly, and its fourth absolute moment is therefore polynomial of degree at most four. Lemma 4.10 isolates the precise ancient-orbit input needed for the noncancellation assertion.

**Lemma 4.11** (Quantitative window and source errors)**.** *Let $\beta > 1$ be fixed, $\mu = \mu_\beta$, and $S, U \geq 0$. Suppose $\mathcal{F}(\mu) \neq 0$ and the source satisfies the bounds of Lemma 4.10. Define the truncation budget*

$$E_{\mathrm{tr}} = \left| \mathcal{F}(\mu) - \int_{-S}^{U} e^{-\mu u} B(u)\, \mathrm{d}u \right|.$$

*It satisfies*

$$E_{\mathrm{tr}} \leq C e^{-(\sigma - \Re\mu)S} + C_\beta (1+U)^4 e^{-\Re\mu U}.$$

*Replacing the moment on that window by $\widetilde{B}$ has budget*

$$E_{\mathrm{rep}} = \int_{-S}^{U} e^{-\Re\mu u} |\widetilde{B}(u) - B(u)|\, \mathrm{d}u.$$

*Set $Q = q^6 |\gamma_\epsilon| e^{-\sigma s_*/\beta}$. If the three budgets satisfy*

$$E_{\mathrm{tr}} < \tfrac{1}{4}|\mathcal{F}(\mu)|, \qquad E_{\mathrm{rep}} < \tfrac{1}{4}|\mathcal{F}(\mu)|, \qquad |a_0| < \tfrac{1}{4} Q |\mathcal{F}(\mu)|,$$

*then adding the independent old growing coefficient $a_0$ to the modified source response leaves a growing coefficient of modulus greater than $Q|\mathcal{F}(\mu)|/4$. If $s_* = \sigma^{-1}\log(\delta/\alpha)$ and both old and child seed amplitudes have size $\alpha = q^{K_*}$, the old-seed condition is algebraically feasible when*

$$K_*(1 - 1/\beta) > 6.$$

*Here $0 < \delta \leq 1$ is fixed or decreases logarithmically in $1/q$; the remaining source and transfer constants are fixed.*

*Proof.* Put $a = \Re\mu = \sigma/\beta \in (0, \sigma)$. The negative tail is bounded by $Ce^{-(\sigma-a)S}/(\sigma - a)$. For the positive tail, substitute $u = U + v$ and use $(1 + U + v)^4 \leq (1+U)^4(1+v)^4$ for $U, v \geq 0$. The integral of $e^{-av}(1+v)^4$ is finite, giving the asserted positive-tail bound. The negative and positive tails together constitute the single budget $E_{\mathrm{tr}}$. By the triangle inequality,

$$\left| \int_{-S}^{U} e^{-\mu u} \widetilde{B}(u)\, \mathrm{d}u \right| \geq |\mathcal{F}(\mu)| - E_{\mathrm{tr}} - E_{\mathrm{rep}} > \tfrac{1}{2}|\mathcal{F}(\mu)|.$$

Equation (58) multiplies this transform by a factor of modulus $Q$. A further triangle inequality subtracting $|a_0|$ leaves a modulus greater than $Q|\mathcal{F}(\mu)|/4$. Finally $e^{-\sigma s_*/\beta} = (\alpha/\delta)^{1/\beta}$, so

$$\frac{\alpha}{q^6(\alpha/\delta)^{1/\beta}} = q^{K_*(1-1/\beta)-6}\delta^{1/\beta} \longrightarrow 0$$

under the stated power inequality. Fixed logarithmic cap factors do not alter this strict inequality. □

#### 4.2.1 Exact heat shielding and the remaining coupled terms

The neutral seed (20) has only sectors $\pm 4$ and $B_4 = 0$. Its velocity is zero outside $B_{1+\epsilon}$: in the exterior angular Green formula its coefficient is exactly its fourth moment. Thus its velocity $V_\vartheta = K * \Phi_\vartheta$ is smooth and compactly supported. Heat commutes with Biot–Savart, so a passive physical seed of amplitude $A\alpha$ and radius $r$ has exterior heat velocity bounded, for $|x| \geq R > 2(1+\epsilon)r$, by

$$|e^{\nu t\Delta}[A\alpha r V_\vartheta(\cdot/r)](x)| \leq CA\alpha r \exp[-cR^2/(\nu t)]. \tag{59}$$

Indeed convolution integrates a bounded function supported in that ball, and the Gaussian mass at distance at least $R/2$ has the displayed bound. No damping estimate of a conserved harmonic moment is invoked. The bound concerns a passive heat evolution; its nonlinear moment generation is an additional term.

For precision, the actual foreign-colour equation yields an exact protected-coordinate identity. Suppose a receiver colour $w$ solves

$$w_t + (K * w + b)\cdot\nabla w = \kappa\Delta w, \qquad \operatorname{div} b = 0,$$

and put $f = w - W_\kappa$. Remove any common rigid rotation first. Testing with Lemma 4.7 gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(f) = \lambda\mathcal{A}(f) - \int T_{\mathrm{adj}} b\cdot\nabla W\,\mathrm{d}x + \mathcal{E}(t), \tag{60}$$

where the exact error is the sum of

$$\kappa\int \Delta T_{\mathrm{adj}} f\,\mathrm{d}x, \qquad -\int T_{\mathrm{adj}}\big[(K*(W_\kappa - W))\cdot\nabla f + (K*f)\cdot\nabla(W_\kappa - W)\big]\,\mathrm{d}x,$$
$$\int f(K*f)\cdot\nabla T_{\mathrm{adj}}\,\mathrm{d}x, \qquad \int fb\cdot\nabla T_{\mathrm{adj}}\,\mathrm{d}x, \qquad -\int T_{\mathrm{adj}} b\cdot\nabla(W_\kappa - W)\,\mathrm{d}x.$$

There is no derivative of the actual $f$ after integrating its transport terms by parts. Smooth heat Taylor and (21) give

$$|\mathcal{E}(t)| \leq C[\kappa(1+t)\|f\|_X + \|f\|_X^2] + \int |f||b||\nabla T_{\mathrm{adj}}|\,\mathrm{d}x + \left|\int T_{\mathrm{adj}} b\cdot\nabla(W_\kappa - W)\,\mathrm{d}x\right|. \tag{61}$$

For the term with $\nabla f$, integration by parts bounds it by $\|\nabla T_{\mathrm{adj}}\|_\infty \|K*(W_\kappa - W)\|_\infty \|f\|_1$; the other heat term uses $\|\nabla(W_\kappa - W)\|_1 = O(\kappa t)$. The quadratic self term uses (21). If $b = \Omega(t)Jx$ is a rigid rotation, its base forcing vanishes and its perturbation contribution is exactly $-4i\Omega(t)\mathcal{A}(f)$; it can be removed by its integrating phase.

Equation (60) identifies the estimates required for coupled use. On the receiver collars one must approximate $b$ by the inverse-quintic child field of Lemma 4.8; its projected error is bounded by the collar sup error times $\|T_{\mathrm{adj}}\nabla W\|_1$. One must also control the two weighted foreign terms in (61) and prove a nonlinear density bound for this actual colour. The canonical source transform must be approximated in the weighted norm of Lemma 4.11, including its post-cap history. The actual foreign-colour estimates and weighted source-history control are additional inputs to a coupled receiver argument.

### 4.2.2 A nonlinear receiver theorem for the concrete canonical input

The following theorem uses the moment of the canonical isolated Euler orbit and realizes its exterior harmonic field through a prescribed smooth divergence-free extension. Its equation has this auxiliary drift; the source and receiver are not asserted here to solve a jointly unforced system.

Fix an ancient orbit $\Omega_{\delta_c}$ from Theorem 4.1, with fixed small $\delta_c > 0$, and put $B(u) = B_4(\Omega_{\delta_c}(u))$. Choose $\beta > 1$ for which

$$C_* := \gamma_\epsilon \mathcal{F}(\mu_\beta) \neq 0.$$

Such a choice is possible in every sufficiently large fixed interval by Lemma 4.10. Take a smooth radial cutoff $\zeta_0$ equal to zero near the origin and equal to one on and outside the two receiver derivative collars. For $c \in \mathbb{C}$ set

$$\Psi_c(r,\theta) = -\frac{\zeta_0(r)}{16\pi r^4}(ce^{4i\theta} + \bar{c}e^{-4i\theta}), \qquad \mathcal{V}_c = \nabla^\perp \Psi_c.$$

This is a smooth real divergence-free field with every fixed spatial derivative bounded by a constant times $|c|$. On the receiver collars it agrees exactly with the inverse-quintic field of a source moment $c$.

**Theorem 4.12** (Transmission for a prescribed canonical harmonic input)**.** *Fix $K_* > 0$ such that $K_*(1 - 1/\beta) > 6$. For small $q$ put*

$$\alpha = q^{K_*}, \quad \varepsilon_q = q^6, \quad \delta_q = [\log(1/q)]^{-4}, \quad t_0 = \frac{1}{\beta\sigma}\log\frac{\delta_c}{\alpha}, \quad S_q = \frac{1}{\sigma}\log\frac{\delta_q}{\varepsilon_q|C_*|}, \quad H_q = t_0 + S_q.$$

*For $0 < \kappa \le q^{3/2}$ define the prescribed foreign drift*

$$b_q(t,x) = \beta\varepsilon_q \mathcal{V}_{e^{4i(t-t_0)/5}B(\beta(t-t_0))}(x), \qquad 0 \le t \le H_q.$$

*Let $w^q$ solve the coloured equation*

$$\partial_t w^q + (K * w^q + b_q)\cdot\nabla w^q = \kappa\Delta w^q, \qquad w^q(0) = W + \alpha\Phi_0.$$

*Then uniformly in $0 < \kappa \le q^{3/2}$,*

$$\sup_{0\le t\le H_q} \|w^q(t) - e^{\kappa t\Delta}W\|_X \le C\delta_q,$$
$$\mathcal{A}(w^q(H_q)) = \delta_q e^{i(\tau S_q + \arg C_*)} + o(\delta_q),$$
$$B_4(w^q(H_q)) = b_\epsilon\delta_q e^{i(\tau S_q + \arg C_*)} + o(\delta_q).$$

*The whole fourth moment is finite. Multiplying the source moment in $b_q$ by a fixed unit complex phase multiplies the leading cap phase by the same phase; all bounds are uniform in that phase.*

*Proof.* For completeness, the positive-viscosity solution with this prescribed drift exists on the whole displayed window. On a short interval its mild equation is a contraction in $C_t(L^1 \cap L^p)$: the heat-gradient estimate is $\|\nabla e^{\kappa t\Delta}\|_{L^r\to L^r} \le C(\kappa t)^{-1/2}$, the Biot–Savart map sends $X$ to $L^\infty$, and $b_q$ is bounded there. Scalar convex-energy estimates give the $L^1, L^p$ contractions and hence a time-uniform velocity bound. Iterating the local construction prevents finite-time loss of the mild solution. Smoothing follows by the same mild equation from the smooth spatial coefficients.

This argument uses only bounded drift on the finite window, not the conclusion of the theorem. Use $s = t - t_0$ and write $b_q = \varepsilon_q b_1(s)$. The ancient and future moment bounds give

$$\|b_1(s)\|_{C^j} \le \begin{cases} C_j e^{\beta\sigma s}, & s \le 0, \\ C_j (1+s)^4, & s \ge 0. \end{cases}$$

In particular this is bounded by $C_j e^{\sigma s}$ for all $s$, and its growing-mode weighted time integral is finite, uniformly in the initial time $-t_0$. The first seed-order coefficient solves

$$(\partial_t - \mathcal{L}) g_{1,0} = -b_1(t - t_0) \cdot \nabla W, \qquad g_{1,0}(0) = (\alpha/\varepsilon_q)\Phi_0.$$

Its growing coordinate is

$$e^{-\lambda s} \mathcal{A}(g_{1,0}(t)) = \frac{\alpha}{\varepsilon_q} e^{\lambda t_0} + \gamma_\epsilon \int_{-\beta t_0}^{\beta s} e^{-\mu_\beta u} B(u)\, \mathrm{d}u.$$

The first term tends to zero in modulus because $K_*(1 - 1/\beta) > 6$. The integral tends to $\mathcal{F}(\mu_\beta)$ as $t_0, s \to \infty$. Thus its growing coefficient tends to $C_*$. The complementary propagator has exponent $2\eta < \sigma$; integrating its ancient exponential forcing and its future polynomial forcing gives a complementary bound $Ce^{2\eta s}$ for $s \ge 0$. For $s \le 0$ the whole first coefficient is bounded by $Ce^{\sigma s}$. Indeed its forcing has ancient rate $\beta\sigma > \sigma$ and $(\alpha/\varepsilon_q)e^{\sigma t_0}$ is bounded. All bounds are uniform in $q$.

Expand in $\varepsilon_q$ and $\kappa$ exactly as in (22), replacing $\alpha$ there by $\varepsilon_q$. In the recursion add $-b_1 \cdot \nabla g_{j-1,\ell}$ for $j \ge 2$ and add $-b_1 \cdot \nabla W_\ell$ for $j = 1$. The initial first coefficient is the one above and the other initial coefficients vanish. The support and finite angular-sector properties are unchanged. Induction gives pure-seed bounds $Ce^{j\sigma s}$ and positive-viscosity bounds $C(1+t)^D e^{j\sigma s}$, uniformly in $t_0$: the first coefficient has just been estimated, and for $j \ge 2$ the external forcing is at most $Ce^{\sigma s} e^{(j-1)\sigma s}$, leaving the same nonresonant gap $j\sigma - \sigma$. Positive-viscosity resonances produce only the finite absolute-time polynomial. Consequently the residual estimate (26) holds with

$$a(t) = \varepsilon_q e^{\sigma(t - t_0)}, \qquad a(0) = q^{6 + K_*/\beta} \delta_c^{-1/\beta}, \qquad a(H_q) = \delta_q / |C_*|.$$

The residual contains no forcing-free term: the exact heat base is retained, and $b_1 \cdot \nabla$ of its heat Taylor remainder has the seed factor $\varepsilon_q$. Its bound is $C\kappa^{N+1}(1+t)^D a(t)$ by the displayed input estimate.

The prescribed drift is identical in the actual and approximate equations, so it disappears from the transport part of their Kato comparison. Choose $M$ using the fixed exponent $C_1$ as before, and choose $N$ with

$$\frac{3}{2}(N+1) > (6 + K_*/\beta) r_1.$$

The exact same $X$ error calculation yields $Ca^{M+1}$ plus a viscosity remainder $o(\delta_q)$ at the cap. It is uniform for $\kappa \le q^{3/2}$. The higher pure-seed terms contribute $O(\delta_q^2)$ and the positive-viscosity correctors contribute $O(\kappa(1 + H_q)^D \delta_q)$. The complementary first-coefficient response at $S_q$ is $\varepsilon_q e^{2\eta S_q} = o(\delta_q)$. Its growing response gives exactly the asserted protected-coordinate cap. These estimates also give the density bound.

For the whole moment, the integrated self-perturbation velocity is $O(\delta_q)$ as in (33). The additional prescribed drift has integral at most $Cq^6(1 + H_q)^5 = o(\delta_q)$, by its ancient and future bounds. The radial exponential Kato tail argument therefore applies unchanged. All formal coefficients are supported in the fixed collars, and the leading growing coefficient has moment $b_\epsilon$. The complementary, nonlinear and viscosity errors estimated above, followed by the Gaussian tail bound, prove the whole-moment cap. This completes the proof. □

For coupled use of Theorem 4.12, the actual Biot–Savart field must satisfy bounds for the harmonic remainder, core leakage and feedback. Over a logarithmic stage, a finite harmonic expansion must be carried to an order whose omitted remainder beats the comparison exponent. The exterior remainder in (56) records the spatial estimate; the time-dependent coupled estimates are additional requirements.

# 5 Actual cores, localization and the cold packet

## 5.1 Actual viscous outer cores and the local harmonic field

This section proves a conditional estimate for the actual coloured Navier–Stokes evolution. Its nonradial-density hypothesis is explicit; it is not deduced from material contours or from named unstable coordinates. We identify $x = (x_1, x_2)$ with $z = x_1 + ix_2$ and identify a velocity with $u_1 + iu_2$. Thus

$$K(z) = \frac{i}{2\pi\bar{z}}, \qquad Jz = iz.$$

Let $P$ denote circular averaging. For $f \in L^1 \cap L^\infty$, splitting the Biot–Savart integral at distance $d$ and optimizing $d$ gives

$$\|K * f\|_\infty \le C\sqrt{\|f\|_1 \, \|f\|_\infty}. \tag{62}$$

Indeed the near integral is at most $Cd\,\|f\|_\infty$ and the far integral is at most $C\,\|f\|_1 / d$.

**Lemma 5.1** (Fourfold growth at the origin)**.** *If $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is invariant under $z \mapsto iz$, then*

$$|K * f(z)| \le C\,\|f\|_\infty \, |z|.$$

*Neither zero circulation nor regularity of $f$ is required.*

*Proof.* For $r = |z| > 0$, the integral over $|y| \le 2r$ is at most $C\,\|f\|_\infty\, r$, by integrating $|z - y|^{-1}$ over $B_{3r}(z)$. For $|y| > 2r$ use the absolutely convergent expansion

$$\frac{1}{\bar{z} - \bar{y}} = -\sum_{n \ge 0} \frac{\bar{z}^n}{\bar{y}^{n+1}}.$$

The inverse moment vanishes unless $n + 1$ is a multiple of four, by changing variables under a quarter turn. The surviving degrees are $3, 7, 11, \ldots$. Their absolute sum is at most

$$Cr^3 \int_{|y|>2r} \frac{|f(y)|}{|y|^4}\,\mathrm{d}y \le C\,\|f\|_\infty\, r^3 \int_{2r}^\infty s^{-3}\,\mathrm{d}s \le C\,\|f\|_\infty\, r.$$

Termwise integration is justified by the integrable majorant $C|f(y)|/|y|$ on the far region. At $z = 0$ the velocity vanishes by rotational covariance and continuity. □

**Lemma 5.2** (Exit controlled only by radial drift)**.** *Let a diffusion satisfy*

$$\mathrm{d}X_s = v(s, X_s)\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}B_s, \qquad X_0 = x,$$

*and let $\sigma$ be its first exit from $B_b$, where $|x| < b$. Suppose, before exit,*

$$|v(s,y) \cdot y/|y|| \le V(s) \quad (0 < |y| < b), \qquad V \ge 0, \qquad D := \int_0^T V(s)\,\mathrm{d}s < b - |x|.$$

*Then*

$$\mathbb{P}(\sigma \le T) \le 2\exp\left(-\frac{(b-|x|-D)^2}{8\nu T}\right). \tag{63}$$

*There is no bound on the tangential component of $v$.*

*Proof.* For $\lambda > 0$ set $\phi(y) = \cosh(\lambda|y|)$. Its even power series shows that it is smooth at zero. In dimension two,

$$\Delta\phi = \lambda^2\cosh(\lambda r) + \frac{\lambda}{r}\sinh(\lambda r) \le 2\lambda^2\phi, \qquad v\cdot\nabla\phi \le \lambda V(s)\phi,$$

using $\sinh a \le a\cosh a$ for $a \ge 0$. Therefore Itô's formula shows that

$$e^{-2\nu\lambda^2(s\wedge\sigma)-\lambda\int_0^{s\wedge\sigma} V(\tau)\,\mathrm{d}\tau}\phi(X_{s\wedge\sigma})$$

is a nonnegative supermartingale. Stopping at $T$ gives

$$\mathbb{P}(\sigma \le T) \le \frac{\cosh(\lambda|x|)}{\cosh(\lambda b)}e^{2\nu\lambda^2 T+\lambda D} \le 2e^{-\lambda(b-|x|-D)+2\nu\lambda^2 T}.$$

Choose $\lambda = (b-|x|-D)/(4\nu T)$. The stopped calculation remains valid for bounded drifts smooth for positive times: stop first before their time endpoint and then pass to that endpoint. No approximation of a radial drift bound is used. □

Consider real, fourfold symmetric colours solving

$$\partial_t w_l + u^\nu\cdot\nabla w_l = \nu\Delta w_l, \qquad u^\nu = K * \sum_l w_l, \tag{64}$$

with geometric scales

$$A_l = A_1\beta^{l-1}, \qquad R_l = R_1 q^{l-1}, \qquad 1 < \beta_{\min} \le \beta \le \beta_{\max}, \qquad q \le \tfrac{1}{4}, \quad \beta q \le \tfrac{1}{2}.$$

Assume

$$\|w_l(0)\|_1 \le C_0 A_l R_l^2, \qquad \|w_l(0)\|_\infty \le C_0 A_l. \tag{65}$$

In the countable case assume that the initial series converges absolutely in $L^1\cap L^p$, $p > 2$, and use its mild Navier–Stokes solution and the associated linear coloured evolutions. The common velocity is bounded and smooth for positive times. Norm contraction identifies its vorticity with the sum of the colours, and (62) implies uniform convergence of the velocity series because $\sum_l A_l R_l < \infty$.

Fix an index $i$, a window $T = H/A_i$, and

$$\kappa_i = \frac{\nu}{A_i R_i^2}.$$

Suppose that for some fixed $a_0 > 0$ every $l \le i$ has an initial constant core,

$$w_l(0) = c_l \quad \text{on } B_{a_0 R_l}, \qquad |c_l| \le C_0 A_l. \tag{66}$$

Write $h_l = (I-P)w_l$ and impose the actual density hypothesis

$$\sup_{0\le t\le T}\|h_l(t)\|_1 \le \delta A_l R_l^2 \qquad (l \le i), \qquad 0 < \delta \le 1. \tag{67}$$

No nonradial hypothesis is imposed on $l > i$.

The scalar stochastic representation used below is

$$w_l(t,x) = \mathbb{E}[w_l(0, X_t)], \qquad \mathrm{d}X_s = -u^\nu(t-s, X_s)\,\mathrm{d}s + \sqrt{2\nu}\,\mathrm{d}B_s, \qquad X_0 = x. \tag{68}$$

Its sign is verified by applying Itô's formula to $w_l(t-s, X_s)$: the drift is $-\partial_t w_l - u^\nu\cdot\nabla w_l + \nu\Delta w_l = 0$. For completeness, first stop at $t-\varepsilon$, where the coefficients are smooth. Boundedness of the drift gives a continuous limiting process at $t$. Its transition densities away from starting time are bounded by $C/[\nu s]$. This bound follows by $L^1$ contraction, the Nash inequality $\|f\|_2^4 \le C\,\|f\|_1^2\,\|\nabla f\|_2^2$, the $L^2$ energy identity, and the corresponding adjoint estimate, composed at the midpoint. Thus $w_l(\varepsilon) \to w_l(0)$ in $L^1$ suffices to pass to the endpoint in expectation. Approximation of the bounded initial colour in $L^1$ by continuous functions also justifies evaluation at the limiting endpoint. This verifies (68) for the actual colours without smoothing an asserted radial bound.

**Proposition 5.3** (Uniform persistence of actual constant cores)**.** *Consider the finite or countable coloured evolution and geometric scales specified above, with $\nu, H > 0$. Assume (65), (66), and (67) on $[0,T]$, where $T = H/A_i$. There is $C \ge 1$, depending only on $C_0, a_0, \beta_{\min}, \beta_{\max}$, with the following property. Set*

$$\mathcal{D} = \sqrt{\delta} + \frac{\delta q^2}{\beta - q^2} + \frac{\beta q}{1-\beta q}, \qquad c_* := \frac{9}{512}, \qquad \varepsilon_* := \frac{a_0(\beta-1)}{16CH}.$$

*If*

$$CH\mathcal{D} \le \frac{a_0}{16}, \qquad 4C_0 e^{-c_* a_0^2/(\kappa_i H)} \le \frac{\varepsilon_*}{2}, \tag{69}$$

*then, with $a = a_0/2$, for $l \le i$ and $0 < t \le T$,*

$$\|w_l(t) - c_l\|_{L^\infty(B_{aR_l})} \le 4C_0 A_l e^{-c_* a_0^2 R_l^2/(\nu t)}. \tag{70}$$

*All constants are independent of $i$, viscosity, and the number of colours.*

*Proof.* The maximum principle gives $\|h_l\|_\infty \le 2C_0 A_l$. The radial average induces a purely tangential velocity, so (62) and (67) imply

$$|(K * w_l)\cdot e_r| \le \|K * h_l\|_\infty \le C\sqrt{\delta} A_l R_l \qquad (l \le i).$$

Every unrestricted inner colour has $\|K * w_k\|_\infty \le C A_k R_k$, whence

$$\sum_{k>l} \|K * w_k\|_\infty \le C A_l R_l \frac{\beta q}{1-\beta q}. \tag{71}$$

Temporarily assume that all probe-core defects satisfy

$$\|w_k - c_k\|_{L^\infty(B_{aR_k})} \le \varepsilon A_k.$$

For $k < l$ and $|z| \le a_0 R_l$, one has $|z| \le aR_k/2$. Decompose the older colour into

$$w_k = c_k \mathbf{1}_{B_{aR_k}} + (w_k - c_k)\mathbf{1}_{B_{aR_k}} + w_k \mathbf{1}_{\mathbb{R}^2\setminus B_{aR_k}}.$$

The first term gives exactly $(c_k/2)Jz$. The second gives a velocity bounded by $C\varepsilon A_k|z|$ by Lemma 5.1. The radial part of the third has zero velocity inside the disk. Its remaining density is $h_k \mathbf{1}_{\mathbb{R}^2\setminus B_{aR_k}}$, whose $L^1$ norm is at most $\delta A_k R_k^2$. Its exterior series starts at degree three, so

$$|K * (w_k \mathbf{1}_{\mathbb{R}^2\setminus B_{aR_k}})(z)| \le C\delta A_k |z|^3/R_k^2. \tag{72}$$

Here and below the density is evaluated at its actual viscous time. Summing the self, inner and older bounds gives

$$|u^\nu(t,z)\cdot e_r| \le CA_lR_l\left(\mathcal{D}+\frac{\varepsilon}{\beta-1}\right) \qquad (|z|\le a_0R_l, l\le i). \tag{73}$$

We used the geometric sums

$$\sum_{k<l}\frac{A_k}{R_k^2}\le\frac{A_l}{R_l^2}\frac{q^2}{\beta-q^2}, \qquad \sum_{k<l}A_k\le\frac{A_l}{\beta-1}.$$

Fix $C$ large enough for (73).

On a bootstrap interval with $\varepsilon\le\varepsilon_*$, the integrated radial drift in the backward representation is at most $a_0R_l/8$, because $A_lT\le H$ and (69) controls both terms. Start at $|x|\le a_0R_l/2$ and stop on exiting $B_{a_0R_l}$. If the backward path does not exit, its endpoint belongs to the initial constant core. The possible discrepancy outside it is at most $2C_0A_l$. Lemma 5.2 therefore gives

$$|w_l(t,x)-c_l|\le 4C_0A_l\exp\left(-\frac{(3a_0R_l/8)^2}{8\nu t}\right),$$

which is (70). Subtracting $c_l$ is legitimate: a constant is itself a bounded scalar solution, and bounded stochastic representation does not require its integrability.

The maximum normalized probe defect over the finite set $l\le i$ is continuous for positive time. It tends to zero at time zero, since the probe has a fixed margin inside the initial constant disk and the common drift is bounded. If it first attained $\varepsilon_*$, the preceding estimate, $R_l\ge R_i$, $t\le H/A_i$, and (69) would instead bound it by $\varepsilon_*/2$. This closes the bootstrap. The finite maximum adds no dependence on $i$; every infinite inner sum was already controlled by (71). □

**Proposition 5.4** (Harmonic exterior field and actual density leak)**.** *Under Proposition 5.3, set*

$$b_i=\sum_{l\le i}K*w_l, \quad \Omega_i=\frac12\sum_{l\le i}c_l, \quad H_i=\sum_{l\le i}K*(w_l\mathbf{1}_{\mathbb{R}^2\setminus B_{aR_l}}), \quad L_i=\sum_{l\le i}K*((w_l-c_l)\mathbf{1}_{B_{aR_l}}).$$

*For $|z|\le aR_i/2$ one has the exact decomposition*

$$b_i(t,z)=\Omega_iJz+H_i(t,z)+L_i(t,z). \tag{74}$$

*The field $H_i$ is divergence free and curl free on $B_{aR_i}$, and*

$$|H_i(t,z)|\le C\delta A_i|z|^3/R_i^2, \qquad |DH_i(t,z)|\le C\delta A_i|z|^2/R_i^2, \tag{75}$$

$$|L_i(t,z)|\le CA_ie^{-c_*a_0^2/(\kappa_iH)}|z|. \tag{76}$$

*The cubic coefficient of $H_i$ is exactly*

$$c_{3,i}(t)=-\frac{i}{2\pi}\sum_{l\le i}\int_{|y|\ge aR_l}\frac{w_l(t,y)}{\bar y^4}\,\mathrm{d}y. \tag{77}$$

*With $\eta_l(t)=\|h_l(t)\|_1/(A_lR_l^2)$,*

$$|c_{3,i}(t)|\le C\frac{A_i}{R_i^2}\sum_{m=0}^{i-1}(q^2/\beta)^m\eta_{i-m}(t)\le C\delta A_i/R_i^2. \tag{78}$$

*Furthermore,*

$$|H_i(t,z) - c_{3,i}(t)\bar{z}^3| \le C\delta A_i |z|^7/R_i^6, \tag{79}$$

$$|D(H_i(t,z) - c_{3,i}(t)\bar{z}^3)| \le C\delta A_i |z|^6/R_i^6. \tag{80}$$

*The derivative denotes the real spatial Jacobian. No derivative bound on $L_i$ is asserted.*

*Proof.* The constant disks induce $(c_l/2)Jz$ in the specified region, proving (74). Exterior densities induce curl free velocities away from their supports, and all Biot–Savart velocities are divergence free. Radial averaging commutes with radial truncation, and an exterior radial density induces zero velocity inside its truncation radius. Consequently, on this region,

$$H_i(t,z) = -\frac{i}{2\pi} \sum_{l\le i} \sum_{m\ge 0} \bar{z}^{4m+3} \int_{|y|\ge aR_l} \frac{h_l(t,y)}{\bar{y}^{4m+4}}\, \mathrm{d}y.$$

The inverse radial moment is zero, so this gives exactly (77). The coefficient integral is at most

$$\frac{\|h_l\|_1}{(aR_l)^4}.$$

This proves (78).

The series and its first spatial derivatives converge uniformly on $|z| \le aR_i/2$. Indeed each series ratio is at most $1/16$, and the derivative majorant adds only the factor $4m+3$. Thus, for each colour, its harmonic field and derivative are bounded by $C\delta A_l|z|^3/R_l^2$ and $C\delta A_l|z|^2/R_l^2$, respectively. Removing $m=0$ gives $C\delta A_l|z|^7/R_l^6$ and $C\delta A_l|z|^6/R_l^6$. Summing uses

$$\sum_{l\le i} \frac{A_l}{R_l^d} \le \frac{A_i/R_i^d}{1-q^d/\beta} \qquad (d=2,6).$$

These prove (75), (79), and (80). Finally Lemma 5.1, (70), and $\sum_{l\le i} A_l \le A_i\beta/(\beta-1)$ prove (76). □

**Scope for the localized tail problem.** If $R_* = \kappa_i^{1/4} R_i \le aR_i/2$, then throughout $B_{R_*}$,

$$|b_i - \Omega_i Jz| \le C\delta A_i |z|^3/R_i^2 + CA_i e^{-c_* a_0^2/(\kappa_i H)}|z|.$$

The integrated linear-leak coefficient is at most $CHe^{-c_* a_0^2/(\kappa_i H)}$. The actual density leak is therefore quantified; no material hole has been retained by assumption. The harmonic derivative estimate (75) is available for a coupled transfer or energy calculation, while the core leak has only the growth estimate (76). The parameter margins are compatible with $H \asymp \log(1/q)$ and $\kappa_i = q^{3/2}$ if $\sqrt{\delta}H$ is sufficiently small. This compatibility does not prove (67); obtaining that hypothesis and controlling the coupled inner aggregate over its whole receiver window remain separate nonlinear tasks. In particular no fixed-data nonunique vanishing-viscosity theorem is concluded here.

## 5.2 A quantitative parabolic localization estimate

The next estimate concerns an actual convection–diffusion equation, and gives a quantitative substitute for compact material support. Its local cubic drift hypothesis must still be verified for a coupled Navier–Stokes cascade.

**Lemma 5.5** (Localized tail exit)**.** *Let $f$ solve*

$$\partial_t f + U \cdot \nabla f = \nu \Delta f, \qquad U(t,x) = \Omega(t) x^{\perp} + V(t,x) + b(t,x), \qquad \operatorname{div} U = 0,$$

*on $0 \le t \le T$, with $\|f(0)\|_1 \le m$ and $\operatorname{supp} f(0) \subset B_r$. Suppose*

$$\begin{aligned} \|V(t)\|_\infty &\le C_0 m (\nu t)^{-1/2}, \\ |b(t,x)| &\le L|x| \quad \text{for all } x, \\ |b(t,x)| &\le C_3 |x|^3 + \ell(t)|x| \quad \text{for } |x| \le R_*. \end{aligned}$$

*Assume $\ell \ge 0$, $\Omega$ is integrable, and*

$$r \le R_*/4, \qquad C_3 R_*^2 T + \int_0^T \ell(t)\, dt \le \tfrac{1}{4}, \qquad C_0 \frac{m}{\nu} \sqrt{\nu T} \le R_*/8.$$

*There are absolute $c, C > 0$ such that*

$$\sup_{t \le T} \int_{|x| \ge R_*} |f(t,x)|\, dx \le C m \exp\left( -\frac{c R_*^2}{\nu T} \right).$$

*For each fixed integer $k \ge 0$ there are constants $c, C_k > 0$ such that*

$$\int_{|x| \ge R_*} |x|^k |f(T,x)|\, dx \le C_k m e^{C_k L T} R_*^k \exp\left( -\frac{c R_*^2}{\nu T} \right),$$

*provided additionally $\sqrt{\nu T} \le c R_*$. The conclusion holds for smooth drifts and for limits for which the parabolic solution and the corresponding diffusion representation are obtained by smooth approximation.*

*Proof.* Kato's inequality bounds $|f|$ by the positive scalar solution from $|f(0)|$. Normalize this initial mass and represent the positive solution by the diffusion

$$dX_t = U(t, X_t)\, dt + \sqrt{2\nu}\, dB_t.$$

A time-dependent rotation removes $\Omega(t) x^{\perp}$; the rotated Brownian motion remains standard and all norm assumptions are unchanged. Let $\sigma$ be the first exit from $B_{R_*}$. Before that exit, deterministic Gronwall applied to the integral stochastic equation gives

$$\sup_{s \le t \wedge \sigma} |X_s| \le e^{C_3 R_*^2 t + \int_0^t \ell} \left[ r + 2 C_0 \frac{m}{\nu} \sqrt{\nu t} + \sqrt{2\nu} \sup_{s \le t} |B_s| \right].$$

The stated margins imply that $\sigma \le T$ requires $\sup_{s \le T} |B_s| \ge c R_* / \sqrt{\nu}$. The coordinate reflection bound for planar Brownian motion yields

$$\mathbb{P}(\sigma \le T) \le C \exp\left( -\frac{c R_*^2}{\nu T} \right).$$

An endpoint outside $B_{R_*}$ requires this exit, proving the mass bound.

For the moment assertion, condition at $\sigma$. The remaining drift has linear growth bounded by $L|x|$ plus $\|V(t)\|_\infty$. Its integrated bounded contribution is at most $2C_0 (m/\nu) \sqrt{\nu T} \le R_*/4$. Gronwall and Brownian moments therefore give, uniformly in the exit time,

$$\mathbb{E}[|X_T|^k \mid \mathcal{F}_\sigma] \le C_k e^{C_k L T} R_*^k \quad \text{on } \{\sigma \le T\}.$$

Multiply by the indicator of this event and use its probability bound. Finally restore the initial mass, which is at most $m$. □

For a receiver with $\kappa_i = \nu/(A_i R_i^2)$ and stage duration $T = H/A_i$, the choice $R_* = \kappa_i^{1/4} R_i$ gives

$$\frac{R_*^2}{\nu T} = \frac{1}{\kappa_i^{1/2} H}, \qquad C_3 R_*^2 T \le C x_i \kappa_i^{1/2} H \quad \text{if } C_3 \le C A_i x_i / R_i^2.$$

A global bound $L \le C A_i$ costs only $e^{C_k H}$, independent of depth. The exponential exit factor dominates that cost as $\kappa_i \downarrow 0$ for logarithmic stage lengths. This parameter calculation does not verify the required local cubic hypothesis for the actual foreign field.

## 5.3 Nonlinear inner-tail suppression through a receiver window

The estimate in this section is conditional on quantitative information about the actual foreign velocity. It closes a longer-time nonlinear subproblem than the isolated-tail and first-diffusion-time estimates. It does not establish the required nonradial-density bound for the active and older outer colours.

Write $J(x_1, x_2) = (-x_2, x_1)$, $z = x_1 + i x_2$, and

$$K(x) = \frac{Jx}{2\pi |x|^2}, \qquad B_4(f) = \int_{\mathbb{R}^2} \bar{z}^4 f(z) \, \mathrm{d}z.$$

A scalar is fourfold symmetric if $f(Jx) = f(x)$; a vector field has the corresponding covariance if $b(t, Jx) = J b(t, x)$. Constants denoted by $C$ can change from line to line.

### 5.3.1 A Gaussian-energy estimate for a globally weak foreign field

**Proposition 5.6** (Weak foreign field through a finite window)**.** *There are universal constants $c, \lambda_0, C > 0$ with the following property. Let $p > 2$, and let $f_0 \in L^1 \cap L^p$ be real, fourfold symmetric, supported in $B_1$, and satisfy*

$$\int f_0 = 0, \qquad B_4(f_0) = 0, \qquad m = \|f_0\|_1 .$$

*Suppose $b$ is divergence free, has fourfold covariance, and, after removal of a time-dependent solid rotation, satisfies*

$$|b(s,x)| \le \ell(s)|x|, \qquad |b(s,x)| \le d|x|^3, \qquad (1+s)\ell(s) \le \lambda_0.$$

*Let $f$ solve*

$$\partial_s f + (K * f + b) \cdot \nabla f = \Delta f.$$

*For $S \ge 2$, assume the displayed hypotheses on $0 \le s \le S-1$ and*

$$m S^{1/25} \sqrt{1 + \log S} \le c. \tag{81}$$

*Then*

$$\sup_{0 \le s \le S-1} |B_4(f(s))| \le C \left[ m^2 S^{7/50} (1 + \log S) + m d S^{102/25} \right]. \tag{82}$$

*The constants are independent of a decomposition of the datum into smaller scales. The assertion holds for smooth drifts and for the parabolic limits obtained by approximation with the stated bounds.*

*Proof.* Scalar contraction and the divergence-free Nash estimate give

$$\|f(s)\|_1 \le m, \qquad \|f(s)\|_\infty \le Cm/s, \qquad \|K * f(s)\|_\infty \le Cm/\sqrt{s}.$$

The last estimate follows by splitting Biot–Savart at a radius and optimizing. Take $s_0 = 1/100$. The positive scalar solution from $|f_0|$ dominates $|f|$. Its diffusion representation, after removing solid rotation, obeys

$$|X_s| \le e^{\lambda_0 s}\left[1 + Cm\sqrt{s} + \sqrt{2}\sup_{a\le s}|B_a|\right] \quad (0 \le s \le s_0).$$

The Gaussian exponential moment of the Brownian supremum is finite at this fixed small time. With $\beta = 3/10$ it follows that

$$\int e^{\beta|x|^2/(1+s_0)}|f(s_0,x)|\,\mathrm{d}x \le Cm.$$

Multiplication by the Nash $L^\infty$ bound gives a weighted $L^2$ norm $O(m)$. Fixed polynomial absolute moments are $O(m)$ on $[0, s_0]$. The exact harmonic-moment equation and symmetrization of self-advection therefore yield

$$|B_4(f(s_0))| \le C(m^2 + md). \tag{83}$$

For clarity, the self contribution is bounded by $Cm\int|x|^2|f|$, because $|\bar z^3 - \bar y^3|/|z-y| \le C(|z|^2 + |y|^2)$; the foreign contribution is bounded by $d\int|x|^6|f|$. These bounds also prove the result directly if the final time is smaller than $s_0$.

For $s \ge s_0$ put

$$\tau = \log(1+s), \quad y = x/\sqrt{1+s}, \quad w(\tau,y) = (1+s)f(s,x), \quad E(\tau,y) = \sqrt{1+s}\,b(s,\sqrt{1+s}\,y).$$

Then

$$w_\tau = \mathcal{L}w - (K*w+E)\cdot\nabla w, \quad \mathcal{L} = \Delta + \tfrac{1}{2}y\cdot\nabla + 1,$$

with $|E| \le \lambda(\tau)|y|$, $\lambda \le \lambda_0$, and $|E| \le a(\tau)|y|^3$, where $a(\tau) = de^{2\tau}$. Let

$$F_\beta = \left\|e^{\beta|y|^2/2}w\right\|_2, \qquad D_\beta = \left\|e^{\beta|y|^2/2}\nabla w\right\|_2^2, \qquad \beta = 3/10.$$

Before using the full Gaussian weight, we establish membership in its energy space. Put $\tau_0 = \log(1+s_0)$ and, for $R \ge 1$, set

$$\Phi_R(t) = \frac{R^2 t}{R^2+t}, \qquad \theta_R(t) = \Phi_R'(t) = \frac{R^4}{(R^2+t)^2}, \qquad \rho_R(y) = e^{\beta\Phi_R(|y|^2)}, \qquad g_R = \rho_R^{1/2}w.$$

These weights are bounded and smooth, $0 \le \theta_R \le 1$, and $\rho_R \uparrow e^{\beta|y|^2}$. The short-time exponential moment and Nash bound proved above give $\int\rho_R w(\tau_0)^2 \le Cm^2$, uniformly in $R$. For each bounded weight, spatial cutoffs and integration by parts give the exact identity

$$\langle w, \mathcal{L}w\rangle_{\rho_R} = -\|\nabla g_R\|_2^2 + \tfrac{1}{2}\|g_R\|_2^2 - \beta\int|y|^2\theta_R(|y|^2)\big(\tfrac{1}{2} - \beta\theta_R(|y|^2)\big)g_R^2\,\mathrm{d}y.$$

Here the second derivatives of $\log\rho_R$ cancel after substituting $w = \rho_R^{-1/2}g_R$. With $c_\beta = \beta(1/2 - \beta) > 0$, the last term is at most $-c_\beta\int|y|^2\theta_R g_R^2$. The divergence-free advection contributes

$$\beta\int(K*w+E)\cdot y\,\theta_R g_R^2\,\mathrm{d}y.$$

For $\tau \geq \tau_0$, the Nash estimate and the similarity change give $\|K * w\|_\infty \leq C_{s_0} m$. The foreign contribution is at most $\beta\lambda_0 \int |y|^2 \theta_R g_R^2$, while Young's inequality gives

$$\beta \|K * w\|_\infty \int |y| \theta_R g_R^2 \leq \frac{c_\beta}{4} \int |y|^2 \theta_R g_R^2 + C_\beta \|K * w\|_\infty^2 \|g_R\|_2^2 .$$

Choose $\lambda_0$ so that $\beta\lambda_0 \leq c_\beta/4$. It follows that, with constants independent of $R$,

$$\tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau} \|g_R\|_2^2 \leq - \|\nabla g_R\|_2^2 - \frac{c_\beta}{2} \int |y|^2 \theta_R g_R^2 + C(1 + m^2) \|g_R\|_2^2 .$$

Gronwall gives a uniform bound on every fixed finite time interval. Monotone convergence and weak lower semicontinuity show that $F_\beta$ is finite there, and that $\nabla(e^{\beta|y|^2/2} w)$ and $|y| e^{\beta|y|^2/2} w$ are square integrable in space-time. Thus $D_\beta$ is locally integrable in time. The transformed equation has its time derivative in the dual of the space with norm $\|g\|_2 + \|\nabla g\|_2 + \||y| g\|_2$: the linear drift and resulting quadratic potential define bounded bilinear forms on that space. The resulting energy identity supplies the continuous weighted-$L^2$ representative. Cutoffs in the now integrable full weight justify the identities and energy inequalities below. This preliminary argument uses no oscillator gap for the bounded weights.

Writing $g = e^{\beta|y|^2/2} w$ gives the exact identities

$$\begin{aligned} \langle w, \mathcal{L} w \rangle_\beta &= \tfrac{1}{2} \|g\|_2^2 - \|\nabla g\|_2^2 - \beta(1/2 - \beta) \||y| g\|_2^2 , \\ D_\beta &= \|\nabla g\|_2^2 + \beta^2 \||y| g\|_2^2 + 2\beta \|g\|_2^2 . \end{aligned}$$

For $a_\beta = \sqrt{\beta(1/2 - \beta)}$, completing squares in $\sum_j \|\partial_j g + a_\beta y_j g\|_2^2$ proves

$$\mathcal{H}_\beta := \|\nabla g\|_2^2 + \beta(1/2 - \beta) \||y| g\|_2^2 \geq 2 a_\beta \|g\|_2^2 = \frac{\sqrt{6}}{5} \|g\|_2^2 .$$

Moreover $D_\beta \leq 3\mathcal{H}_\beta$: one can first bound it by $\frac{3}{2}\mathcal{H}_\beta + \frac{3}{5}\|g\|_2^2$ and then use the last inequality. Splitting $\mathcal{H}_\beta$ into $\frac{3}{50}\mathcal{H}_\beta + \frac{47}{50}\mathcal{H}_\beta$ yields

$$\langle w, \mathcal{L} w \rangle_\beta \leq -\tfrac{1}{50} D_\beta + \tfrac{1}{25} F_\beta^2 . \tag{84}$$

Here $\frac{1}{2} - \frac{47}{50}\frac{\sqrt{6}}{5} < \frac{1}{25}$.

The Hardy–Littlewood–Sobolev estimate and Gaussian weighted Hölder give

$$\|K * w\|_4 \leq C \|w\|_{4/3} \leq C_\beta F_\beta .$$

The two-dimensional Gagliardo–Nirenberg inequality gives

$$\|g\|_4 \leq C \|g\|_2^{1/2} \|\nabla g\|_2^{1/2} .$$

Since $\||y| g\|_2 + \|\nabla g\|_2 \leq C_\beta \sqrt{D_\beta}$, self-advection contributes at most

$$C F_\beta^{3/2} D_\beta^{3/4} \leq C_\beta F_\beta D_\beta$$

to the weighted energy. The foreign contribution is $\beta \int E \cdot y \, e^{\beta|y|^2} w^2$, bounded by $C_\beta \lambda D_\beta$. Thus, while $F_\beta$ is sufficiently small,

$$\tfrac{1}{2}(F_\beta^2)' \leq -\tfrac{1}{100} D_\beta + \tfrac{1}{25} F_\beta^2 .$$

The initial weighted norm and a continuity bootstrap using (81) prove

$$F_\beta(\tau) \le Cme^{\tau/25}, \qquad \int_{\log(1+s_0)}^{\log S} D_\beta(\tau)\,\mathrm{d}\tau \le Cm^2 S^{2/25}. \tag{85}$$

Let $P$ be angular averaging, $G = Pw$, and $h = (I-P)w$. Every ordinary moment of $h$ of degree at most three vanishes: angular averaging removes radial moments, and fourfold symmetry removes all other moments in these degrees. In

$$X = L^2(e^{|y|^2/4}\,\mathrm{d}y), \qquad H = \|h\|_X\,, \qquad D_h = \|\nabla h\|_X^2\,,$$

one consequently has

$$D_h \ge 3H^2. \tag{86}$$

Indeed $\mathcal{L}$ is self-adjoint in $X$, and its Hermite basis is $\partial^\alpha \mathcal{G}$, with $\mathcal{G} = (4\pi)^{-1}e^{-|y|^2/4}$ and eigenvalue $-|\alpha|/2$. Orthogonality to degrees at most three is equivalent to the stated ordinary-moment cancellations, because the corresponding dual Hermite polynomials span these polynomial degrees. Expansion in this basis gives $\langle h, \mathcal{L}h\rangle_X \le -2H^2$; since the left side is $-D_h + H^2$, (86) follows. This elementary Hermite expansion also follows from the tensor one-dimensional harmonic-oscillator factorization.

Angular averaging is orthogonal for all the radial weights used here, and commutes with $\mathcal{L}$. Writing $V_a = K * a$, the angular equation is

$$h_\tau = \mathcal{L}h - (I-P)\,\mathrm{div}[V_G h + V_h G + V_h h + EG + Eh].$$

The radial field $V_G$ is tangential, so its weighted energy contribution is zero. The $h$–$h$ and $E$–$h$ contributions are bounded by $CHD_h$ and $C\lambda D_h$. Put $D_G = \|\nabla G\|_X^2 \le D_\beta$. Weighted integration by parts, unweighted HLS, and GN give

$$|\langle h, -\mathrm{div}(V_h G)\rangle_X| \le CHF_\beta^{1/2} D_G^{1/4} D_h^{1/2} \le \eta D_h + C_\eta H^2 F_\beta D_G^{1/2}.$$

The strict Gaussian margin $\beta - 1/4 = 1/20$ absorbs cubic multiplication:

$$\left\| e^{|y|^2/8} EG \right\|_2 \le Ca(\tau)F_\beta.$$

Consequently the $E$–$G$ contribution is bounded by $\eta D_h + C_\eta a(\tau)^2 F_\beta^2$. Take $\eta$ and the bootstrap constants sufficiently small. Using (86) yields

$$(H^2)' \le -\tfrac{19}{5}H^2 + CF_\beta D_G^{1/2} H^2 + Ca(\tau)^2 F_\beta^2.$$

By (85) and Cauchy–Schwarz,

$$\int F_\beta D_G^{1/2}\,\mathrm{d}\tau \le Cm^2 S^{2/25}\sqrt{1+\log S} \le Cc^2.$$

Variation of constants therefore gives

$$H(\tau) \le Cm\left[e^{-19\tau/10} + de^{51\tau/25}\right]. \tag{87}$$

Factors depending on the fixed initial time are absorbed in $C$.

Finally $B_4(f(s)) = e^{2\tau} B_4(w(\tau))$. Harmonicity cancels diffusion, and differentiating $e^{2\tau}$ cancels the dilation term. Radial self-advection vanishes. HLS and Gaussian weighted Hölder show that

$$\left| \frac{\mathrm{d}}{\mathrm{d}\tau} B_4(f(e^\tau - 1)) \right| \le C e^{2\tau} [F_\beta H + a(\tau) F_\beta].$$

Substitution of (85) and (87), followed by integration, gives a first term $Cm^2 S^{1/25+1/10}(1+\log S)$ and remaining terms at most $CmdS^{102/25}$, using $m \le 1$. Together with (83), this proves the proposition.

The bounded-weight argument and the subsequent uniform estimates also justify passage from smooth data and drifts to the stated parabolic limits. No initial $L^\infty$ norm enters a constant. □

### 5.3.2 Comparison with the actual local foreign field

**Proposition 5.7** (Local harmonic field and a rough small leak)**.** *Let $\nu, r, T, R_* > 0$, $R_* \ge 4r$, and $p > 2$. Let $f_0 \in L^1 \cap L^p$ be real, fourfold symmetric, supported in $B_r$, and satisfy $\int f_0 = 0$ and $B_4(f_0) = 0$. Put $M = \|f_0\|_1$, and let $f$ solve on $[0,T]$*

$$f_t + (K * f + b) \cdot \nabla f = \nu \Delta f.$$

*After removal of a solid rotation, suppose $b$ is divergence free, has fourfold covariance, and admits, on $B_{2R_*}$, the decomposition*

$$\begin{aligned} b = b_{\mathrm{h}} + e, \qquad & |b(t,x)| \le L|x| \quad \textit{globally}, \\ |b_{\mathrm{h}}(t,x)| \le C_3|x|^3, \quad & |Db_{\mathrm{h}}(t,x)| \le C_3|x|^2, \\ |e(t,x)| \le \epsilon L|x|. & \end{aligned} \tag{88}$$

*where $b_{\mathrm{h}}$ is divergence free with fourfold covariance and $0 \le \epsilon \le 1$. All bounds hold on $0 \le t \le T$. Its streamfunction is defined on $B_{2R_*}$. Assume*

$$C_3 R_*^2 T + \epsilon L T \le \tfrac{1}{4}, \qquad C(M/\nu)\sqrt{\nu T} \le R_*/8, \qquad \sqrt{\nu T} \le c R_*. \tag{89}$$

*Take a fixed radial cutoff of that streamfunction, equal to one in $B_{R_*}$ and zero outside $B_{2R_*}$, and denote its velocity by $\widetilde{b}$. Let $g$ solve the same self-consistent equation as $f$, with foreign field $\widetilde{b}$ and the same initial datum. Set $\ell = CC_3R_*^2$. If $M/\nu$ and $\ell(r^2/\nu + T)$ are sufficiently small, then*

$$\begin{aligned} \sup_{t \le T} |B_4(f(t)) - B_4(g(t))| \le CMR_*^4 e^{CLT} \Big[ & \epsilon L (T + r\sqrt{T/\nu}) \\ & + (1 + LR_*\sqrt{T/\nu}) e^{-cR_*^2/(\nu T)} \Big]. \end{aligned} \tag{90}$$

*No derivative bound on the leak $e$ is required. The assertion holds for smooth drifts and for parabolic limits obtained by approximation with the stated bounds.*

*Proof.* Choose the streamfunction $\psi_{\mathrm{h}}$ zero at the origin. Integration along radial segments gives $|\psi_{\mathrm{h}}(x)| \le CC_3|x|^4$. A streamfunction cutoff gives

$$|\widetilde{b}| \le CC_3|x|^3, \qquad |\widetilde{b}| \le \ell|x|, \qquad \left\| D\widetilde{b} \right\|_\infty \le \ell.$$

It is divergence free, has fourfold covariance, and equals $b_{\mathrm{h}}$ in $B_{R_*}$. A second useful bound is $|\widetilde{b}(x)| \le CL|x|$: from $\epsilon \le 1$ and (88), $|b_{\mathrm{h}}| \le 2L|x|$ on $B_{2R_*}$, so $|\psi_{\mathrm{h}}| \le CL|x|^2$ there, and cutoff derivatives preserve this linear growth. Thus this bound does not require $C_3R_*^2 \le L$.

We first verify the velocity-gradient estimate needed for comparison. Normalize $r = \nu = 1$ and write $m = M/\nu$; the cutoff growth and Lipschitz constant are normalized in the same way. Nash gives

$$\|g(t)\|_\infty \le Cm/t, \qquad \|u_g(t)\|_\infty \le Cm/\sqrt{t}, \qquad u_g = K * g.$$

For $M_k(t) = \int |x|^k |g(t,x)| \, \mathrm{d}x$, Kato's inequality gives

$$\begin{aligned} M_2' &\le 4m + Cmt^{-1/2}\sqrt{mM_2} + 2\ell M_2, \\ M_4' &\le 16M_2 + Cmt^{-1/2} m^{1/4} M_4^{3/4} + 4\ell M_4. \end{aligned}$$

Since $M_2(0), M_4(0) \le m$, a continuity bootstrap gives

$$M_2(t) \le Cm(1+t), \qquad M_4(t) \le Cm(1+t)^2$$

when $m$ and $\ell(1+T)$ are small. On these trial bounds the integrated self terms are respectively $O(m^2(1+t))$ and $O(m^2(1+t)^2)$; $\ell T \ll 1$ absorbs the foreign terms. Hence

$$\|xg(t)\|_1 \le Cm\sqrt{1+t}, \qquad \|xg(t)\|_4 \le Cmt^{-3/4}\sqrt{1+t}.$$

The second follows from $\|xg\|_4^4 \le \|g\|_\infty^3 M_4$.

With $H_t$ the heat kernel, scaling and the smooth kernels' exterior decay give

$$\begin{aligned} \|\nabla K * H_t\|_\infty &\le Ct^{-1}, \\ \|\nabla K * \operatorname{div} H_t\|_\infty \le Ct^{-3/2}, \quad \|\nabla K * \operatorname{div} H_t\|_1 &\le Ct^{-1/2}, \quad \|\nabla K * \operatorname{div} H_t\|_{4/3} \le Ct^{-3/4}. \end{aligned}$$

For the nonlinear differentiated kernel the tail is $O(|x|^{-3})$, so the last three norms are finite before scaling. The self flux $u_g g$ has $L^1$ norm $Cm^2 t^{-1/2}$ and $L^\infty$ norm $Cm^2 t^{-3/2}$. Split the heat-mild time integral at $t/2$, using the kernel's $L^\infty$ norm first and its $L^1$ norm second; its contribution to $\nabla u_g$ is $Cm^2/t$. The foreign flux has $L^1$ norm $C\ell m\sqrt{1+t}$ and $L^4$ norm $C\ell m t^{-3/4}\sqrt{1+t}$. Use the kernel's $L^\infty$ norm first and its $L^{4/3}$ norm second. This contribution is

$$C\ell m t^{-1/2}\sqrt{1+t} \le Cm/t,$$

because $\ell(1+T)$ is small. The initial heat contribution is $Cm/t$. Undoing normalization proves

$$\|\nabla u_g(t)\|_\infty \le C(M/\nu)/t. \tag{91}$$

Put $\delta = f - g$, $v = K * \delta$, and $F = (b - \widetilde{b})f$. Then

$$\delta_t - \nu\Delta\delta + (u_f + \widetilde{b}) \cdot \nabla\delta + v \cdot \nabla g = -\operatorname{div} F.$$

Let $\psi$ be a streamfunction of $\delta$. Testing in $H^{-1}$ gives

$$\tfrac{1}{2}(\|v\|_2^2)' + \nu \|\delta\|_2^2 \le (\|\nabla u_g\|_\infty + \left\|D\widetilde{b}\right\|_\infty) \|v\|_2^2 + \|v\|_2 \|F\|_2 . \tag{92}$$

Here $v \cdot \nabla\psi = 0$. Thus the term $v \cdot \nabla g$ vanishes when paired with $\psi$, and the same identity replaces $u_f = u_g + v$ by $u_g$ in the other self transport term. Integration by parts bounds that term by $\|\nabla u_g\|_\infty \|v\|_2^2$. The identical calculation applies to $\widetilde{b}$. The circulation of $\delta$ is zero, and its polynomial moments are finite, so $v \in L^2$ and these tests are justified by approximation.

Both solutions converge to their common datum in $L^1 \cap L^p$ as $t \downarrow 0$. Interpolation, using $p > 2$, therefore gives $\|\delta(t)\|_2 \to 0$. Their first absolute moments are uniformly bounded near zero by the Kato moment estimates and the linear-growth bounds. Since $\int \delta = 0$,

$$|\widehat{\delta}(t,\xi)| \le |\xi| \int |x|\,|\delta(t,x)|\,\mathrm{d}x.$$

In the Fourier identity $\|v\|_2^2 = C\int |\xi|^{-2}|\widehat{\delta}(t,\xi)|^2\,\mathrm{d}\xi$, the part with $|\xi| < a$ is $O(a^2)$ uniformly near zero. The part with $|\xi| \ge a$ is at most $Ca^{-2}\,\|\delta(t)\|_2^2$. First taking $t \downarrow 0$ and then $a \downarrow 0$ proves $\|v(t)\|_2 \to 0$.

The true coefficient in (92) is integrable at time zero, although the upper bound (91) is singular there. Indeed $g_0 \in L^p$, $p > 2$, gives from the same mild formula

$$\|\nabla u_g(t)\|_\infty \le C_0(t^{-1/p} + t^{1/2-1/p}), \qquad 0 < t \le 1.$$

The constant may depend on the actual initial $L^p$ norm and on the bounded cutoff drift. The kernel norms are $Ct^{-1/p}$ and $Ct^{-1/2-1/p}$ in $L^{p'}$; the latter time singularity is integrable, and $(u_g + \widetilde{b})g$ has bounded $L^p$ norm near zero by contraction and the initial $L^1$–$L^p$ Biot–Savart bound. Together with the initial trace just proved, this starts the integrating factor in (92) from $v(0) = 0$. Only after writing its source integral do we use (91) on $[s,t]$, obtaining

$$\|v(t)\|_2 \le \int_0^t (t/s)^{CM/\nu} e^{\ell(t-s)}\,\|F(s)\|_2\;\mathrm{d}s. \tag{93}$$

Common smooth approximations have bounded initial $L^p$ norms, so no arbitrary initial term at a positive time is discarded in this argument.

The localization estimate, applied at each time $s \le T$, gives

$$\int_{|x|\ge R_*} |x|^2 |f(s,x)|\,\mathrm{d}x \le CMe^{CLs}R_*^2 e^{-cR_*^2/(\nu s)}.$$

The cutoff field and the actual field have global growth $CL|x|$. Combining this with Nash yields

$$\left\|(b-\widetilde{b})f\,\mathbf{1}_{|x|\ge R_*}\right\|_2 \le CLM\frac{R_*}{\sqrt{\nu s}}e^{CLs}e^{-cR_*^2/(\nu s)}.$$

Inside $B_{R_*}$ the difference of drifts is exactly $e$. A weighted Kato estimate gives $\int |x|^2|f(s)| \le CM(r^2+\nu s)e^{CLs}$. Consequently

$$\left\|ef\,\mathbf{1}_{|x|<R_*}\right\|_2 \le C\epsilon LM\sqrt{\frac{r^2+\nu s}{\nu s}}e^{CLs}.$$

Substitute these bounds into (93). Since $CM/\nu < 1/4$, integration of $s^{-1/2-CM/\nu}$ is legitimate, and yields, uniformly for $t \le T$,

$$\|v(t)\|_2 \le CMe^{CLT}\left[\epsilon L(T + r\sqrt{T/\nu}) + LR_*\sqrt{T/\nu}\,e^{-cR_*^2/(\nu T)}\right]. \tag{94}$$

The bounded factor $e^{\ell T}$ is absorbed into $C$.

For a radial cutoff $\chi_*$ equal to one in $B_{R_*}$ and zero outside $B_{2R_*}$,

$$\left\|\nabla(\chi_*\bar{z}^4)\right\|_2 \le CR_*^4, \qquad \left|\int \chi_*\bar{z}^4\delta\right| \le CR_*^4\,\|v\|_2\,.$$

Both $f$ and $g$ satisfy the localized exit estimate, with the margins (89). Their exterior absolute fourth moments are at most $CMR_*^4e^{CLT}e^{-cR_*^2/(\nu T)}$. Combining this with (94) proves (90). □

### 5.3.3 Physical normalization, the available power margin, and scope

Set

$$m = M/\nu, \qquad d = CC_3 r^4/\nu, \qquad S = 2 + \nu T/r^2.$$

To supply the full time interval required by Proposition 5.6, extend the cutoff comparison field $\widetilde{b}$ by zero on $(T, T + r^2/\nu]$ and continue its solution $g$ there. This extension preserves divergence, fourfold covariance, and the cubic and linear growth bounds. It is a bounded measurable-in-time, spatially Lipschitz drift and is covered by the parabolic approximation formulation. On the original interval the normalized weak-field coefficient is $\ell_{\mathrm{norm}} = (r^2/\nu)\ell$ and satisfies

$$(1+s)\ell_{\mathrm{norm}}(s) \le \ell(r^2/\nu + T),$$

and it is zero afterwards. Consequently the same smallness hypothesis supplies the normalized drift through $s = S - 1$; the comparison on $[0, T]$ is unchanged. The space-time change $x = ry$, $t = r^2 s/\nu$, and $f(t,x) = \nu r^{-2} F(s,y)$ gives these quantities exactly; moreover $B_4(f) = \nu r^4 B_4(F)$. Applying Proposition 5.6 to the cutoff-field solution and Proposition 5.7 to the actual solution gives

$$\begin{aligned}\sup_{t\le T} |B_4(f(t))| \le CMr^4 \left[\frac{M}{\nu} S^{7/50}(1+\log S) + \frac{C_3 r^4}{\nu} S^{102/25}\right] \\ + CMR_*^4 e^{CLT}\Big[\epsilon L(T + r\sqrt{T/\nu}) + (1 + LR_*\sqrt{T/\nu}) e^{-cR_*^2/(\nu T)}\Big]. \end{aligned} \tag{95}$$

This requires (81) and the small normalized linear-growth bound of the cutoff field in addition to the local hypotheses. No cutoff has been applied to the actual nonlinear solution.

For this geometric cascade budget fix $\beta$ and $C_H > 0$, assume $0 < \delta \le 1$ and $1 \le H \le C_H \log(1/q)$, and take a receiver $(A_i, R_i)$ with

$$r = qR_i, \quad M \le C\beta A_i q^2 R_i^2, \quad \kappa = \frac{\nu}{A_i R_i^2} = q^{3/2}, \quad T = H/A_i, \quad R_* = \kappa^{1/4} R_i.$$

Suppose the actual outer-colour estimates furnish

$$L \le CA_i, \qquad C_3 \le C\delta A_i/R_i^2, \qquad \epsilon \le Ce^{-c/(\kappa H)}.$$

Then $m = O(\beta q^{1/2})$, $d = O(\delta q^{5/2})$, and $S = O(q^{-1/2}H)$. Dividing the first line of (95) by $A_i R_i^6$ gives

$$O\Big(\beta^2 q^{643/100} H^{7/50}(1+\log S) + \beta\delta q^{323/50} H^{102/25}\Big). \tag{96}$$

The powers are 6.43 and 6.46. The localization terms are smaller than every $q$-power for logarithmic $H$, since

$$\frac{R_*^2}{\nu T} = \frac{1}{\sqrt{\kappa} H}, \qquad LT = O(H), \qquad C_3 R_*^2 T = O(\delta\sqrt{\kappa} H).$$

The remaining tail margins follow from

$$\begin{aligned} \ell(r^2/\nu + T) &= O\Big(\delta[q^{5/4} + q^{3/4}H]\Big), \\ \frac{(M/\nu)\sqrt{\nu T}}{R_*} &= O(\beta q^{7/8}\sqrt{H}), \qquad \frac{\sqrt{\nu T}}{R_*} = q^{3/8}\sqrt{H}, \\ mS^{1/25}\sqrt{1+\log S} &= O\Big(\beta q^{12/25} H^{1/25}\sqrt{1+\log S}\Big). \end{aligned}$$

Also $r/R_* = q^{5/8}$ and $\epsilon LT = O(He^{-c/(q^{3/2}H)})$. All these quantities tend to zero uniformly in the stated horizon range. Writing $L_q = \log(1/q)$, the two terms of (96) are bounded by

$$C\beta^2 q^{643/100} L_q^{57/50} + C\beta\delta q^{323/50} L_q^{102/25} = o(q^6),$$

uniformly for $0 < \delta \le 1$ and $1 \le H \le C_H L_q$. A fixed prefactor in the support radius changes only the constants.

The actual outer-core argument supplies the required harmonic-plus-leak hypothesis conditionally on the density bound

$$\sup_{t\le T} \|(I-P)w_l(t)\|_1 \le \delta A_l R_l^2, \qquad l \le i.$$

Its harmonic exterior multipole series can be differentiated to give $|Db_{\mathrm{h}}(x)| \le CC_3|x|^2$ throughout $B_{2R_*}$ when this disk lies strictly inside the constant-core probe region; the exponentially small core density defect contributes the rough leak. When these inputs are obtained from Proposition 5.3, its core margins (69) must also hold. Thus the present section closes full-window nonlinear inner-tail suppression under this explicit outer-density input. It does not prove that input for the active colour and older seeds, a signed outward transfer, or distinct vanishing-viscosity accumulation paths. The zero initial fourth moment used here is the one of the prescribed initial inner stack: termwise integration is justified by $\sum_l A_l R_l^6 < \infty$. A restart from a later state retains its actual fourth moment. If only $B_4(f_0) = 0$ is dropped while all other hypotheses of Proposition 5.6 remain in force, the same proof adds $|B_4(f_0)|$ to (82). A noncompact restarted state additionally requires a separate Gaussian-space entry estimate.

*Remark* 5.8 (Selected-cutoff scope). The displayed $q^{6.43}$ and $q^{6.46}$ budget uses $\kappa_i = q^{3/2}$ at the selected cutoff level. At later, larger receivers the same physical viscosity has a smaller normalized value; their aggregate of smaller colours includes the intended active child. The budget is not an erasure estimate for that entire aggregate at every later stage. After a source leaves its small cap neighbourhood, its constant-core probe on a canonical source window requires a separate comparison; it is not a consequence of the small-density hypothesis over that post-cap window.

# 6 The actual inactive prefix

## 6.1 Fixed seeds: separate initial and forced prefix contributions

The following scalar comparison separates the common initial seeds from the forced response. Its coupling inequality is a hypothesis.

**Lemma 6.1** (Two prefix norms). *Let $\beta \ge 4$, $0 < q < 1$, $S \ge 0$, $b \ge 0$, and $A_i = \beta^{i-1}$. Fix a receiving index $j \ge 2$. Suppose nonnegative continuous functions $x_i$, $1 \le i < j$, satisfy, on $0 \le \theta \le S$,*

$$x_i(\theta) \le Mbe^{\lambda_0\theta} + M\int_0^\theta e^{\lambda_0(\theta-s)} \left(\sum_{l<j} K_{il}x_l(s) + f_i(s)\right) \mathrm{d}s, \qquad \lambda_0 = \frac{\sigma+\eta}{\beta}, \tag{97}$$

*where $M, C, \sigma > 0$, $\eta, r_* > 0$, $f_i \ge 0$ are integrable, and the nonnegative matrix $K$ obeys*

$$K_{ii} \le Cr_*/\beta, \qquad K_{il} \le \begin{cases} C\beta^{l-j}q^{2(i-l)}, & l < i, \\ C\beta^{l-j}q^{6(l-i)}, & i < l < j. \end{cases}$$

*Assume $\beta q^6 < 1$. Define*

$$r_B = \frac{C}{\beta}\left(r_* + \frac{\beta q^6}{1-\beta q^6} + \frac{q^2/\beta}{1-q^2/\beta}\right), \qquad \lambda_B = \lambda_0 + M r_B.$$

*Then $x_i \le h_i + p_i$, where the initial contribution satisfies*

$$\max_{i<j} h_i(\theta) \le M b e^{\lambda_B \theta}. \tag{98}$$

*For $D > \beta$ with $Dq^8 < \beta$, put $w_i = (Dq^6)^{j-i}$ and*

$$r_P = \frac{C}{\beta}\left(r_* + \frac{\beta}{D-\beta} + \frac{Dq^8}{\beta - Dq^8}\right), \qquad \lambda_P = \lambda_0 + M r_P.$$

*The forced contribution satisfies*

$$P(\theta) := \max_{i<j} \frac{p_i(\theta)}{w_i} \le M \int_0^\theta e^{\lambda_P(\theta - s)} \max_{i<j} \frac{f_i(s)}{w_i}\, \mathrm{d}s. \tag{99}$$

*All constants are independent of $j$. One can choose $\eta, r_*$ small, then $D$ large and $q$ small, so both $\lambda_B$ and $\lambda_P$ are strictly smaller than $\sigma/2$.*

*Proof.* The row sums of $K$ are bounded by $r_B$: for $l = i + d$ the inner coefficient is at most $C\beta^{i-j}(\beta q^6)^d$, and for $l = i - d$ the outer coefficient is at most $C\beta^{i-j}(q^2/\beta)^d$. Use $\beta^{i-j} \le 1/\beta$ and sum the geometric series.

Solve the positive comparison system

$$Y' = (\lambda_0 I + MK)Y + Mf, \qquad Y(0) = Mb\mathbf{1}.$$

Iteration of (97), or its integral comparison principle, gives $x \le Y$. The variation-of-constants formula splits $Y = h + p$, with

$$h = e^{(\lambda_0 I + MK)\theta} Mb\mathbf{1}, \qquad p = M\int_0^\theta e^{(\lambda_0 I + MK)(\theta - s)} f(s)\, \mathrm{d}s.$$

The unweighted maximum norm gives (98). For the weighted maximum norm, the inner row acquires $(\beta/D)^d$ and the outer row acquires $(Dq^8/\beta)^d$. Their geometric sums give $r_P$ and hence (99). Since $\lambda_0$ is close to $\sigma/\beta \le \sigma/4$, first make the diagonal term small. Increasing $D$ makes the remaining $q$-independent term in $r_P$ small; subsequently decreasing $q$ makes all other terms small. □

**Corollary 6.2** (The fixed-seed baseline is small in its actual field)**.** *Assume the hypotheses of Lemma 6.1, with $\lambda_B \le \lambda_* < \sigma/2$, in the clock $\theta = A_j t$. Suppose $H \ge 0$ and the completed cap clocks satisfy*

$$T_{j,N} \le S/A_N + H\sum_{m=j}^{N-1} A_m^{-1}, \qquad N \ge j, \qquad S \ge H\beta/(\beta - 1).$$

*Let $0 \le \delta \le \delta_*$, where $\delta_* < \infty$ is fixed, and put $b = \delta e^{-\sigma S}$. Then through $T_{j,N}$ the initial contribution is bounded uniformly by*

$$B = M\delta e^{-(\sigma - \lambda_*)S}.$$

*Its normalized inward feedback on the receiver $j$ is at most*

$$C\sum_{i<j}\beta^{i-j}q^{2(j-i)}h_i \le CB\frac{q^2/\beta}{1-q^2/\beta}. \tag{100}$$

*The forced contribution has feedback at most*

$$CP\frac{Dq^8/\beta}{1-Dq^8/\beta}.$$

*In particular, for a fixed $h_0 \ge 0$ and $H = h_0\log(1/q)$ one can take $S = C_0\log(1/q)$ so that $B = O(q^6)$ and its feedback is $O(q^8)$, with constants independent of depth and uniform for $0 \le \delta \le \delta_*$.*

*Proof.* Writing $a = \beta^{j-N} \in (0,1]$ gives

$$A_jT_{j,N} \le Sa + \frac{H\beta}{\beta-1}(1-a) \le S.$$

Apply (98). The actual inward row in (100) is a geometric series with ratio $q^2/\beta$. For $p_i \le w_iP$ its ratio is $Dq^8/\beta$. Choose a fixed $C_0 \ge h_0\beta/(\beta-1)$ with $C_0(\sigma-\lambda_*) > 6$. Then $B \le M\delta_*q^{C_0(\sigma-\lambda_*)} = O(q^6)$, and its receiving-field row adds a factor $O(q^2)$. If $Dq^6 < 1$, the local chart bound follows from $x_i \le B + w_iP \le B + Dq^6P$ whenever the latter quantity is below $r_*$. □

For the coloured Navier–Stokes evolution, the spatial $q^2$ and $q^6$ rows and the forced term must be estimated separately; see Proposition 6.4 and Section 6.3.

*Remark* 6.3 (A generic smooth-profile self rate). The same positive-matrix proof works with $\lambda_0 = \gamma/\beta$ for any fixed $\gamma > 0$. Choose $\beta > 4\gamma/\sigma$, then choose the foreign-row parameters so $Mr_B$ and $Mr_P$ are less than $\sigma/4 - \gamma/\beta$. For a differential inequality whose diagonal rate is at most $\gamma\beta^{i-j}$, use $\gamma/\beta$ as its common upper bound. Thus a fixed-profile Kato rate can be used independently of the unstable eigenvalue. The transform selection in Lemma 4.10 permits choosing a good $\beta$ in any fixed sufficiently large interval, so this parameter change causes no conflict with the isolated source transform.

## 6.2 Actual inactive densities and a large amplitude ratio

Fix the smooth radial reference $W$ and $p > 2$. The inactive comparison uses the actual coloured Navier–Stokes equation without differentiating an inactive colour. Its fixed-profile transport-stability rate is made small in the receiver clock by choosing the amplitude ratio sufficiently large. The first two closure corollaries assume an explicit packet bound; Section 6.3 derives a bound from the actual packet.

Let $P$ denote angular averaging. Use

$$\begin{aligned} A_l &= A_1\beta^{l-1}, & R_l &= R_1q^{l-1},\\ G_l(t,x) &= A_l\left[e^{\nu tR_l^{-2}\Delta}W\right](x/R_l), & &\\ \zeta_l &= w_l - G_l, & h_l &= (I-P)w_l = (I-P)\zeta_l. \end{aligned}$$

For any density $v$ put

$$\|v\|_{X_l} = \frac{\|v\|_1}{A_lR_l^2} + \frac{\|v\|_p}{A_lR_l^{2/p}}, \qquad x_l(t) = \|\zeta_l(t)\|_{X_l}.$$

Assume the initial colours are real, fourfold symmetric, have zero circulation, are supported in $B_{c_0R_l}$, equal a constant in $B_{a_0R_l}$, and satisfy

$$\|w_l(0)\|_1 \le C_0 A_l R_l^2, \qquad \|w_l(0)\|_\infty \le C_0 A_l.$$

Here $a_0, c_0, C_0$ are fixed. For the smooth seeded family these facts follow from the fixed compact derivative collars. The series of colours is absolutely convergent in $L^1 \cap L^p$, and $\beta q < 1$, so its common velocity and coloured evolution are well defined as before.

Choose a fixed dimensionless annulus $\mathcal{A}$ and $d_* > 0$ such that

$$\operatorname{dist}(\operatorname{supp} \nabla W, \mathbb{R}^2 \setminus \mathcal{A}) \ge d_*.$$

Set $\mathcal{A}_l = R_l \mathcal{A}$. Choose $q$ afterwards so that these annuli lie inside the older constant-core probes and are separated from the small receiver truncation balls used below.

**Proposition 6.4** (Stopped inactive row with explicit parabolic errors)**.** *There are constants $\gamma, C, c > 0$, depending only on the fixed smooth profile, $p$, and the displayed initial prefactors, with the following property. Fix $j \ge 2$ and*

$$T \le S/A_j, \qquad \kappa_j = \frac{\nu}{A_j R_j^2} \le q^{3/2}, \qquad 1 \le S \le C_1 \log(1/q).$$

*Stop at the first time any $i < j$ reaches $x_i = \delta$. Assume $\beta \ge 4$, $0 < q < 1$, $\beta q \le 1/2$, $0 < \delta \le 1$, $x_i(0) < \delta$ for $i < j$, $q$ sufficiently small, and*

$$CS(\sqrt{\delta} + \beta q) \le c. \tag{101}$$

*In the clock $\theta = A_j t$, every inactive $i < j$ obeys*

$$\frac{\mathrm{d}^+ x_i}{\mathrm{d}\theta} \le \gamma \beta^{i-j} x_i + C \sum_{l<i} \beta^{l-j} q^{2(i-l)} x_l + C \sum_{i<l<j} \beta^{l-j} q^{6(l-i)} x_l + f_i(\theta) + E_{j-i}, \tag{102}$$

*where*

$$f_i(\theta) = \frac{1}{A_j} \left\| \mathbf{1}_{\mathcal{A}_i} \left( \sum_{l \ge j} K * h_l \right) \cdot \nabla G_i \right\|_{X_i} \qquad \text{at } t = \theta/A_j$$

*is the actual current/deep nonradial forcing. With*

$$e_m = C e^{CS} \exp\left[ -\frac{c q^{-2m}}{\kappa_j S} \right], \qquad m \ge 1,$$

*one can take*

$$E_k = C\Big[ \beta^{-k} e_k + \sum_{r>k} \beta^{-r} e_r + \sum_{1 \le r < k} \beta^{-r} q^{6(k-r)} e_r + e_k \sum_{1 \le r < k} \beta^{-r} q^{k-r} \Big]. \tag{103}$$

*The sums may be enlarged to infinite sums of nonnegative terms. No small nonradial-density assumption is imposed on colours $l \ge j$.*

*Proof.* The exact radial heat profile is an isolated Navier–Stokes solution. For the actual common velocity $u$, subtraction gives

$$(\partial_t + u\cdot\nabla - \nu\Delta)\zeta_i = -\left[K * \zeta_i + \sum_{l\neq i} K * w_l\right]\cdot\nabla G_i. \tag{104}$$

Every radial foreign velocity is tangential and annihilates $\nabla G_i$, so the foreign fields on the right can be replaced by $K * h_l$. In particular diffusion produces no separate angular source. If every initial colour equals its radial reference $G_l(0)$, then $\zeta_i = h_i = 0$ solves the comparison system with the prescribed initial data. For arbitrary radial initial colours the common velocity remains tangential, and

$$h_i(t) = 0, \qquad \zeta_i(t) = e^{\nu t\Delta}(w_i(0) - G_i(0)).$$

This difference is radial and need not vanish. In the all-radial zero-seed member of the reference family the colours equal their references.

The scaled Biot–Savart split gives $\|K * \zeta_i\|_\infty \le C_p A_i R_i x_i$. Heat contraction of the fixed reference gradient in $L^1$ and $L^p$, and the scalar Kato inequalities, bound the self term by $\gamma A_i x_i$. For example $\gamma$ can be chosen as a constant multiple of $C_p(\|\nabla W\|_1 + \|\nabla W\|_p)$. No derivative of $\zeta_i$ enters this estimate; the actual nonlinear transport term disappears from the norm inequalities.

We verify the localized foreign-field estimates under the stop. For any fixed index $l$, the global radial component obeys

$$|u(t,x)\cdot e_r| \le CA_l|x| + CA_lR_l\beta q. \tag{105}$$

Indeed every radial average has tangential velocity. For $k \le l$, the fourfold origin-growth bound gives $|K * h_k(x)| \le C\|h_k\|_\infty |x| \le CA_k|x|$, and $\sum_{k\le l} A_k \le CA_l$. For $k > l$ use $\|K * w_k\|_\infty \le CA_kR_k$ and $\sum_{k>l} A_kR_k \le CA_lR_l\beta q$. The same estimate holds for only the foreign radial field. The constants in these sums are uniform for $\beta \ge 4$ and $\beta q \le 1/2$.

Apply the actual outer-core bootstrap to the stopped finite prefix $l \le j-1$. Its nonradial-density hypothesis follows from $\|h_l\|_1 \le 2A_lR_l^2x_l \le 2\delta A_lR_l^2$. All colours $k \ge j$ enter that bootstrap through the unrestricted, convergent velocity sum $\sum_{k>l} A_kR_k$. The parameter margin (101) and small $q$ therefore give the individual core estimate

$$\|w_l(t) - c_l\|_{L^\infty(B_{aR_l})} \le CA_l e^{-cR_l^2/(\nu t)}, \qquad a = a_0/2. \tag{106}$$

The bootstrap error threshold is of order $S^{-1}$; its improvement follows from $\kappa_j S \ll 1$ just as in the outer-core proof. Retaining the individual radius in that proof, rather than its maximum over $l$, gives (106).

Fix $c_2 > c_0$. On $B_{c_2R_l}$ the own nonradial velocity is bounded by $C\sqrt{\delta}A_lR_l$. Older constant cores contribute only rotation; their harmonic remainders contribute $C\delta A_lR_lq^2/\beta$ and their core leaks are exponentially small. All inner velocities contribute at most $CA_lR_l\beta q$. Choose $q$ so that this ball is strictly inside every older core probe disk. By (101), the integrated radial drift is less than a fixed fraction of $(c_2-c_0)R_l$. The radial exit estimate for the positive scalar evolution from $|w_l(0)|$ gives

$$\int_{|x|>c_2R_l} |w_l(t,x)|\,\mathrm{d}x \le CA_lR_l^2 e^{-cR_l^2/(\nu t)}. \tag{107}$$

For its post-exit fourth moment use $\phi(x) = (R_l^2 + |x|^2)^2$. Equation (105) gives $u\cdot\nabla\phi \le CA_l\phi$, while $\nu\Delta\phi \le C\nu R_l^{-2}\phi$. Conditioning at exit from $B_{c_2R_l}$ consequently gives $\mathbb{E}[\phi(X_t) \mid \mathcal{F}_{\text{exit}}] \le$

$CR_l^4 e^{C(A_l+\nu/R_l^2)T} \le CR_l^4 e^{CS}$. Multiplication by the exit probability proves

$$\int_{|x|>c_2R_l} |x|^4 |w_l(t,x)| \, \mathrm{d}x \le CA_l R_l^6 e^{CS} e^{-cR_l^2/(\nu t)}. \tag{108}$$

Angular averaging preserves radial integrals. Inside $B_{c_2R_l}$ use $\|h_l\|_1 \le 2A_l R_l^2 x_l$; outside use (108). It follows that

$$\int |x|^4 |h_l(t,x)| \, \mathrm{d}x \le CA_l R_l^6 [x_l(t) + e_{j-l}]. \tag{109}$$

For an inner colour $i < l < j$, take a fixed small $\eta > 0$ such that $B_{\eta R_i}$ is inside the initial constant disk of colour $i$ and $c_0 q < \eta/4$. In this ball the radial velocity estimate based on index $i$ has the same small integrated bound: the own nonradial term is $C\sqrt{\delta} A_i R_i$, older remainders are controlled by (106), and all inner colours enter by $CA_i R_i \beta q$. Starting the $l$th positive colour in $B_{c_0R_l}$, radial exit from $B_{\eta R_i}$ therefore gives

$$\int_{|x|>\eta R_i} |w_l(t,x)| \, \mathrm{d}x \le CA_l R_l^2 e^{-cR_i^2/(\nu t)}. \tag{110}$$

This receiver-radius estimate is essential.

On $\mathcal{A}_i$, for $l < i$, split $h_l$ into its core and exterior. Equation (106) bounds the core density by $CA_l e^{-cR_l^2/(\nu t)}$; the origin-growth estimate bounds its velocity by $CA_l|x|e^{-cR_l^2/(\nu t)}$. The exterior part has mass at most $2A_l R_l^2 x_l$ and a fourfold inverse-moment expansion starting at degree three. Thus

$$\frac{|K * h_l(x) \cdot e_r|}{R_i} \le CA_l [q^{2(i-l)} x_l + e_{j-l}]. \tag{111}$$

For $i < l < j$, truncate $h_l$ at radius $\eta R_i$. Every radial truncation has zero circulation because $Ph_l = 0$, and fourfold symmetry cancels harmonic moments of degrees one to three. Its exterior field is bounded by $CR_i^{-5} \int |y|^4 |h_l(y)| \, \mathrm{d}y$. For the remaining density use (110), $\|h_l\|_\infty \le CA_l$, and Biot–Savart interpolation. This gives

$$\frac{|K * h_l(x) \cdot e_r|}{R_i} \le CA_l \left[ q^{6(l-i)} (x_l + e_{j-l}) + q^{l-i} e_{j-i} \right]. \tag{112}$$

The fixed exponent $c$ in $e_m$ has been reduced to absorb square roots and polynomial factors.

Away from $\mathcal{A}_i$, $\nabla G_i$ is the Gaussian convolution of the compactly supported reference gradient. The fixed margin $d_* R_i$ gives, with $\tau_i = \nu t/R_i^2$, scaled $L^1$ and $L^p$ tails after multiplication by $|x| + R_i$ bounded by $Ce^{-cd_*^2/\tau_i}$. The Gaussian kernel has this decay whenever its argument is at least $d_* R_i$; integration and fixed polynomial weights are absorbed by decreasing $c$. The foreign version of (105) consequently bounds the total omitted forcing by $CA_i e_{j-i}$. Combining this with the self estimate, (111), (112), and the explicitly retained $f_i$ proves (102)–(103). □

**Corollary 6.5** (Inactive density closure under packet history)**.** *Let $\sigma > 0$ be the growth rate of the fixed smooth reference. Choose $\beta$ sufficiently large that $\gamma/\beta < \sigma/4$, then a fixed $D > \beta$ sufficiently large, then $q$ sufficiently small. Set $w_i = (Dq^6)^{j-i}$ and $b = \delta e^{-\sigma S}$. Assume $x_i(0) \le Mb$, and the packet-history bound*

$$\sup_{\theta \le A_j T} \int_0^\theta e^{\lambda_P(\theta - s)} \max_{i<j} \frac{f_i(s)}{w_i} \, \mathrm{d}s \le C_* \delta, \tag{113}$$

*where $\lambda_P < \sigma/2$ is the weighted comparison rate below. Take $S = C_2 \log(1/q)$, with $C_2$ sufficiently large and fixed. Fix a finite $L_\delta > 0$ and choose the cap so that*

$$q^{L_\delta} \le \delta(q) \le 1, \qquad CS(\sqrt{\delta(q)} + \beta q) \le c. \tag{114}$$

*All constants and exponents are fixed before $q$ is decreased. Then the stopped solution remains strictly below its stopping threshold:*

$$x_i(t) \le B + w_i\left[CC_*\delta + o(q^K)\right] < \delta/4, \qquad B \le M\delta e^{-(\sigma-\lambda_B)S},$$

*for every $i<j$ and $t \le T$. Here $K$ is any fixed prescribed power, the error in the forced weighted norm is also $o(\delta q^6)$, and the constants are independent of depth. Alternatively, the lower bound in (114) may be replaced by the direct relative-error condition*

$$\eta(q)Se^{\lambda_P S} = o(\delta(q)q^6), \qquad \eta(q) = \sup_{m\ge 1}\frac{e_m}{(\beta Dq^6)^m}. \tag{115}$$

*The conclusion is conditional on (113).*

*Proof.* Bound each self coefficient in (102) by $\lambda_0 = \gamma/\beta$. This replaces, rather than identifies with, the earlier sharp spectral rate $(\sigma+\varepsilon)/\beta$. The unweighted off-diagonal row sum is at most

$$r_B = \frac{C}{\beta}\left[\frac{\beta q^6}{1-\beta q^6} + \frac{q^2/\beta}{1-q^2/\beta}\right].$$

The row sum in the forced weights $w_i$ is at most

$$r_P = \frac{C}{\beta}\left[\frac{\beta}{D-\beta} + \frac{Dq^8}{\beta - Dq^8}\right].$$

Thus $\lambda_B = \lambda_0 + r_B$ and $\lambda_P = \lambda_0 + r_P$ are both below $\sigma/2$ in the prescribed parameter order. The positive comparison equation splits exactly into initial, packet, and parabolic-error contributions. Its initial part is bounded by $Mbe^{\lambda_B\theta}$, giving the stated $B$.

For the explicit error put

$$\eta(q) = \sup_{m\ge 1}\frac{e_m}{(\beta Dq^6)^m}.$$

This is smaller than every power of $q$, uniformly when $\kappa_j \le q^{3/2}$ and $S = O(\log(1/q))$. Indeed successive ratios in this supremum are less than $1/2$ for small $q$, and its first term has Gaussian exponent of order $q^{-7/2}/S$. A direct summation of the four terms of (103) gives

$$\sup_{k\ge 1}\frac{E_k}{(Dq^6)^k} \le C\eta(q)\left[1 + \frac{Dq^6}{1-Dq^6} + \frac{1}{D-1} + \frac{\beta q}{1-\beta q}\right].$$

For example the third sum becomes $D^{-k}\sum_{r<k}\beta^{-r}q^{-6r}e_r \le \eta(q)D^{-k}\sum_{r<k}D^r$. The fourth uses $\sum_{r<k}\beta^{-r}q^{k-r} \le \beta^{-k}\beta q/(1-\beta q)$. Its weighted convolution is therefore at most

$$H_G(q) := C\eta(q)Se^{\lambda_P S} = o(q^L) \quad \text{for every fixed } L>0.$$

Choose $L > L_\delta + 6$. Then (114) gives

$$\frac{H_G(q)}{\delta(q)q^6} \le \frac{o(q^L)}{q^{L_\delta+6}} \longrightarrow 0.$$

Thus the error is $o(\delta q^6)$ in the forced weighted norm. Under the alternative (115), this relative bound holds directly.

The positive-matrix variation of constants and (113) yield the asserted bound. Choose $C_2$ so $C_2(\sigma-\lambda_B) > 6$; then $B \ll \delta q^6$. Finally $\max_{i<j} w_i = Dq^6$, so small $q$ makes the complete right side less than $\delta/4$, contradicting a first stopping time. The estimate bounds the comparison error; radial data have no angular forcing by (104). □

**Corollary 6.6** (A static packet bound suffices for inactive density)**.** *For the inactive-density conclusion alone, the packet-history hypothesis* (113) *may be replaced by*

$$\sup_{\theta\le A_jT}\max_{i<j}\frac{f_i(\theta)}{w_i}\le C_*\delta.$$

*Fix $C_2>6/\sigma$. Choose $\beta,D$, and then $q$, so that*

$$C_2(\sigma-\lambda_B)>6,\qquad C_2\lambda_P<1.$$

*With $S=C_2\log(1/q)$ and the cap condition* (114)*, or the alternative relative condition* (115) *together with the radial margin, the actual inactive densities remain below $\delta/4$. Their normalized inward nonradial velocity on the current receiver's fixed collars is*

$$O\Big(\delta Sq^{8-C_2\lambda_P}\Big)+o(\delta q^6)=o(\delta q^6).$$

*The static packet bound is an explicit input. The packet estimate in Section 6.3 closes the inactive stop directly, without requiring this static input to be $O(\delta)$.*

*Proof.* The stated parameter choices are possible since $\lambda_B,\lambda_P$ can be made arbitrarily small by taking the fixed $\beta,D$ large and then $q$ small. The packet contribution to the forced weighted norm is at most $CC_*\delta Se^{\lambda_P S}$. Thus

$$x_i\le B+(Dq^6)^{j-i}\left[CC_*\delta Sq^{-C_2\lambda_P}+o(q^K)\right].$$

The baseline $B=o(\delta q^6)$ and the nearest forced bound $C\delta Sq^{6-C_2\lambda_P}$ are both below the stopping threshold. The parabolic errors are $o(\delta q^6)$ in the forced weighted norm by the preceding relative-error calculation, so they are below the stopping threshold as well. For the inward field use the actual older-core multipole estimate:

$$\sum_{i<j}\beta^{i-j}q^{2(j-i)}x_i\le B\frac{q^2}{\beta-q^2}+CC_*\delta Sq^{-C_2\lambda_P}\frac{Dq^8}{\beta-Dq^8}+o(q^K).$$

The error field is at most $O(q^8)H_G=o(\delta q^6)$, and $C_2\lambda_P<1$ makes the principal packet term $o(\delta q^6)$. □

### 6.3 An actual packet bound closes the inactive density stop

All colours in this section are the actual scalar colours of the common unforced Navier–Stokes velocity. Use the notation and fixed profile hypotheses of Proposition 6.4. In addition choose $q$ so small that $\beta q^{2/p}\le 1/2$.

Let $\mathcal{S}(t,s)$ be the positive scalar transport–diffusion propagator for that common velocity and put

$$P_j(t)=\mathcal{S}(t,0)\Big(\sum_{l\ge j}|w_l(0)|\Big),\qquad v_j=\sum_{l\ge j}w_l.$$

The symbol $P_j$ here denotes a positive envelope, not angular averaging. The initial sum is in $L^1\cap L^p$ and is supported in $B_{c_0R_j}$. Positivity and monotone convergence imply

$$\sum_{l\ge j}|w_l(t)|\le P_j(t),\qquad \|P_j\|_1\le CA_jR_j^2,\qquad \|P_j\|_p\le CA_jR_j^{2/p}.$$

The positive aggregate is used only in $L^1\cap L^p$. The actual packet velocity is controlled by the individual colour contractions.

**Lemma 6.7** (Actual packet moments)**.** *For every fixed integer $n \geq 2$, with $\theta = A_j t$ and $\kappa_j = \nu/(A_j R_j^2)$, one has*

$$\int (R_j^2 + |x|^2)^{n/2} P_j(t,x)\,\mathrm{d}x \leq C_n A_j R_j^{n+2}(1 + \theta + \sqrt{\kappa_j \theta})^n e^{C_n \theta/\beta}. \tag{116}$$

*Constants are independent of $j$, depth, viscosity and all seed phases. The moment bound holds without the inactive stop.*

*Proof.* Write $u = u_{<j} + u_{\geq j}$. Individual colour contractions and Biot–Savart interpolation give

$$\|u_{\geq j}\|_\infty \leq C \sum_{l \geq j} A_l R_l \leq C A_j R_j.$$

The fourfold origin-growth inequality applied to each older colour yields $|u_{<j}(x)| \leq C A_j |x|/\beta$ for $\beta \geq 4$. The constants in these two sums are uniform for $\beta \geq 4$ and $\beta q \leq 1/2$. Thus the actual radial drift is bounded by $L|x| + V$, with $L = CA_j/\beta$ and $V = CA_j R_j$. Tangential rotation does not affect radial weights.

The positive scalar propagator preserves mass. Let $M = \int P_j(0) = \int P_j(t)$ and $Q_n = \int (R_j^2 + |x|^2)^{n/2} P_j$. Differentiation of the weight, positivity, and Hölder with respect to $P_j\,\mathrm{d}x$ give

$$Q_n' \leq C_n L Q_n + C_n V M^{1/n} Q_n^{(n-1)/n} + C_n \nu M^{2/n} Q_n^{(n-2)/n}.$$

If $M = 0$ there is nothing to prove. Otherwise $Y = (Q_n/M)^{1/n}$ satisfies $Y \geq R_j$ and

$$Y' \leq C_n L Y + C_n V + C_n \nu / Y.$$

A supersolution is $e^{C_n L t}(C_n R_j + C_n V t + C_n \sqrt{\nu t})$, after enlarging $C_n$: the derivative of the square-root term controls $C_n \nu / Y$ for $t > 0$. The initial support bounds $Y(0)$ by $C_n R_j$, proving the claim. To justify the moment calculation without assuming moments in advance, start with a finite truncation of the positive initial aggregate. The common velocity is bounded on the physical window, so its positive diffusion representation gives finite moments. Apply the displayed uniform calculation and pass monotonically to the full positive datum. Equivalently one may use truncated radial weights. The constants in the refined estimate use only $L, V$ above, not the crude global drift bound. □

**Proposition 6.8** (Actual force on a stopped inactive prefix)**.** *Under the stopped hypotheses and parameter margins of Proposition 6.4, set*

$$\mathcal{M}(S) = (1 + S + \sqrt{\kappa_j S})^4 e^{C_4 S/\beta}.$$

*For every $i < j$, $k = j - i$, and $0 \leq \theta \leq A_j T \leq S$, the actual packet force in that proposition obeys*

$$f_i(\theta) \leq C q^{6k} \mathcal{M}(S) + C q^k \exp\left[-\frac{c_p q^{-2k}}{\kappa_j S}\right]. \tag{117}$$

*In particular, for the weights $w_i = (D q^6)^{j-i}$ with fixed $D > 1$,*

$$\sup_{\theta \leq A_j T} \max_{i<j} \frac{f_i(\theta)}{w_i} \leq C\mathcal{M}(S) + \epsilon(q),$$

*where $\epsilon(q)$ is smaller than every fixed power of $q$, uniformly for $\kappa_j \leq q^{3/2}$ and $S = O(\log(1/q))$.*

*Proof.* Let $h_{\ge j} = (I-P)v_j$, where $P$ again denotes angular averaging. Its fourth absolute moment is bounded by twice that of $P_j$. Choose a fixed $\eta > 0$ so that $B_{\eta R_i}$ is inside the constant core probe of colour $i$, and is separated from $\mathcal{A}_i$. On this ball the own radial field is bounded by $C\sqrt{\delta}A_iR_i$, the older fields by their actual core/harmonic bounds, and all inner fields by $CA_iR_i\beta q$. The stop and its radial margin therefore give a displacement strictly smaller than $\eta R_i/2$ up to $T$. Every positive packet constituent starts in $B_{c_0R_j} \subset B_{\eta R_i/4}$. The radial exit estimate consequently gives

$$\int_{|x|>\eta R_i} P_j(t,x)\,\mathrm{d}x \le CA_jR_j^2 e^{-cR_i^2/(\nu t)}.$$

This estimate uses the unrestricted convergent inner velocity sum, not a small-density assumption on the packet.

Radial truncation of $h_{\ge j}$ has zero circulation and vanishing harmonic moments of degrees one to three. Its field on $\mathcal{A}_i$ is therefore bounded by $CR_i^{-5}\int |x|^4|h_{\ge j}|$. For the exterior leakage use the last exit estimate and the $L^p$ contraction of $P_j$. The elementary kernel split gives, for $p > 2$,

$$\|K * f\|_\infty \le C_p \|f\|_1^{\frac{p-2}{2(p-1)}} \|f\|_p^{\frac{p}{2(p-1)}}.$$

Indeed the near integral is at most $C_p d^{1-2/p}\|f\|_p$, and the far integral at most $C\|f\|_1/d$; optimize $d$. Angular averaging preserves radial tail masses and contracts $L^p$. Thus the leaked nonradial packet has velocity at most $CA_jR_je^{-c_pR_i^2/(\nu t)}$. Combining this with the fourth moment and the uniform scaled heat-gradient bound on $G_i$ proves (117): the force normalization is velocity divided by $A_jR_i$.

After division by $w_i$, the first term is at most $CD^{-k}\mathcal{M}(S)$. The second is at most

$$CD^{-k}q^{-5k}e^{-c_pq^{-2k}/(\kappa_jS)}.$$

For small $q$ successive such terms decrease, and the first is smaller than every $q$-power because $\kappa_jS \le q^{3/2}C\log(1/q)$. This proves the weighted assertion. □

**Corollary 6.9** (Unconditional stopped inactive closure)**.** *Fix $K > 6$ and any fixed $C_2 > 6/\sigma$. Suppose $x_i(0) \le Mq^K$ for every $i < j$. Choose $\beta$ sufficiently large, then $D$ sufficiently large, and then $q$ sufficiently small so that*

$$C_2\lambda_B < K-6, \qquad C_2(\lambda_P + C_4/\beta) < 1, \qquad \beta q^{2/p} \le 1/2.$$

*Use a comparison-row horizon constant at least $C_2$. Put*

$$S = C_2\log(1/q), \qquad \delta = (\log(1/q))^{-4}.$$

*For every $T \le S/A_j$ with $\kappa_j \le q^{3/2}$ the actual older densities satisfy*

$$x_i(t) \le Mq^{K-C_2\lambda_B} + C(Dq^6)^{j-i}S(1+S)^4e^{(\lambda_P+C_4/\beta)S} + o(q^K) < \delta/4. \tag{118}$$

*The core persistence conclusion and the harmonic older-field estimates hold on this window, with the packet force supplied by Proposition 6.8. On the current colour's fixed collars, the normalized older inward nonradial field is*

$$O(q^{K+2-C_2\lambda_B}) + O\left(S(1+S)^4q^{8-C_2(\lambda_P+C_4/\beta)}\right) + o(q^K).$$

*These statements control magnitudes; they give no nonzero signed impulse.*

*Proof.* Use the positive comparison matrix from the proof of Corollary 6.5. Its initial unweighted part is bounded by $Mq^K e^{\lambda_B S}$. Its packet part is bounded in the forced weights by $CS\mathcal{M}(S)e^{\lambda_P S}+o(q^K)$, using Proposition 6.8. The parabolic error sum is the same sum as in that corollary. The cap $\delta=(\log(1/q))^{-4}$ satisfies $q^{L_\delta}\le\delta$ for every fixed $L_\delta>0$ once $q$ is small, so its error is also $o(\delta q^6)$. Since $\kappa_j S\ll 1$, this proves (118). Write $a=C_2(\lambda_P+C_4/\beta)<1$. The nearest forced density is bounded by $CDq^{6-a}S(1+S)^4$, and the initial term has exponent $K-C_2\lambda_B>6$. Their sum and the Gaussian error are $o(\delta)$, giving the strict $\delta/4$ improvement. The radial margin holds since $S\sqrt{\delta}=O((\log(1/q))^{-1})$ and $S\beta q\to 0$. A first stopping time is thus impossible. Finally sum the actual older core harmonic row $\sum_{i<j}\beta^{i-j}q^{2(j-i)}x_i$; the initial and packet parts give the two displayed field powers. The choices of $\beta, D$ are possible because $\lambda_B,\lambda_P,C_4/\beta$ can be made arbitrarily small in this parameter order. This closes the density stop and the older-field magnitude estimates. A signed transfer or protected active cap requires additional information. □

# 7 Actual source activation and discounted history

## 7.1 A global weighted foreign-field estimate for the adjoint error

The following estimate controls an exact foreign pairing in the scalar growing-coordinate equation. It does not assume that the foreign velocity is uniformly small throughout the plane in the current colour's units.

Let $T_{\mathrm{adj}}$ be the smooth growing adjoint test from Section 4.2. Only its already proved bounds

$$|\nabla T_{\mathrm{adj}}(y)|\le C\min\{|y|^3,|y|^{-5}\}$$

are used here, with the usual bounded extension across $|y|=1$. Normalize a current colour to $(A_j,R_j)=(1,1)$. Let $b$ be the actual sum of all foreign fields, after subtracting the constant rotation $\Omega_{<j}Jy$ of the older cores.

**Lemma 7.1** (Global weighted foreign field)**.** *Assume the actual older core/harmonic conclusions of Propositions 5.3 and 5.4, with older nonradial density parameter $\delta\le 1$ and normalized core defect at most $\varepsilon$. Assume $\beta\ge 4$, $\beta q\le 1/2$, and $q$ is small enough that the current unit ball is inside every older probe core. Then*

$$\sup_y |b(y)|\,|\nabla T_{\mathrm{adj}}(y)|\le C\left(\beta q+\frac{\delta q^2+q^4+\varepsilon}{\beta}\right). \tag{119}$$

*Consequently, for every $f\in L^1$,*

$$\int |f|\,|b|\,|\nabla T_{\mathrm{adj}}|\le C\left(\beta q+\frac{\delta q^2+q^4+\varepsilon}{\beta}\right)\|f\|_1.$$

*All constants are independent of the current index and packet depth.*

*Proof.* The younger fields have global norm at most $C\sum_{l>j}(A_l/A_j)(R_l/R_j)\le C\beta q$ by the individual colour contractions. Multiplication by the bounded $\nabla T_{\mathrm{adj}}$ gives their term.

On $|y|\le a/(2q)$, the actual older harmonic/core-leak decomposition, with the rigid rotation removed, gives

$$|b_{<j}(y)|\le C\frac{\delta q^2}{\beta}|y|^3+C\frac{\varepsilon}{\beta}|y|.$$

The geometric sums are $q^2/(\beta - q^2)$ for the cubic coefficient and $1/(\beta - 1)$ for the core leak. For $|y| \le 1$, multiplying this bound by $C|y|^3$ gives at most $C(\delta q^2 + \varepsilon)/\beta$. For $1 \le |y| \le a/(2q)$ use $C|y|^{-5}$, giving the same bound.

On the complementary region, the individual fourfold origin-growth bound gives $|\sum_{l<j} K * w_l(y)| \le C|y|/\beta$ in current units. The subtracted constant rotation has the same bound, since its coefficient is at most $C\sum_{l<j} A_l/A_j \le C/\beta$. Thus $|b_{<j}||\nabla T_{\mathrm{adj}}| \le C|y|^{-4}/\beta \le Cq^4/\beta$ there. This uses the sum of older amplitudes, which is finite and uniformly bounded in current units; the sum of older velocity maxima in current units would not give a depth-uniform estimate and is not used. Combining the regions proves the result. □

**Corollary 7.2** (A harmless linear scalar loss)**.** *Suppose on an actual stage of length $H \le C_0 \log(1/q)$ its perturbation satisfies $\|f(t)\|_X \le Ca(t)$, where $a(t) = a(0)e^{\sigma t}$, and the preceding lemma holds uniformly. Write*

$$\chi_q = C\left(\beta q + \frac{\delta q^2 + q^4 + \varepsilon}{\beta}\right).$$

*Then the exact foreign cross-advection pairing in the growing-coordinate identity contributes at most $C\chi_q H$ to the error divided by the growing envelope at any time in the stage. If $\beta$ is fixed, $\delta \le 1$ and $\varepsilon$ is superpolynomially small, this loss tends to zero uniformly in depth.*

*Proof.* The scalar identity and variation of constants weight the pairing at time $t$ by $e^{\sigma(s-t)}$. The lemma bounds it by $C\chi_q a(t)$. Division by $a(s)$ cancels the exponential envelope, leaving $C\chi_q s \le C\chi_q H$. □

This is a linear loss, not a quadratic error in $f$. The foreign radial-base forcing and its heat-profile correction must still be estimated separately on the derivative collars, with their actual retained coefficients. The corollary proves neither a signed lower bound nor the density envelope assumed in its statement. It removes the need for an unjustified global foreign-field bound of order $a(t)$ in this one scalar term.

## 7.2 Actual postcap source moments under a stopped-parent core

The following proposition is a conditional estimate for an actual scalar colour in the common Navier–Stokes velocity. It requires the displayed actual parent-core and positive cap-envelope hypotheses. It does not assume that the signed colour has already been identified with a canonical Euler profile. In particular, it supplies a uniformly integrable future history, not convergence or noncancellation of that history.

Use dimensionless child coordinates, and let $f$ solve

$$\partial_u f + v(u, y) \cdot \nabla f = \kappa \Delta f, \qquad 0 \le u \le U,$$

where $v$ is the actual common velocity expressed in these coordinates. Let $P$ be its positive scalar evolution with $P(0) = P_0 \ge |f(0)|$. Then positivity gives $|f(u)| \le P(u)$. A rigid rotation of the coordinates may be made at any time; all assumptions and conclusions below concern radial quantities and are unaffected by it.

**Proposition 7.3** (Discounted actual future-tail estimate)**.** *Fix $\beta, \sigma, C_0, C_a, C_d > 0$. Suppose $R \geq 8$, $H_0 \geq 1$, $0 < \kappa \leq 1$, $M = \int P_0 > 0$, and*

$$\int (1+|y|)^4 P_0(y)\,\mathrm{d}y \leq C_0 M, \tag{120}$$

$$\int_{|y|>R/4} (1+|y|)^4 P_0(y)\,\mathrm{d}y \leq C_0 M \exp\left[-\frac{c_0 R^2}{\kappa H_0}\right] \tag{121}$$

*for a fixed $c_0 > 0$. Assume the actual radial drift satisfies*

$$|v(u,y)\cdot e_r| \leq a|y| + D \qquad (|y| < R), \tag{122}$$

$$|v(u,y)\cdot e_r| \leq a_0|y| + D \qquad (y \in \mathbb{R}^2), \tag{123}$$

*where*

$$0 \leq a \leq C_a\delta/\beta, \qquad 0 \leq a_0 \leq C_a/\beta, \qquad 0 \leq D \leq C_d.$$

*There is a fixed constant $C_*$ depending only on the displayed constants such that, if*

$$e^{aU}(R/4 + DU) \leq R/2, \qquad C_*\delta \leq \sigma/4, \tag{124}$$

*then, for every $0 \leq u \leq U$,*

$$\int |y|^4 P(u,y)\,\mathrm{d}y \leq CM(1+u+\sqrt{\kappa u})^4 e^{C_*\delta u/\beta} + \mathcal{E}, \tag{125}$$

*where one can take*

$$\mathcal{E} = CMe^{C_*U/\beta}(1+R+U+\sqrt{\kappa(H_0+U)})^4 \exp\left[-\frac{cR^2}{\kappa(H_0+U)}\right]. \tag{126}$$

*The constants $C, c > 0$ depend only on the fixed constants and $c_0$. For every $0 \leq J \leq U$ and every real measurable phase $\vartheta(u)$,*

$$\int_J^U e^{-\sigma u/\beta}\left|e^{i\vartheta(u)}B_4(f(u))\right|\,\mathrm{d}u \leq C_{\beta,\sigma}M(1+J)^4 e^{-\sigma J/(2\beta)} + \frac{\beta}{\sigma}\mathcal{E}. \tag{127}$$

*In particular the same bound controls any complex moment integral with weight $e^{-\mu u}$ and $\operatorname{Re}\mu = \sigma/\beta$.*

*Proof.* All statements can first be proved for the actual smooth positive-time Navier–Stokes velocity and then by approximation. Its positive scalar representation is the diffusion with drift $v$ and covariance $2\kappa I$. For radial estimates a predictable rotation eliminates the tangential component of its drift: if $Y_u = O_u X_u$, then the martingale $\int O_u\,\mathrm{d}B_u$ is again a planar Brownian motion and $|Y_u| = |X_u|$. Equivalently one may use the radial exit estimate proved above. The local drift bound and Gronwall give, up to the first exit time $\tau_R$ from $B_R$,

$$\sup_{s\leq u\wedge\tau_R}|X_s| \leq e^{au}\left[|X_0| + Du + \sqrt{2\kappa}\sup_{s\leq u}|B_s|\right].$$

The stopped process is kept at its exit point in this inequality. For $|X_0| \leq R/4$, (124) and the Brownian supremum tail therefore imply

$$\Pr_{X_0}(\tau_R \leq U) \leq C\exp\left[-\frac{cR^2}{\kappa U}\right]. \tag{128}$$

For $U = 0$ this means zero exit probability. The predictable rotation can be regularized at the origin; for a smooth fourfold Navier–Stokes drift $v(0) = 0$ and its tangential angular speed is locally bounded, so removal of that regularization causes no difficulty. No angular drift bound enters the constants.

Split $P_0$ into its restrictions to $B_{R/4}$ and its complement. On unexited trajectories of the first part, the path bound and the fourth moment of the Brownian supremum give

$$\mathbb{E}_{X_0}[(1+|X_u|)^4 \mathbf{1}_{\{\tau_R > u\}}] \le Ce^{4au}\left[(1+|X_0|)^4 + (Du)^4 + (\kappa u)^2\right].$$

Integration against the initial positive density and (120) bound this contribution by the first term of (125), after enlarging $C_*$.

The global radial bound also gives, conditionally from any starting time $s$ and point $x$,

$$\mathbb{E}[(1+|X_u|)^4 \mid X_s = x] \le Ce^{C_*(u-s)/\beta}\left[(1+|x|)^4 + (D(u-s))^4 + (\kappa(u-s))^2\right].$$

This follows from the same radial rotation and Gronwall argument, now using $a_0$. Conditional application at $\tau_R$, where $|X_{\tau_R}| = R$, and (128) bound all exited trajectories of the first initial part by

$$CMe^{C_*U/\beta}(1+R+U+\sqrt{\kappa U})^4 e^{-cR^2/(\kappa U)}.$$

For the second initial part, the global conditional bound and (121) give at most

$$CMe^{C_*U/\beta}(1+U+\sqrt{\kappa U})^4 e^{-c_0R^2/(\kappa H_0)}.$$

Reducing $c$ and combining these estimates gives (126), and hence (125). This proves in particular that the entire fourth moment of $f$ is well defined.

Since $|f| \le P$, one has $|B_4(f)| \le \int |y|^4 P$. The cap-smallness condition in (124) makes its discounted first term bounded by

$$CM(1+u)^4 e^{-3\sigma u/(4\beta)},$$

because $\kappa \le 1$. Integrating from $J$ to infinity yields $C_{\beta,\sigma}M(1+J)^4 e^{-\sigma J/(2\beta)}$. The constant-error term integrates to at most $(\beta/\sigma)\mathcal{E}$. Unit complex phases do not change any absolute value, proving (127). □

**Corollary 7.4** (Uniform scale margin for an actual source)**.** *Fix all profile constants and a fixed finite $\beta$. Let $q \downarrow 0$ and assume*

$$R = c_R/q, \qquad U \le C_U\beta\log(1/q), \qquad H_0 \le C_H\log(1/q), \qquad \kappa \le q^{3/2}, \qquad \delta = [\log(1/q)]^{-4},$$

*with the preceding proposition's explicit actual drift and cap-envelope hypotheses. For small $q$, the margins in that proposition hold, and $\mathcal{E}/M$ is smaller than every fixed power of $q$, uniformly in viscosity, depth and seed phases. The actual discounted source histories then have uniformly integrable future tails:*

$$\int_J^U e^{-\sigma u/\beta}|B_4(f(u))|\,\mathrm{d}u \le C_{\beta,\sigma}M(1+J)^4 e^{-\sigma J/(2\beta)} + M\,o(q^K)$$

*for any fixed prescribed $K > 0$.*

*Proof.* Here $aU = O(\delta \log(1/q)) \to 0$ and $DU/R = O(q \log(1/q)) \to 0$, so the displacement margin holds. The exponential in (126) has an exponent bounded above by

$$-\frac{cq^{-7/2}}{\log(1/q)},$$

whereas its remaining factors are bounded by a fixed power of $q^{-1}$ times a fixed power of $\log(1/q)$. This proves the superpolynomial error estimate, uniformly for every smaller $\kappa$. The discounted estimate follows from the proposition. □

*Remark* 7.5 (How the actual drift hypotheses arise). For a child colour at scale $(A_j, R_j)$, its own field and all descendant fields have normalized velocity bounded by a fixed $D$, by individual $L^1, L^\infty$ contractions and $\sum_{l \geq j} A_l R_l \leq C A_j R_j$. The global older fourfold fields obey a radial bound $C|y|/\beta$. Inside a fixed fraction of the parent's protected constant core, the stopped actual older densities and their harmonic/core-leak estimate give instead a radial bound $C\delta|y|/\beta$, after the rigid rotation has been removed and exponentially small leaks have been absorbed into that coefficient. These facts supply (122)–(123) when that stopped core estimate is available. The positive cap-envelope moment and parent-radius leak (120)–(121) remain explicit inputs here. They must be obtained from the actual preparation-stage localization; they are not consequences of a signed cap-coordinate estimate alone.

### 7.3 The actual deepest selected source

We retain the fixed smooth profile $W$, its unstable eigenvalue $\lambda = \sigma + i\tau$, its normalized growing coordinate $\mathcal{A}$, and its neutral seeds $\Phi_\vartheta$. All colours below solve the scalar equation with the same actual unforced Navier–Stokes velocity.

**Lemma 7.6** (Weighted off-diagonal Nash estimate). *Let $\mathcal{S}(t,s)$ be the positive scalar propagator for the actual divergence-free Navier–Stokes velocity $u$, or its rigidly rotating frame, and diffusion coefficient $\kappa > 0$ in two dimensions. Let $\phi$ be a fixed bounded smooth function satisfying $|\nabla\phi| \leq 1$, $|\Delta\phi| \leq C_\phi$. Suppose $|u \cdot \nabla\phi| \leq V(t)$, where $V \geq 0$, and put $D(t,s) = \int_s^t V$. For $\ell \geq 0$ one has*

$$\|e^{\ell\phi}\mathcal{S}(t,s)e^{-\ell\phi}\|_{L^1 \to L^\infty} \leq \frac{C}{\kappa(t-s)} \exp\big(\ell D(t,s) + \kappa\ell^2(t-s) + C_\phi\kappa\ell(t-s)\big). \tag{129}$$

*If a positive datum of mass $M$ is supported where $\phi = 0$, while $\phi = d_0 > 0$ on a set $E$, and $D(t,0) < d_0/4$, then, provided $\kappa t$ is sufficiently small,*

$$\|\mathbf{1}_E \mathcal{S}(t,0) f_0\|_\infty \leq \frac{CM}{\kappa t} e^{-c/(\kappa t)}, \qquad \|\mathbf{1}_E \mathcal{S}(t,0) f_0\|_1 \leq CM e^{-c/(\kappa t)}.$$

*The constants do not depend on a bound for the tangential velocity or on derivatives of $u$.*

*Proof.* For $F = e^{\ell\phi} f$, direct conjugation gives the drift $u + 2\kappa\ell\nabla\phi$ and zeroth-order coefficient

$$\ell u \cdot \nabla\phi + \kappa\ell^2|\nabla\phi|^2 - \kappa\ell\Delta\phi.$$

Integration of its positive equation bounds its $L^1$ growth by $\Lambda_1 = \ell V + \kappa\ell^2 + C_\phi\kappa\ell$. In the $L^2$ energy identity half the drift divergence cancels the Laplacian term, so its growth coefficient is at most $\Lambda_2 = \ell V + \kappa\ell^2$. Apply the planar Nash inequality to $e^{-\int_s^t \Lambda_1} F$: its $L^1$ norm does not increase, and its $L^2$ dissipation is at least $\kappa\|\nabla F\|_2^2$ after the same scalar integrating factor.

Solving the resulting scalar differential inequality gives the $L^1 \to L^2$ bound $C[\kappa(t-s)]^{-1/2} e^{\int_s^t \Lambda_1}$. The adjoint calculation has the same bounds, with the signs of $u$ and $\ell$ reversed. Composition at the midpoint proves (129). The $L^1$ bound before Nash is $M \exp(\ell D + \kappa \ell^2 t + C_\phi \kappa \ell t)$. Multiply both bounds by $e^{-\ell d_0}$ and optimize $\ell > 0$. Since $D < d_0/4$ and $\kappa t$ is small, this proves the last assertions. The energy calculations can first start at positive time and use smooth data. For the actual initial $L^1 \cap L^p$ vorticity, $p > 2$, the physical velocity is bounded by the Biot–Savart split; its rotating-frame addition is tangential to radial cutoff boundaries. Thus the integrations by parts are justified by cutoff limits. Their estimates use only $V$, not that possibly large velocity bound. Truncating the positive initial datum and passing monotonically gives the $L^1$ version, with the same constants. The vorticity generating $u$ may have either sign: divergence freedom is the only property of that velocity used in these energy identities. For any signed transported aggregate $g$, scalar positivity gives $|g| \le \mathcal{S}(t,0)|g(0)|$, or domination by the sum of the absolute initial colours. That latter positive datum need only be in $L^1$; no summed initial $L^\infty$ norm appears in any step. □

Choose once and for all

$$6 < K < \frac{643}{100}, \qquad C_2 > K/\sigma + 1.$$

Use the actual packet estimate to choose $\beta, D$ so large that

$$e_B = C_2 \lambda_B < K - 6, \qquad e_P = C_2(\lambda_P + C_4/\beta) < \min\{1, 8 - K\}. \tag{130}$$

Here the comparison rates are those of Corollary 6.9; they tend to zero in its stated parameter order. Choose $q$ sufficiently small after these fixed choices. In particular $\beta q^{2/p} < 1/2$ and $\beta q < 1/2$.

**Theorem 7.7** (Actual selected-source activation and protected first cap)**.** *Let $A_l = A_1 \beta^{l-1}$, $R_l = R_1 q^{l-1}$, and*

$$w_l(0,x) = A_l[W(x/R_l) + q^K \eta_l \Phi_{\vartheta_l}(x/R_l)], \quad 0 \le \eta_l \le 1.$$

*Fix any $j \ge 1$, impose $\eta_j = 1$, and set $\nu = q^{3/2} A_j R_j^2$. The phases and all other amplitudes are arbitrary. Let $w_l$ be the actual scalar colours of the common unforced Navier–Stokes solution. Normalize the $j$th source to radius and amplitude one, and remove the rigid rotation*

$$\Omega_{<j} = \frac{W(0)}{2} \sum_{l<j} \frac{A_l}{A_j}.$$

*Denote its normalized, rotated vorticity by $\widetilde{w}_j(t)$ and put $f_j = \widetilde{w}_j - e^{\kappa t \Delta} W$, $\kappa = q^{3/2}$, $\alpha = q^K$, $a(t) = \alpha e^{\sigma t}$. Fix one sufficiently large finite expansion order $M$, with $M > C_W/\sigma$ if a reference-state reset is required below. There are constants $B, \delta_* > 0$, allowed to depend on this finite order and independent of $j$ and all phases, such that through $H_* = \sigma^{-1} \log(\delta_*/\alpha)$,*

$$\|f_j(t)\|_X \le B a(t), \qquad \int_0^{H_*} \|K * f_j(t)\|_\infty \, dt \le CB\delta_*. \tag{131}$$

*The first hitting time $t_j$ of $|\mathcal{A}(f_j)| = \delta_q$, $\delta_q = (\log(1/q))^{-4}$, is well defined and protected: in a fixed neighbourhood of that crossing, almost everywhere,*

$$\frac{d}{dt} |\mathcal{A}(f_j(t))| \ge \sigma \delta_q / 2.$$

*It obeys*

$$t_j = \sigma^{-1}\log(\delta_q/\alpha) + O(\delta_q + q^\gamma(1+\log(1/q))^{D_0})$$

*for some $\gamma > 0$ and finite $D_0$.*

*For that fixed finite $M$, let $s_q$ be the exact $\delta_q$ cap on one fixed canonical reference orbit. Choose $\phi_j$ so that $\mathcal{A}(\mathcal{T}_{\phi_j}\Omega_{\mathrm{ref}}(s_q)) = \mathcal{A}(f_j(t_j))$. Then the full vorticity state satisfies*

$$\|\widetilde{w}_j(t_j) - \mathcal{T}_{\phi_j}\Omega_{\mathrm{ref}}(s_q)\|_X \le C_M\delta_q^{M+1} + C_M q^\gamma(1+\log(1/q))^{D_M}\delta_q + C\kappa t_j. \tag{132}$$

*The last, radial term is the explicit difference between the exact Navier–Stokes heat base and the Euler base. It is not a seeded error. One can choose $M > C_W/\sigma$, where $C_W$ is the one-sided $X$ comparison constant of a fixed small ancient segment. All fixed positive-envelope moments at $t_j$ are bounded uniformly; outside one fixed ball they are smaller than every $q$-power. This theorem concerns the deepest source selected by diffusion. It does not assert inductive transmission to all larger receiving annuli.*

*Proof.* The initial series converges absolutely in $L^1 \cap L^p$. Individual scalar contractions give $\|w_l\|_1 \le CA_lR_l^2$, $\|w_l\|_\infty \le CA_l$ and hence $\sum_{l>j}\|K * w_l\|_\infty \le CA_jR_j\beta q$. These are actual-colour bounds and require no hypothesis on the deep colours' nonradial densities.

Work in the normalized rotating frame. Let $C_\Phi$ bound the fixed autonomous first coefficient in $X$ divided by $e^{\sigma t}$, and fix $B > 4(C_\Phi + 1)$ now. All later field estimates use this fixed $B$; their constants may depend on it. Stop provisionally when $\|f_j(t)\|_X = Ba(t)$. The terminal cap $\delta_*$ is a provisional small number, to be fixed after the finite expansion orders, before the final choice of $q$. By Corollary 6.9, all larger colours already obey their unconditional inactive-density estimate through $S = C_2\log(1/q)$; their cores and harmonic estimates hold there. Their nonrotational field on any fixed source collar is, in every fixed spatial derivative norm,

$$O(q^{K+2-e_B}) + O(q^{8-e_P}(1+S)^5) + o(q^P) \tag{133}$$

for every fixed $P$. Derivatives here are taken only on the harmonic part; the core leak has no asserted derivative bound and is included in the last velocity remainder.

The stopped source core persists directly, without a source-core hypothesis in the bootstrap. Write its own velocity exactly as $K * e^{\kappa t\Delta}W + K * f_j$; the first term is tangential everywhere and the second is bounded by $CBa(t)$. Larger cores have already been proved independently by the inactive-prefix result. Their harmonic and leaked nonrotational contributions on the fixed source-core disk have integral tending to zero by (133). Every smaller velocity contributes at most $C\beta q$. Thus the integral of the absolute radial drift on that disk is at most

$$CB\delta_*/\sigma + C\beta qS + o(1).$$

Choose $\delta_*$ and then $q$ so this is less than $a_0/8$. In the exact backward representation the sign of the drift is reversed, but the same absolute radial bound holds. Starting in $B_{a_0/2}$ and stopping on exit from the initial constant core $B_{a_0}$ gives

$$\|\widetilde{w}_j(t) - W(0)\|_{L^\infty(B_{a_0/2})} \le Ce^{-c/(\kappa t)}$$

by the radial exit lemma and the individual colour maximum principle. Consequently, in the source core all foreign fields acting on the aggregate $g = \sum_{l>j}\widetilde{w}_l$ are harmonic plus a rough exponentially small leak; after rotation their cubic constant is at most $CB\delta_*$. Their global growth is $C|x|$, uniformly in $j$.

The aggregate $g$ has mass norm $M_g \le C\beta q^2$, initial support in $B_{Cq}$, zero circulation and zero initial $B_4$. Apply the already proved full nonlinear cold-tail estimate with $r = Cq$, $R_* = \kappa^{1/4}$ and window $S$. Its margins hold since $M_g/\kappa = O(\beta q^{1/2})$, $\kappa S \to 0$, and $CB\delta_*\sqrt{\kappa}S \to 0$. It yields

$$\sup_{t \le H_*} |B_4(g(t))| \le Cq^{643/100}(1+S)^{D_1}. \tag{134}$$

Every constant in this stopped calculation is independent of $j$.

For completeness, higher moments require no signed cancellation. Nash for the positive passive envelope gives $\|K * g(t)\|_\infty \le CM_g/\sqrt{\kappa t}$. Inside $B_{2R_*}$, foreign radial drift is bounded by $(CB\delta_* R_*^2 + \epsilon(t))|x|$. Its integrated linear-growth coefficient tends to zero. The elementary radial moment calculation, or the integrating-factor diffusion comparison, therefore gives for each fixed integer $n \ge 2$, before exit,

$$\int |x|^n |g(t,x)|\, dx \le C_n M_g (q + \sqrt{\kappa S} + M_g\sqrt{S/\kappa})^n.$$

The last displacement is smaller than $\sqrt{\kappa S}$ since $M_g/\kappa \to 0$. Radial exit from $B_{R_*}$ has probability $Ce^{-cR_*^2/(\kappa S)}$, because both integrated self displacement and the cubic displacement are small compared with $R_*$. After exit the global foreign linear growth and integrable self displacement amplify a fixed moment by at most $C_n e^{C_n S}$, absorbed by that probability. Thus the whole-moment bound is

$$\sup_{t \le H_*} \int |x|^n |g(t,x)|\, dx \le C_n q^{2+3n/4}(1+S)^{n/2} + o(q^P). \tag{135}$$

In particular $B_8(g) = O(q^8(1+S)^4)$. Using only the absolute fourth moment instead of (134) would fail, since its available power is merely $q^5$.

There is also an exponentially small actual density leak on the source collars. Choose in Lemma 7.6 a fixed radial $\phi$ which rises from zero to $d_0$ entirely inside the protected source core, and is constant thereafter. On the support of its derivative the integrated radial drift is at most $CB\delta_* + C\beta qS + o(1) < d_0/4$. Apply that lemma to the actual positive envelope of $g$, not to a new Navier–Stokes solution. It bounds its mass and $L^\infty$ norm beyond the rise by $CM_g e^{-c/(\kappa t)}$ and $CM_g(\kappa t)^{-1}e^{-c/(\kappa t)}$. Kernel interpolation makes the corresponding leaked velocity smaller than every $q$-power through $S$.

Take finitely many ACTUAL coefficients of the old interior harmonic series and of the cold exterior multipole series, retaining degrees $3, 7, \dots$ and $4, 8, \dots$, respectively. Extend their stream-functions by fixed smooth radial cutoffs, vanishing near zero and outside one fixed ball, so the fields agree on a broad annulus containing both derivative collars. The result $b_{\mathrm{app}}$ is real, smooth, divergence free and has a fixed finite set of sectors. For every finite derivative order required below,

$$\|b_{\mathrm{app}}(t)\|_{C^k} \le \varepsilon_q \alpha e^{\sigma t}, \qquad \varepsilon_q \le Cq^{\gamma_0}(1+S)^{D_2}, \tag{136}$$

$$\|b - b_{\mathrm{app}}\|_{L^\infty(\mathrm{collars})} \le Cq^P, \tag{137}$$

where $\gamma_0 > 0$. Indeed one may take any number smaller than

$$\min\{643/100 - K,\ 8 - K - e_P,\ 2 - e_B\} > 0.$$

The fourth cold coefficient uses (134); the higher coefficients use (135); the old coefficients use (133). Retaining enough moments makes the remainder exponent $P$ arbitrarily large: the next

cold moment has power $2+3n/4$, while old Taylor remainders gain arbitrarily many factors of $q$. All density/core leaks are superalgebraic. This step is valid for each fixed finite $P,k$ and does not assert a derivative bound for a rough leak.

Expand the receiving/source colour in $\alpha,\kappa$ as in the isolated proof, adding the ACTUAL coefficient functions $b_1 = b_{\mathrm{app}}/\alpha$ to the triangular recursion. Explicitly, in addition to the right-hand side of (23), add

$$-\mathbf{1}_{m=1} b_1(t)\cdot\nabla W_\ell(t) - \mathbf{1}_{m\geq 2} b_1(t)\cdot\nabla g_{m-1,\ell}(t), \qquad W_\ell(t) = t^\ell \Delta^\ell W/\ell!.$$

Here $g_{1,0}(0) = \Phi_{\vartheta_j}$ and all other initial coefficients vanish. Their dependence on this actual run does not require analyticity in either parameter: $b_1(t)$ is one fixed coefficient function for that run, and the recursion cancels the actual equation algebraically.

We make the finite derivative hierarchy explicit. Fix an integer $r \geq 4$ and estimate coefficient $(m,\ell)$ at Sobolev order

$$d_{m,\ell} = r + 2(M-m) + 2(N-\ell), \qquad 1\leq m\leq M, \quad 0\leq \ell\leq N.$$

The Laplacian input $(m,\ell-1)$ has two more assigned derivatives. Every heat coupling $(m,\ell-s)$, $s\geq 1$, has at least two more; each nonlinear input has strictly smaller seed order and therefore at least two more assigned derivatives, sufficient for its one differentiated factor. The external differentiated input $(m-1,\ell)$ also has two more. Use the already established $C^k$ bounds for $b_1$ up to the finite maximum assigned order, and smoothness of $W$ up to that order plus the finitely many heat Laplacians. The finite-sector propagator acts at each of these assigned orders without radial derivative loss: base transport is angular multiplication on each sector. In two dimensions $H^d$, $d\geq 4$, is an algebra; on the fixed coefficient support, zero circulation and the Fourier low-frequency estimate give $\|K*g\|_{H^d}\leq C_d\|g\|_{H^d}$. These facts prove all coefficient estimates by induction in $m+\ell$. There is no infinite derivative iteration and no regularity assumption on a rough actual core leak. Coefficients remain smooth, supported in the fixed collars, and of zero circulation, since every forcing is a Laplacian or the divergence of a compactly supported flux. If the retained input stream frequencies are at most $4N_{\mathrm{h}}$, coefficient $(m,\ell)$ has frequencies at most $4mN_{\mathrm{h}}$; thus only one fixed finite frequency set is used throughout this hierarchy.

The first coefficient differs from the isolated coefficient by $C\varepsilon_q(1+t)e^{\sigma t}$, by the sharp finite-sector semigroup. For orders $m\geq 2$, the forcing has exponent $m\sigma$ and the gap $m\sigma-\sigma>0$; induction bounds the coefficient differences by $C_m\varepsilon_q(1+S)e^{m\sigma t}$. The finite coefficient bounds are uniform since $\varepsilon_q S\to 0$. Positive-viscosity terms have only finite time polynomials. The source factor is retained in the heat Taylor remainder.

Use the ACTUAL foreign drift in the transport part of the comparison between the actual source and this expansion. It cancels from the Kato norm calculation. The additional residual is $(b-b_{\mathrm{app}})\cdot\nabla\omega_{\mathrm{app}}$. All nonheat coefficient gradients have compact collar support, so (137) controls them. Away from a broad collar annulus the exact heat gradient is Gaussian; its product with the foreign radial field is uniform because $|b\cdot e_r|\leq C(|x|+\beta q)$ globally, by origin growth for old colours and the unrestricted deep velocity sum. Thus the residual obeys

$$\|R\|_X \leq Ca^{M+1} + C\kappa^{N+1}(1+t)^D a + Cq^P + o(q^{P'}),$$

where $P'$ can also be arbitrarily large. The Kato coefficient is the same fixed $C_1 = C_p(\|\nabla W\|_X+1)$ as before, after a small terminal cap. For clarity the parameter choices and bootstrap improvement can be made in the following order. Fix $C_1$ before the expansion orders. Choose $(M+1)\sigma > C_1$, then $N$ with

$$\tfrac{3}{2}(N+1) > K(C_1/\sigma-1)_+ + 1, \qquad P > K\max\{1, C_1/\sigma\} + 1,$$

then choose the finite harmonic truncation and derivative hierarchy above. Use the $B > 4(C_\Phi + 1)$ fixed at the start of the density bootstrap. The finite pure-coefficient bounds are uniform when $\varepsilon_q(1+S)$ is small. Choose $\delta_*$ sufficiently small that the summed higher pure coefficients are at most $Ba/32$, the pure gradient sum is at most $1/2$, the pure residual comparison is at most $Ba/32$, and the stopped source core has its fixed radial margin. All remaining constants are finite and fixed. Now choose $q$ small so the forced first-coefficient correction, positive-viscosity sums and comparisons, spatial mismatch comparison, and remaining gradients are each within their respective $Ba/32$ or gradient-$1/2$ budgets. In particular $\|\nabla\omega_{\mathrm{app}}\|_X \le \|\nabla W\|_X + 1$, so the coefficient is precisely the previously fixed $C_1$. The explicit comparison error is bounded by

$$Ca^{M+1} + C\kappa^{N+1}(1+t)^{D+1}a(t)e^{(C_1-\sigma)_+ t} + Cq^P(1+t)e^{C_1 t} + o(q^{P'}e^{C_1 t}),$$

uniformly through $H_*$. Dividing by $a(t)$, the second term is bounded by $Cq^{\frac{3}{2}(N+1)-K(C_1/\sigma-1)_+}(1+S)^{D+1}$; the third by $Cq^{P-K\max\{1,C_1/\sigma\}}(1+S)$. Both tend to zero, with explicit positive powers. Choose $P'$ large enough for the last superalgebraic term. The approximate density and comparison error sum to less than $Ba/2$. Continuity of $\|f_j\|_X/a$ therefore rules out its first stop and proves (131). The core and cold-tail estimates held on that sole provisional density stop and were obtained before its improvement. Therefore no conditional source-density input remains.

The autonomous pure coefficients approach their universal ancient homological coefficients at an exponential rate. To check the rate, use the rotating generator $\mathcal{L}_{\mathrm{rot}}$ of the ancient-profile construction and normalize coefficient $m$ by its factor $e^{m\sigma t}$ and the common angular rotation. The first normalized coefficient differs from its growing limit by $O(e^{-2\sigma t})$. For $m \ge 2$ its equation is

$$F_m'(t) = (\mathcal{L}_{\mathrm{rot}} - m\sigma)F_m(t) - \sum_{m_1+m_2=m} (K * F_{m_1}(t)) \cdot \nabla F_{m_2}(t).$$

The finite-sector propagator of the first operator decays at least as $Ce^{-(m-1)\sigma t}$. Its limiting constant is the unique homological resolvent coefficient. Subtract that constant; the differentiated-input hierarchy above controls the forcing difference. Induction gives a bound $C_m e^{-ct}$, with any fixed $0 < c < \sigma$, in all the finitely required orders. Taking $c = \sigma/2$ turns its cap-time remainder into a positive q-power times a finite logarithmic loss. The above actual-input coefficient difference is $O(\varepsilon_q S)$, so at a cap with linear amplitude $\delta_q$ the actual source is the universal ancient polynomial up to

$$C_M\delta_q^{M+1} + C_M q^\gamma (1+S)^{D_M}\delta_q.$$

This displayed bound concerns $f_j$ relative to the radial heat base. For the full vorticity state used in (132), one must additionally use the exact heat identity

$$\|e^{\kappa t_j \Delta}W - W\|_X \le \kappa t_j \|\Delta W\|_X.$$

It gives the explicit unseeded radial term $C\kappa t_j$ in that equation. The radial heat term has zero protected coordinate, so it does not alter the first-cap phase or the coordinate crossing estimate below. The protected scalar identity has quadratic self error $O(a^2)$, heat error $O(\kappa(1+t)a)$ and projected base forcing $O(\varepsilon_q a)$. Its other foreign error is small: on a fixed ball the old field is (133), the cold global velocity is $O(\beta q)$, and outside that ball the source's positive envelope has Gaussian tails. It follows that near $a = \delta_q$ its radial crossing derivative is at least $\sigma\delta_q/2$ for small $q$. More explicitly the same expansion and comparison, using that the protected test kills the stable first coefficient, give on preparation

$$|\mathcal{A}(f_j(t)) - \alpha e^{i\vartheta_j}e^{\lambda t}| \le Ca(t)\big[a(t) + \varepsilon_q(1+S) + q^\gamma(1+S)^D\big].$$

For $a(t) \le 2\delta_q$ this estimate excludes any cap before the indicated small neighbourhood of the linear crossing. Within that neighbourhood the positive radial derivative gives one crossing. This proves protection and the time estimate. Matching the canonical exact coordinate removes its amplitude and phase bias; local invertibility of the canonical radial coordinate proves (132).

Finally, the integrated radial drift on any fixed ball surrounding the initial source support is at most $CB\delta_* + C\beta qS + o(1)$, small compared with a fixed boundary gap. Forward radial exit of its actual positive passive envelope is therefore bounded by $Ce^{-c/(\kappa t)}$. After exit, the uniform global radial growth amplifies each fixed moment by at most $C_n e^{C_n S}$, absorbed by that probability. This proves the cap-time positive-envelope assertions and completes the theorem. □

The universal finite polynomial above is a jet of the same canonical orbit for every finite order, by Theorem 4.5. Its finite-order constants and its range of small amplitudes may depend on that order; no infinite formal series is used.

**Lemma 7.8** (Actual cap-to-reference comparison and two-scale envelope reset)**.** *Fix the profile, $p > 2$, and one sufficiently small canonical ancient segment before choosing the later finite expansion orders. On this segment fix a reference orbit with $\mathcal{A}(\Omega_{\mathrm{ref}}(0)) = \delta_{\mathrm{ref}} > 0$ and write*

$$\mathcal{A}(\Omega_{\mathrm{ref}}(u)) = c_{\mathrm{ref}} e^{\lambda u} + O(e^{2\sigma u}), \qquad a_{\mathrm{ref}} = |c_{\mathrm{ref}}|.$$

*Fix $C_0 < C_2$, the parameter margins of Corollary 6.9, and $\beta q^{2/p}, \beta q \le 1/2$. Put $L = \log(1/q)$, $\delta = L^{-4}$, and let $s_q < 0$ be the reference orbit's exact $\delta$-cap time. Normalize an actual child colour of index $j \ge 2$ to unit amplitude and radius and remove the solid rotation of its older cores. All actual colours here belong to the geometric stack of Lemma 3.1, with $\alpha = q^K$, $A_l = \beta^{l-1}$, $R_l = q^{l-1}$, and the allowed seed phases and amplitude reductions. The clock and parameter margins stipulated above are those of Corollary 6.9 in these child units. Suppose $0 < \kappa \le q^{3/2}$, its actual cap time is $\tau \le C_0 L$, and, in the exact-coordinate phase gauge,*

$$\|\widetilde{w}(t) - e^{\kappa t \Delta} W\|_X \le C_{\mathrm{past}} \delta e^{\sigma(t-\tau)} \qquad (0 \le t \le \tau), \tag{138}$$

$$\|\widetilde{w}(\tau) - \Omega_{\mathrm{ref}}(s_q)\|_X \le E_q. \tag{139}$$

*Here $C_{\mathrm{past}}$ is fixed independently of depth. Assume*

$$\delta^{-r}(E_q + q) \longrightarrow 0, \qquad r = C_W/\sigma,$$

*where $C_W > 0$ is a fixed one-sided $X$ comparison exponent on the chosen canonical segment. Set $t_{\mathrm{ref}} = \tau - s_q$.*

*The actual older-field and descendant hypotheses needed on the advance follow from Corollary 6.9, Proposition 5.4, and the individual scalar contractions. Consequently*

$$\sup_{0 \le v \le -s_q} \|\widetilde{w}(\tau+v) - \Omega_{\mathrm{ref}}(s_q+v)\|_X + \int_0^{-s_q} \|\widetilde{w}(\tau+v) - \Omega_{\mathrm{ref}}(s_q+v)\|_X \, \mathrm{d}v \le C\delta^{-r}(E_q+q). \tag{140}$$

*There is no hypothesis comparing $\delta_{\mathrm{ref}}$ with a terminal finite-expansion amplitude. If*

$$\delta^{-r-1/\beta}(E_q + q) \longrightarrow 0, \tag{141}$$

*the actual signed fourth-moment history on $[s_q, 0]$ converges to the canonical history in the weighted absolute norm with weight $e^{-\sigma u/\beta}$.*

*For sufficiently small fixed $\delta_{\mathrm{ref}}$ and subsequently small $q$, the actual positive envelope of the child, and that of its entire source/deep packet, have uniformly bounded moments of every fixed*

*order at $t_{\mathrm{ref}}$. They have superalgebraic moment tails outside one fixed ball. More strongly, choose a fixed $c_R > 0$ so $R = c_R/q$ lies in a fixed fraction of the parent's protected core. If $P(t)$ denotes either positive envelope and $M = \int P(0)$, then for every fixed integer $n \geq 2$,*

$$\int_{|y|>R/4} (1+|y|)^n P(t_{\mathrm{ref}}, y)\, \mathrm{d}y \leq C_n M \exp\left[-\frac{c_n R^2}{\kappa C_2 L}\right]. \tag{142}$$

*All constants are independent of depth and of fixed trial past/reset constants; the threshold for $q$ may depend on those trial constants.*

*Proof.* The canonical first jet and exact-cap monotonicity give

$$s_q = \sigma^{-1} \log(\delta/a_{\mathrm{ref}}) + O(\delta), \qquad -s_q = O(\log L). \tag{143}$$

Thus $t_{\mathrm{ref}} + J < C_2 L$ for every fixed $J$ and small $q$. The canonical support and its $H^k$ bounds on the fixed negative segment are supplied by Theorem 4.1. With $k \geq 3$ they give fixed $X$ bounds for both $\nabla\Omega_{\mathrm{ref}}$ and $\Delta\Omega_{\mathrm{ref}}$, as in the proof of Corollary 4.2.

We verify the actual foreign-field input independently of (138) after $\tau$. Write

$$e_B = C_2\lambda_B < K - 6, \qquad e_P = C_2(\lambda_P + C_4/\beta) < 1.$$

Corollary 6.9 protects every older core and its harmonic field through the full child-clock interval $[0, C_2 L]$, without an active-child density assumption. On any fixed child ball, its weighted harmonic row and the core-leak decomposition give, after removing the solid rotation,

$$\|b_{<j}\|_\infty \leq C\{q^{K+2-e_B} + q^{8-e_P}(1+C_2L)^5\} + o(q^P).$$

The decomposition holds on the entire older probe core by (74)–(76); every fixed child ball lies there for small $q$. Individual colour $L^1, L^\infty$ contractions give the unrestricted descendant bound $\|b_{>j}\|_\infty \leq C\beta q$, as in Lemma 6.7. Therefore on the canonical support

$$\|b\|_\infty \leq \rho_q := C\{\beta q + q^{K+2-e_B} + q^{8-e_P}(1+C_2L)^5\} + o(q^P) = O(q). \tag{144}$$

In particular neither an actual child-core hypothesis nor a continuation of the finite-expansion envelope is used here.

Set $\zeta(v) = \widetilde{w}(\tau+v) - \Omega_{\mathrm{ref}}(s_q+v)$ and $E(v) = \|\zeta(v)\|_X$. Its exact equation is

$$\partial_v\zeta + (K * \widetilde{w} + b)\cdot\nabla\zeta = \kappa\Delta\zeta - (K*\zeta + b)\cdot\nabla\Omega_{\mathrm{ref}} + \kappa\Delta\Omega_{\mathrm{ref}}.$$

The actual foreign field stays in divergence-free transport. Lemma 2.5 and (21) imply

$$E'(v) \leq C_W E(v) + C(\rho_q + \kappa).$$

Hence $E(v) \leq Ce^{C_W v}(E_q + q)$, and integration of this exponential, using $C_W > 0$ and (143), proves (140). This proof never uses a derivative of the actual colour or of a rough density leak.

We next reset the positive envelopes at a fixed radius. Choose a fixed ball enclosing their original support with a fixed radial margin. Before the cap, (138) and (21) bound the child's integrated nonradial speed by $CC_{\mathrm{past}}\delta$. On the advance, the radial component of its own velocity is at most

$$\|K * (\Omega_{\mathrm{ref}}(u) - W)\|_\infty + C_p E(u - s_q).$$

The fixed canonical negative envelope, which follows from its first finite jet in (51), gives

$$\int_{s_q}^{0} \|K * (\Omega_{\mathrm{ref}}(u) - W)\|_\infty \, \mathrm{d}u \le C a_{\mathrm{ref}} = O(\delta_{\mathrm{ref}}).$$

The comparison contribution tends to zero by (140). Older nonrotational fields contribute $o(1)$ and descendants contribute $C\beta q C_2 L = o(1)$. The resulting absolute radial displacement is

$$O(C_{\mathrm{past}}\delta) + O(\delta_{\mathrm{ref}}) + o(1).$$

Fix $\delta_{\mathrm{ref}}$ sufficiently small for the prescribed margin, and then choose $q$ for the fixed trial constants. The radial exit estimate of Lemma 5.2 gives the fixed-radius Gaussian exit bound. Conditional global moment growth, using (105), multiplies it by at most $C_n e^{C_n C_2 L}(1 + C_2 L)^n$, which is absorbed by its Gaussian. Thus all fixed envelope moments are bounded and their fixed-radius moment tails are smaller than every prescribed $q$-power. The packet has bounded initial mass and the same fixed initial support by the positive-envelope construction preceding Lemma 6.7; it uses the same actual common drift, so the argument applies without a summed initial $L^\infty$ norm.

For completeness, the parent-radius estimate is proved directly, rather than deduced from the fixed-radius tail. Choose $c_R$ so $B_R$ is inside a fixed fraction of the parent's older probe core. The actual older harmonic and leak estimates give there

$$|b_{<j} \cdot e_r| \le C\frac{\delta q^2}{\beta}|y|^3 + C\frac{\epsilon_{\mathrm{core}}}{\beta}|y| \le a_q|y|, \qquad a_q \le C\delta/\beta + o(q^P).$$

The child's own and descendant signed velocities have total speed at most a fixed $D$, by the individual contractions and the convergent sum of $A_l R_l$. The full actual radial drift on $B_R$ is therefore bounded by $a_q|y| + D$. Outside this ball its unrestricted radial bound is $C|y|/\beta + D$, as in Lemma 6.7.

Let $H = t_{\mathrm{ref}} \le C_2 L$ and enclose the original packet support in the fixed $B_{R_{\mathrm{in}}}$. Since $a_q H = o(1)$ and $DH = O(L) \ll R$,

$$e^{a_q H}(R_{\mathrm{in}} + DH) \le R/8$$

for small $q$. Run the radial diffusion calculation at radius $R/4$. After predictable removal of tangential drift, its Brownian supremum must exceed a fixed multiple of $R/\sqrt{\kappa}$ to exit. Thus

$$\Pr\{\tau_{R/4} \le H\} \le C\exp[-cR^2/(\kappa H)].$$

Condition at that exit and use the unrestricted global radial bound. The same moment calculation as in the proof of Proposition 7.3 gives

$$\int_{|y|>R/4} (1 + |y|)^n P(H, y) \, \mathrm{d}y \le C_n M e^{C_n H/\beta}(1 + R + H + \sqrt{\kappa H})^n e^{-cR^2/(\kappa H)}.$$

Its prefactor is a fixed power of $q^{-1}$ times a fixed logarithmic power, whereas $R^2/(\kappa C_2 L) \ge cq^{-7/2}/L$. Reducing the Gaussian exponent constant proves (142). Truncation of the positive initial packet and monotone passage justify the calculation when that initial packet has no finite $L^\infty$ norm.

Finally, cut the fourth moment off at the fixed reset radius. Its difference is bounded by the cutoff norm times $E$, plus superalgebraic tails. Since $e^{-\sigma u/\beta} \le C\delta^{-1/\beta}$ on $[s_q, 0]$, (140) gives

$$\int_{s_q}^{0} e^{-\sigma u/\beta}|B_{\mathrm{actual}}(u) - B_4(\Omega_{\mathrm{ref}}(u))| \, \mathrm{d}u \le C\delta^{-r-1/\beta}(E_q + q) + o(q^P).$$

This proves the weighted assertion under (141). □

*Remark* 7.9 (The outer source). When $j = 1$, the older field is empty. The comparison and fixed-radius reset in Lemma 7.8 remain valid with that term zero. No parent-radius handoff is needed at the outermost scale; the $R = c_R/q$ estimate is asserted only for $j \geq 2$.

**Corollary 7.10** (Actual reference-state reset and fixed source windows)**.** *Choose the order in Theorem 7.7 with $M > C_W/\sigma$, and normalize one sufficiently small canonical reference state in time and phase so that*

$$\mathcal{A}(\Omega_{\mathrm{ref}}(0)) = \delta_{\mathrm{ref}} > 0, \qquad \mathcal{A}(\Omega_{\mathrm{ref}}(s)) = c_{\mathrm{ref}} e^{\lambda s} + O(e^{2\sigma s}) \quad (s \to -\infty).$$

*Write $a_{\mathrm{ref}} = |c_{\mathrm{ref}}|$; then $a_{\mathrm{ref}} = \delta_{\mathrm{ref}} + O(\delta_{\mathrm{ref}}^2)$, but the two parameters are not identified. Let $s_q < 0$ be the exact canonical hitting time of $|\mathcal{A}| = \delta_q$ on $\Omega_{\mathrm{ref}}$, and remove the actual source's constant phase so that its first-cap coordinate equals $\mathcal{A}(\Omega_{\mathrm{ref}}(s_q))$. Put $L_q = -s_q$ and $t_{\mathrm{ref}} = t_j + L_q$. For every fixed $J < \infty$, after the same phase and rigid-rotation removal,*

$$\widetilde{w}_j(t_{\mathrm{ref}} + \cdot) \longrightarrow \Omega_{\mathrm{ref}} \quad \text{in } C([-J, J]; L^1 \cap L^p),$$

*uniformly in cascade depth and all other seed phases. Whole fourth moments converge uniformly on each such window. At $t_{\mathrm{ref}}$ the actual source's positive passive envelope, and the positive envelope of its entire source/deep packet, have all fixed moments bounded uniformly and have superalgebraic moment tails outside one fixed ball. In particular no power of $\log\log(1/q)$ is lost in this reference-time moment reset.*

*Proof.* Theorem 7.7 is used only through its shrinking exact cap $t_j$. Its full-state reset, including the radial heat-base defect, gives

$$E_q \leq C_M \delta^{M+1} + C_M q^{\gamma} (1 + C_2 L)^{D_M} \delta + C\kappa t_j.$$

Its preparation envelope supplies (138) with a fixed $C_{\mathrm{past}}$. The choice $M > C_W/\sigma$ makes $\delta^{-r}(E_q + q) \to 0$, so Lemma 7.8 (with its empty-older-prefix variant when $j = 1$) proves convergence at the fixed reference and on every fixed earlier reference window, together with both positive-envelope resets. It is immaterial whether the selected source's terminal finite-expansion amplitude exceeds $\delta_{\mathrm{ref}}$.

The clock lies inside the required inactive horizon directly. Indeed the selected-source cap clock and (143) give

$$t_{\mathrm{ref}} = t_j - s_q = \frac{K}{\sigma} L + \frac{1}{\sigma} \log a_{\mathrm{ref}} + o(1).$$

The choice $C_2 > K/\sigma + 1$ therefore places every fixed reference window inside $[0, C_2 L]$ for small $q$. On a fixed later window the canonical Euler support and its $X$ gradient/Laplacian bounds depend only on that window. The independently available actual older-field bound and descendant velocity sum still apply there. The same one-sided comparison extends convergence to $C([-J, J]; X)$.

To remove the fourth-moment cutoff on such a later window, the child and descendant total velocity is uniformly bounded by scalar contractions. On a sufficiently large fixed ball the actual older radial field is small; the initial reference envelope has the fixed-radius Gaussian tail just proved. Radial exit and the unrestricted global moment bound give uniform fourth-moment tightness on that fixed window. Strong $L^1$ convergence on a cutoff ball then proves uniform whole-fourth-moment convergence. This completes the corollary without extending the selected-source expansion envelope beyond its proved endpoint. □

**Lemma 7.11** (Gaussian handoff at the parent radius)**.** *Fix any $j \geq 2$ and any viscosity with normalized value $0 < \kappa \leq q^{3/2}$ at this child. The child may have been activated by its incoming packet; it need not be the deepest source selected by diffusion. Use only the common geometric initial colour bounds and constant cores, and assume every actual larger nonradial density is at most $\delta_q A_l R_l^2$ through a normalized horizon $0 < H \leq C_H \log(1/q)$. Let $F_j(t)$ be the actual positive scalar envelope transported from the absolute value of the normalized source datum. The same conclusion holds for the positive envelope of the entire source/deep packet, transported from the sum of its initial absolute colours. At every prepared cap or reference handoff time $0 < t \leq H$, there is a fixed $r_0 > 0$ such that, for every fixed integer $n \geq 0$,*

$$\int_{|x|>r_0/q} (1+|x|^n) F_j(t,x)\,dx \leq C_n \exp\left[-\frac{c_n q^{-2}}{\kappa H}\right]. \tag{145}$$

*This estimates positive-envelope leakage at the parent radius independently of any fixed-radius cap reset.*

*Proof.* Take $r_0/q$ strictly inside the parent's protected core, for example $r_0 = a_0/4$. The actual larger cores follow from the stipulated density stop by applying the outer-core proposition at the parent index. Its horizon in parent units is $H/\beta = O(\log(1/q))$, its normalized viscosity is $\beta q^2 \kappa \leq \beta q^{7/2}$, and its radial margin tends to zero because $(H/\beta)\sqrt{\delta_q} = O(1/\log(1/q))$ and $qH \to 0$. Thus no selected-source cold-tail theorem or child small-density assumption is being invoked in this handoff lemma. The actual old core and harmonic decomposition gives on this disk a radial bound $C\delta_q|x|/\beta + \epsilon(t)|x|$, with superalgebraic leak integral. The own source and all smaller velocities together are at most $C$, by the convergent individual scalar $L^1, L^\infty$ bounds. Consequently the radial displacement bound for exit from this disk is at most

$$CH + (C\delta_q H/\beta + \text{superalgebraic error})r_0/q.$$

Since $H \ll q^{-1}$ and $\delta_q H = o(1)$, this is less than a fixed fraction of its radius. The initial source support has fixed radius. The radial exit lemma bounds its exit probability by $C\exp[-cq^{-2}/(\kappa H)]$. Globally all older colours satisfy the origin-growth bound $|u_{<j}(x)| \leq C|x|/\beta$, while the source/deep packet velocity is bounded by $C$. Conditioning at exit therefore bounds its later fixed moment by $C_n q^{-n} e^{C_n H/\beta}$. Both this polynomial prefactor and the fixed q-power from $H = O(\log(1/q))$ are absorbed by reducing the constant in the exit exponential. Positivity of the actual envelope and its fixed initial mass complete (145). □

**Proposition 7.12** (Actual positive source history before the parent cap)**.** *Start the source history at the fixed reference reset of Corollary 7.10. The same statement applies to any other actual source colour having the same scalar contractions and the following uniform reference-envelope bounds: for every fixed integer $n \geq 4$,*

$$\int (1+|y|)^n P_0(y)\,\mathrm{d}y \leq C_n M, \qquad M = \int P_0,$$

*with constants independent of $q$ and depth. For the prepared geometric children these bounds are supplied by Lemma 7.8. Their separate Gaussian parent-radius handoff is also available and is used in the logarithmically shrinking-cap application below. The present general future-moment bound uses all the displayed fixed moments; a fourth moment bound alone is not assumed to supply them. On a following interval $0 \leq u \leq U_q \leq C_0\beta\log(1/q)$ assume that every larger colour is still in its actual nonradial stop $\|(I-P)w_l\|_1 \leq \delta A_l R_l^2$, and that its actual constant core persists on*

*the appropriate probe disk with the Gaussian defect estimate. These are explicit hypotheses for this following interval; the preceding selected-source theorem alone does not establish the whole parent interval. Assume the normalized viscosity satisfies* $0 < \kappa \le q^{3/2}$. *For sufficiently small fixed* $\delta > 0$ *one has*

$$|B_4(\widetilde{w}_j(t_{\mathrm{ref}} + u))| \le C(1+u)^4 e^{C\delta u/\beta} + o(q^P),$$

*for every fixed* $P$, *uniformly in depth. Choose* $\delta$ *additionally so* $C\delta < \sigma/2$. *Then this is an integrable uniform positive-history bound under the receiving discount* $e^{-\sigma u/\beta}$.

*Proof.* On the disk $|z| \le a_0/(4q)$ the actual old harmonic series gives $|b_{\mathrm{old}} \cdot e_r| \le C\delta|z|/\beta + \epsilon(t)|z|$. Indeed the series is bounded by $C\delta \sum_{k\ge1} \beta^{-k} q^{2k}|z|^3$, and $q^2|z|^2$ is bounded on this disk. The core leak has superalgebraic time integral. At every time the source's own velocity is bounded by $C$, while the sum of all smaller velocities is bounded by $C\beta q$, without any nonradial stop. Thus actual radial drift there is at most $C(1+\beta q) + (C\delta/\beta + \epsilon(t))|z|$. For its actual positive passive envelope use the weight $(1+|z|^2)^2$. Its diffusion contribution is at most $C\kappa(1+|z|^2)$, its additive-drift contribution at most $C(1+|z|^2)^{3/2}$, and its linear-drift contribution at most $C\delta(1+|z|^2)^2/\beta$. Hölder and scalar Gronwall for the fourth root give the displayed polynomial times weak exponential bound before exit.

Write $R = c_R/q$, $L = \log(1/q)$, and let $U \le C_U \beta L$. On $B_R$ use $|v \cdot e_r| \le a|y| + D$, with $a = C_a\delta/\beta$ after absorbing the Gaussian core leak, while globally use $|v \cdot e_r| \le a_0|y| + D$ with $a_0 \le C_g/\beta$ and $D \le C_d$. Choose the fixed $\delta$ sufficiently small that

$$\theta := C_a C_U \delta < \tfrac{1}{4}, \qquad 4C_a\delta < \tfrac{\sigma}{2}.$$

The first condition gives $e^{aU} \le q^{-\theta}$. Put $b = Re^{-aU}$ and split the initial positive envelope at $r_{\mathrm{cut}} = b/8$. Then $r_{\mathrm{cut}} \ge cq^{-1+\theta} \to \infty$, and $DU \le b/8$ for small $q$.

For paths beginning in $B_{r_{\mathrm{cut}}}$, let $X_u$ be the actual diffusion and set $Y_u = e^{-au}X_u$. Its radial drift, until $X$ exits $B_R$, is at most $De^{-au}$ and its diffusion coefficient is $\sqrt{2\kappa}e^{-au}$. If $X$ exits $B_R$ by $U$, then $Y$ has already exited $B_b$; until this latter exit the local drift hypothesis applies because $|X_u| = e^{au}|Y_u| \le R$. Applying the cosh test from Lemma 5.2, using only the upper radial bound, gives

$$\Pr\{\tau_R \le U\} \le 2\exp\left[-\frac{cb^2}{\kappa U}\right] = 2\exp\left[-\frac{cR^2e^{-2aU}}{\kappa U}\right].$$

Indeed the test generator is bounded by $[2\kappa e^{-2au}\lambda^2 + De^{-au}\lambda]\cosh(\lambda|Y|)$, and the exit gap is at least $b - r_{\mathrm{cut}} - DU \ge 3b/4$. No tangential drift bound is used. At $U = 0$ the exit probability is zero.

The stopped fourth-moment calculation gives, for the unexited part,

$$\int |y|^4 P(u,y)\, \mathrm{d}y \le CM(1 + u + \sqrt{\kappa u})^4 e^{4au} \quad \text{before adding exited and initially exterior paths.}$$

This follows by testing $(1+|y|^2)^2$ and taking the fourth root; the initial fourth moment is at most $C_4M$. Conditional global moment growth after exit gives a remainder bounded by

$$CMe^{C_g'U/\beta}(1 + R + U + \sqrt{\kappa U})^4 \exp\left[-\frac{cR^2e^{-2aU}}{\kappa U}\right],$$

where $C_g'$ depends only on the global growth and additive speed. This is smaller than $Mq^P$ for every fixed $P > 0$: with $\kappa \le q^{3/2}$, its Gaussian exponent has size at least $cq^{-7/2+2\theta}/L$, and all its other factors are fixed powers of $q^{-1}$ and $L$.

For the initially exterior part, the $n$th reset moment gives

$$\int_{|y|>r_{\mathrm{cut}}} (1+|y|)^4 P_0(y)\, \mathrm{d}y \le C_n M r_{\mathrm{cut}}^{-(n-4)}.$$

Propagating this part with the global radial bound costs at most $Ce^{C_g' U/\beta}(1+U)^4$. For any prescribed $P$, choose one finite $n$ such that $(1-\theta)(n-4) > P + C_g' C_U + 1$. The resulting exterior contribution is $Mo(q^P)$. Thus, uniformly on $0 \le u \le U$,

$$\int |y|^4 P(u,y)\, \mathrm{d}y \le CM(1+u)^4 e^{4C_a \delta u/\beta} + Mo(q^P).$$

The second cap-smallness condition makes this bound integrable under $e^{-\sigma u/\beta}$, with uniformly vanishing discounted future tails. This proves the stated generic future-history estimate. In the actual application $\delta = \delta_q$, the strict displacement margin of Proposition 7.3 also holds, and its sharper Gaussian-error estimate is used by Lemma 9.1.

□

*Remark* 7.13 (The core part of a stopped parent environment). If every larger colour is stopped at the sharper density level $\delta_q = (\log(1/q))^{-4}$, the core hypothesis in Proposition 7.12 follows from the already proved outer-core proposition, applied at the parent index. Its total time horizon in parent units is $O(\log(1/q))$, its normalized viscosity is $\beta q^2 \kappa \le \beta q^{7/2}$, and its radial margin is $O(\log(1/q)\sqrt{\delta_q} + \beta q \log(1/q)) = o(1)$. Thus only the actual parent/larger density stop is a substantive input in this usual logarithmic window; a separate post-cap small-density assumption on the source is unnecessary.

**Corollary 7.14** (The actual whole-history source transform)**.** *Use the fixed reference gauge of Corollary 7.10 and the actual post-reference environment of Proposition 7.12. Assume also $U_q \to \infty$. Extend the actual source moment by zero before its physical initial time and after $U_q$. Then, with $\mu_\beta = [\sigma + i(\tau - 4/5)]/\beta$,*

$$\int_{\mathbb{R}} e^{-\mu_\beta u} B_{\mathrm{actual,ref}}(u)\, du \longrightarrow \int_{\mathbb{R}} e^{-\mu_\beta u} B_4(\Omega_{\mathrm{ref}}(u))\, du,$$

*uniformly in depth and all other seed phases. The same conclusion holds for the weighted absolute difference of the two moment functions. The actual history is used only through $U_q$; the zero extension is a definition of the truncated history, not a claim about later actual dynamics. If the canonical transform is nonzero at the chosen $\beta$, the actual one is nonzero for all sufficiently small $q$, with a uniform lower bound in this fixed reference gauge.*

*Proof.* On the preparation interval, the actual source norm envelope and fixed-radius Gaussian tails imply $|B_4(\widetilde{w}_j(t))| \le C\alpha e^{\sigma t}$. The exact reference shift satisfies $\alpha e^{\sigma t_{\mathrm{ref}}} = a_{\mathrm{ref}}(1+o(1))$. Thus the actual reference moment is bounded by $Ce^{\sigma u}$ for $u < 0$, including the part between the shrinking cap and the reference state. Its tail outside the fixed source radius is superalgebraic and smaller than $\alpha$, so it is included in that same envelope. The post-reference bound gives $C(1+u)^4 e^{C\delta u/\beta}$ for $u > 0$ up to its endpoint, with superalgebraic errors uniformly on a logarithmic horizon. The canonical moment has the corresponding negative exponential and positive polynomial bounds. Since $\beta > 1$ and $C\delta < \sigma/2$, both are integrable under $e^{-\sigma u/\beta}$. The extra superalgebraic errors times the whole logarithmic horizon and its possible fixed q-power amplification still tend to zero. The fixed-window whole-moment convergence now proves the weighted absolute-difference assertion by truncating at $[-J, J]$, first choosing $J$ and then $q$. This gives the transform assertion and its nonzero conclusion. □

*Remark* 7.15 (Why the fixed reference gauge matters). If the transform is instead centred at the shrinking exact cap, the time and phase shift multiplies both actual and canonical transforms by the same factor whose modulus is comparable to $(\delta_q/a_{\mathrm{ref}})^{1/\beta}$. The absolute lower bound then shrinks by that factor, while the relative transform error remains small. Using the fixed reference state as the receiving force origin avoids that artificially shrinking coefficient. Merely comparing a fixed window around the shrinking cap would omit the later nonlinear source pulse.

*Remark* 7.16 (The reference inputs in the invariant class). For the inductive reset write

$$E_q \le C_{\mathrm{reset}}\delta^{M+1} + q^{\chi}L^{D}.$$

The fixed choice $M+1 > C_W/\sigma + 2$ implies (141), including the extra factor $1+|s_q|$ used in Proposition 9.2. For each fixed trial reset constant the reference comparison error and its integrated radial contribution tend to zero. The reference and its localization radius are fixed before that later finite order.

The reference past before its small cap has weighted norm at most $CC_{\mathrm{past}}\delta^{1-1/\beta}$. In small-cap time $v \le 0$, this follows from the envelope $C_{\mathrm{past}}\delta e^{\sigma v}$ and the factor $e^{-\sigma s_q/\beta} = O(\delta^{-1/\beta})$. The advance has fixed canonical history norm plus the vanishing error just proved.

At reference time (142), with $n = 4$, supplies (121) with $H_0 = C_2 L$, while the fixed-radius moment reset supplies (120). Under the current parent's protected-core stop, its actual older harmonic/leak field satisfies (122); its unrestricted origin-growth bound gives (123); the child's own and descendant velocity sum gives the fixed $D$. Thus Proposition 7.3 supplies the prospective future history without an assumed signed source waveform.

The upper history prefactor can be chosen independently of a fixed trial parent stop. In the proof of that proposition the local stop constant occurs in $e^{CC_{\mathrm{stop}}\delta u/\beta}$, while the prefactor uses only the reset moment and $D$. On $u \le C_2\beta L$, this factor is at most two after choosing $q$ for the fixed trial stop; its discounted exponent is at most $-\sigma u/(2\beta)$. Conditional global moment growth in the Gaussian error uses the unrestricted bound, independently of that stop. The future weighted norm is consequently bounded by a fixed reference constant. Hence, for the finite derivative orders selected before the class constants,

$$M_{\mathrm{input}} \le M_{\mathrm{ref}} + CC_{\mathrm{past}}\delta^{1-1/\beta} + o(1),$$

where $M_{\mathrm{ref}}$ may depend on those finite orders, fixed stream cutoffs, profile, reference, clocks, and $\beta$, but not on trial past/reset/stop constants or depth. Orders can be chosen using the already fixed comparison exponents, then their input constant computed, then the output class constants fixed, and finally $q$ decreased.

Fixed-window comparison and the two discounted tails also give uniform $o(1)$ closeness of the signed transform to the fixed canonical transform. This provides its nonzero lower bound once the full prospective receiver interval is reached; the independent upper input bound is available on every shorter stop. These are the reference and handoff inputs needed by Proposition 10.1.

# 8 Finite nonlinear expansions for retained actual forcing

## 8.1 Finite actual forcing: density closure and exact-cap reset

A finite expansion with the actual foreign-field coefficients gives an actual density envelope and an exact-cap state reset. The recursion uses those coefficients as measurable functions of time. A nonzero growing coefficient and pointwise bounds on the protected late window supply the first-cap conclusion.

### 8.1.1 Spaces, operators and the derivative reserve

Let $\mathcal{C}$ be the union of the two closed derivative collars of the fixed smooth radial profile $W$. Choose a closed radial interval $I \Subset (0,\infty)$ containing these collars in its interior. For an integer $J \geq 1$, let $E_j^r$ denote the real finite angular-sector space of functions supported in $\mathcal{C}$ whose sectors are $m \in 4\mathbb{Z}$, $|m| \leq 4Jj$, with radial coefficients in $H^r(I)$. The norm is the sum of their radial Sobolev norms. On these supported finite-sector functions it is equivalent to the full-plane $H^r$ norm, with constants depending on $I, J, j, r$. Initially and in every recursion below the functions are smooth, flat at the collar endpoints, and have zero integral. The coefficient equations therefore use a common closed supported subspace on each finite Sobolev level; there is no assertion of an unrestricted all-angular-mode spectral estimate.

Write

$$\mathcal{L}f = -(K * W) \cdot \nabla f - (K * f) \cdot \nabla W, \qquad Q(f,g) = -(K * f) \cdot \nabla g.$$

The inherited finite-sector spectral lemma gives

$$e^{t\mathcal{L}} = e^{\lambda t} P_+ + e^{\bar{\lambda} t} P_- + \mathcal{R}(t), \qquad \|\mathcal{R}(t)\|_{H^r \to H^r} \leq C_r e^{2\eta t}, \quad 2\eta < \sigma, \quad \lambda = \sigma + i\tau. \tag{146}$$

This is used only in one of the finitely many spaces $E_j^r$. The growing eigenfunction $F_+$ lies in sector $+4$ and is normalized by $\mathcal{A}(F_+) = 1$; its real conjugate lies in sector $-4$. The global test defining $\mathcal{A}$ selects sector $+4$. For the pattern rotation set

$$(\mathcal{T}_\theta f)(r,\varphi) = f(r, \varphi + \theta/4), \qquad \mathcal{L}_{\mathrm{rot}} = \mathcal{L} - (\tau/4)\partial_\varphi.$$

For vector fields its definition is the pushforward $\mathcal{T}_\theta V(x) = \mathsf{R}_{-\theta/4} V(\mathsf{R}_{\theta/4} x)$, where $\mathsf{R}$ is the planar rotation matrix. Equivalently its polar components are shifted by $\theta/4$. This gives $K * (\mathcal{T}_\theta f) = \mathcal{T}_\theta (K * f)$ and $\mathcal{T}_\theta [V \cdot \nabla f] = (\mathcal{T}_\theta V) \cdot \nabla (\mathcal{T}_\theta f)$. This rotation is applied to the entire function, separately in every angular sector. It is not replaced by the scalar factor $e^{-ij\tau s}$ on an order-$j$ coefficient. The operators commute, and rotations are isometries in radial Sobolev norms. Thus $e^{t\mathcal{L}_{\mathrm{rot}}} \leq C e^{\sigma t}$ and the growing real space has eigenvalue $\sigma$.

Fix truncation orders $M, N \geq 1$ and a desired final Sobolev order $k$. Set

$$k_* := \max\{4, k\}, \qquad r_{j\ell} = k_* + 2(M - j) + 2(N - \ell), \qquad R = k_* + 2M + 2N + 3. \tag{147}$$

The reserve is explicit. A diffusion source uses $g_{j,\ell-1}$, whose order exceeds $r_{j\ell}$ by two. A quadratic or foreign-advection source uses a coefficient with strictly smaller seed order, whose order exceeds the requested order by at least two. A heat-interaction source uses $g_{j,\ell-r}$, again with at least two additional derivatives. All profile derivatives used below are fixed derivatives of $W$; $W$ is smooth in every finite order.

For zero-integral functions supported in one fixed ball,

$$\|K * f\|_{H^{r+1}(\mathbb{R}^2)} \leq C_r \|f\|_{H^r}, \qquad \|Q(f,g)\|_{H^r} \leq C_r \|f\|_{H^r} \|g\|_{H^{r+1}}, \quad r \geq 2. \tag{148}$$

To check the first low-frequency estimate, compact support and zero mass give $|\widehat{f}(\xi)| \leq C|\xi| \|f\|_1$ for $|\xi| \leq 1$. The Biot–Savart multiplier has order $-1$, so it is bounded there by this cancellation and is of order $-1$ at high frequency. Plancherel proves the first estimate. For the second, $H^r(\mathbb{R}^2)$ is an algebra for $r > 1$, and $\nabla g \in H^r$; multiplication and the first estimate prove the assertion. On the supported spaces $H^{k_*}$ also controls all the $L^1, L^p$ product norms and the two extra derivatives in residual Laplacians. These elementary bounds justify, rather than hide, every transport derivative in the finite induction.

### 8.1.2 The finite recursion and its exact residual

Put $X = L^1 \cap L^p$, $p > 2$, with its sum norm. In a scale family write $L = \log(1/q)$, $\delta = L^{-4}$, with $0 < q < 1$; all error exponents $\chi > 0$ are fixed before $q$ is decreased. Splitting Biot–Savart at radius one gives $\|K * f\|_\infty \le C_p \|f\|_X$. Let $0 \le H < \infty$, $t_0 \ge 0$, $\alpha \ge 0$, $\varepsilon > 0$, $0 < \kappa \le 1$, and

$$s = t - t_0, \qquad a(t) = \varepsilon e^{\sigma(t-t_0)}, \qquad W_\kappa(t) = e^{\kappa t \Delta} W, \qquad W_r(t) = \frac{t^r}{r!}\Delta^r W.$$

Let $B(t,x)$ be real and divergence free, spatially $C^\infty$ for almost every time, and locally integrable in time with values in every fixed finite $C^m$ norm. Only its $C^R$ norm is used in the quantitative bounds. This qualitative regularity is automatic for the finite smooth stream basis with actual moment coefficients. Its polar velocity components have finitely many angular sectors $m \in 4\mathbb{Z}$, $|m| \le 4J$. The retained field is $b_J = \varepsilon B$. Assume

$$\frac{\alpha}{\varepsilon} e^{\sigma t_0} \|\Phi\|_{H^{r_{10}}} \le A_0, \qquad D(t) := e^{-\sigma(t-t_0)} \|B(t)\|_{C^R}, \qquad \int_0^H D(t)\,\mathrm{d}t \le B_0, \tag{149}$$

where $\Phi \in E_1^{r_{10}}$ is a real smooth zero-integral seed. The bounded constants $A_0, B_0$ are the input ball; no pointwise upper bound on $D$ is imposed.

Set $g_{10}(0) = (\alpha/\varepsilon)\Phi$, with all other initial coefficients zero. For $1 \le j \le M$, $0 \le \ell \le N$, solve

$$\begin{aligned}(\partial_t - \mathcal{L}) g_{j\ell} = \mathbf{1}_{\ell \ge 1} \Delta g_{j,\ell-1} - \sum_{r=1}^{\ell} \left[ (K * W_r) \cdot \nabla g_{j,\ell-r} + (K * g_{j,\ell-r}) \cdot \nabla W_r \right] \\ + \sum_{\substack{j_1+j_2=j,\ j_1,j_2 \ge 1 \\ \ell_1+\ell_2=\ell}} Q(g_{j_1\ell_1}, g_{j_2\ell_2}) - \mathbf{1}_{j=1} B \cdot \nabla W_\ell - \mathbf{1}_{j \ge 2} B \cdot \nabla g_{j-1,\ell}.\end{aligned} \tag{150}$$

Every coefficient on the right precedes $(j,\ell)$ in the ordering by $j + \ell$. The angular sectors are at most $4Jj$. All outputs are supported in $\mathcal{C}$: Laplacians preserve the support of smooth flat functions; $\Delta^r W$ with $r \ge 1$ has this support; and every advective term differentiates a supported function. Each forcing has zero integral by divergence freedom or the Laplacian identity. The bounded finite-sector generator and Duhamel therefore construct a unique smooth coefficient, absolutely continuous in time in every required lower Sobolev order. Measurable time coefficients are sufficient.

Define

$$F(t) = \sum_{j=1}^{M} \sum_{\ell=0}^{N} \varepsilon^j \kappa^\ell g_{j\ell}(t), \qquad w_{\mathrm{app}} = W_\kappa + F.$$

The recursion is an algebraic identity at the given run's $B, \kappa$. Dependence of $B$ on the actual solution and on $\kappa$ does not change that identity.

**Lemma 8.1** (Coefficients and finite residual)**.** *Under* (149)*, with the reserve* (147)*,*

$$\|g_{j0}(t)\|_{H^{r_{j0}}} \le C e^{j\sigma s}, \qquad \|g_{j\ell}(t)\|_{H^{r_{j\ell}}} \le C(1+t)^{d_{j\ell}} e^{j\sigma s}, \quad d_{j\ell} = 2\ell(j+\ell), \quad \ell \ge 1. \tag{151}$$

*The constants depend on the fixed finite orders and input ball, not on $t_0, H, \varepsilon, \kappa$ or the particular waveform. One may take $d_R = 4N(M+N) + N + 3$. If $a \le 1$ and $\kappa(1+H)^{d_R} \le c$, the residual in the retained field satisfies*

$$\|\partial_t w_{\mathrm{app}} + (K * w_{\mathrm{app}} + \varepsilon B) \cdot \nabla w_{\mathrm{app}} - \kappa \Delta w_{\mathrm{app}}\|_X \le C\big[a^{M+1} + \kappa^{N+1}(1+t)^{d_R} a\big](1 + D(t)). \tag{152}$$

*Proof.* For $g_{10}$, (146) and the weighted initial bound show that its homogeneous contribution is at most $Ce^{\sigma s}$. Its forcing convolution is at most

$$C\int_0^t e^{\sigma(t-r)}\,\|B(r)\|_{C^R}\,\mathrm{d}r = Ce^{\sigma s}\int_0^t D(r)\,\mathrm{d}r.$$

The reserve supplies its requested derivatives. Suppose preceding coefficients have the claimed bounds. A quadratic pure-seed source has size $Ce^{j\sigma(r-t_0)}$ in the requested Sobolev order, by (148). For $j\ge 2$ its convolution has the gap $(j-1)\sigma$, so its bound is $Ce^{j\sigma s}$ with no polynomial. A pure external-advection source has size $Ce^{j\sigma(r-t_0)}D(r)$; dividing its convolution by $e^{j\sigma s}$ leaves $C\int_0^t e^{-(j-1)\sigma(t-r)}D(r)\,\mathrm{d}r \le CB_0$. This proves the pure-seed estimate.

For positive viscosity order, the heat terms carry $(1+r)^r$ from $W_r$, the diffusion term has the preceding polynomial, and the quadratic terms add their two polynomial degrees. A convolution at $j=1$ can add at most one degree; for $j\ge 2$ the strict gap adds none. The choice $d_{j\ell}=2\ell(j+\ell)$ dominates all these degrees:

$$d_{j\ell}\ge d_{j,\ell-1}+1,\quad d_{j\ell}\ge r+d_{j,\ell-r}+1,\quad d_{j\ell}\ge d_{j-1,\ell},\quad d_{j\ell}\ge d_{j_1\ell_1}+d_{j_2\ell_2}.$$

The second inequality is used for $r\ge 1$ and $\ell\ge r$; the third for $j\ge 2$; the last for the displayed quadratic indices. For the direct $B\nabla W_\ell$ term its degree $\ell$ is smaller. The weighted $D$ convolution adds no degree beyond the preceding factor. This proves (151) by the finite induction, using precisely the reserve already specified.

For completeness we give the residual identity. Put $G_N = W_\kappa - \sum_{r=0}^N \kappa^r W_r$ and $Q_r h = -[(K*W_r)\nabla h + (K*h)\nabla W_r]$. Write $Q_G h$ for the same expression with $G_N$ replacing $W_r$. After cancellation of the retained monomials the residual is exactly

$$\begin{aligned} R^J = &-\sum_{j=1}^{M}\varepsilon^j\kappa^{N+1}\Delta g_{jN} - \sum_{\substack{1\le j\le M,\ 0\le\ell\le N,\ 1\le r\le N\\ \ell+r>N}}\varepsilon^j\kappa^{\ell+r}Q_r g_{j\ell}\\ &-\sum_{\substack{1\le j_1,j_2\le M,\ 0\le\ell_1,\ell_2\le N\\ j_1+j_2>M\ \text{or}\ \ell_1+\ell_2>N}}\varepsilon^{j_1+j_2}\kappa^{\ell_1+\ell_2}Q(g_{j_1\ell_1},g_{j_2\ell_2})\\ &+\varepsilon^{M+1}\sum_{\ell=0}^{N}\kappa^\ell B\cdot\nabla g_{M\ell}+\varepsilon B\cdot\nabla G_N - Q_G F. \end{aligned} \tag{153}$$

Here juxtaposition with a gradient means a dot product. The exact radial heat base has zero residual on its own. The integral heat Taylor formula gives

$$\|G_N\|_X + \|\nabla G_N\|_X \le C\kappa^{N+1}t^{N+1}.$$

Use the Biot–Savart $X$ bound and (151) to estimate $Q_G F$. Every remaining factor in (153), other than the heat remainder, is supported on the fixed collars. Pure terms with seed order exceeding $M$ are $Ca^{M+1}$; their pure coefficient bounds have no polynomial. A positive viscosity power multiplied by any of the finitely many polynomial factors is absorbed by increasing $d_R$ and requiring $\kappa(1+H)^{d_R}\le c$. Terms with viscosity order exceeding $N$ are $C\kappa^{N+1}(1+t)^{d_R}a$, since $a^j\le a$. The external top-seed term uses $\|B\| = e^{\sigma s}D$ and is $Ca^{M+1}D$. The external heat remainder is $C\kappa^{N+1}(1+t)^{d_R}aD$. This proves (152), including all viscosity terms and without assuming a pointwise bound on $D$. □

### 8.1.3 Comparison with the actual foreign velocity

Suppose the actual colour solves

$$\partial_t w + (K * w + b) \cdot \nabla w = \kappa \Delta w, \qquad w(0) = W + \alpha \Phi.$$

The actual $b$ is divergence free; it may depend on the entire interacting solution. Assume the usual positive-time smoothness and $C_t X$ initial trace, sufficient for the scalar Kato inequalities. A removed rigid rotation is allowed. Let $\mathcal{U}$ be a fixed annular neighbourhood of $\mathcal{C}$, separated from its complement by a positive radial heat-gradient gap. Assume

$$\sup_{x \in \mathcal{U}} |b - b_J| \le m_q, \qquad |b(t,x) \cdot e_r| \le C_b(1 + |x|), \tag{154}$$

with fixed $C_b$. The second is used only against the exact radial heat gradient outside $\mathcal{U}$. No derivative of $b$ is assumed. The stream extensions giving $B$ agree with the retained harmonics on $\mathcal{U}$. These are exactly the collar omission and actual global radial-growth inputs supplied by the packet/core estimates.

**Lemma 8.2** (Actual density comparison)**.** *Fix $C_1 = C_p(\|\nabla W\|_X + 1)$ before choosing $M, N, J$. Choose $(M+1)\sigma > C_1$. There is a later threshold $a_* > 0$ such that if $a(t) \le a_*$ and the finite viscosity smallness conditions hold, then $\|\nabla w_{\mathrm{app}}\|_X$ gives the fixed exponent $C_1$ and*

$$\begin{aligned} \|w - w_{\mathrm{app}}\|_X \le{}& C a^{M+1} + C\kappa^{N+1}(1+t)^{d_R+1} a e^{(C_1-\sigma)_+ t} \\ &+ C m_q (1+t) e^{C_1 t} + C(1+H) e^{C_1 H}(1 + \varepsilon e^{\sigma H}) e^{-c/(\kappa H)}. \end{aligned} \tag{155}$$

*The last Gaussian term is understood as zero at $H = 0$. Consequently in a scale family*

$$\varepsilon = q^r, \qquad \sigma t_0 \le \theta L, \qquad H \le C_H L, \qquad \kappa \le q^\gamma, \qquad m_q \le q^P L^{d_m},$$
$$r, \gamma, \chi > 0, \qquad \theta \ge 0, \qquad 0 < C_H < \infty.$$

*if*

$$\gamma(N+1) > C_H(C_1 - \sigma)_+ + \chi, \qquad P > r + \theta + C_H(C_1 - \sigma)_+ + \chi, \tag{156}$$

*then*

$$\|w - w_{\mathrm{app}}\|_X \le C a^{M+1} + C q^\chi L^{d_*} a, \qquad \|w - W_\kappa\|_X \le C a. \tag{157}$$

*These estimates require no nonzero transmission coefficient.*

*Proof.* The finite coefficient bounds make the pure perturbation gradient at most $1/2$ after choosing $a_*$ sufficiently small. Choose the later viscosity threshold so the remaining finite gradients are at most $1/2$. Heat contracts $\|\nabla W\|_X$. This yields the already fixed $C_1$; the coefficient constants change thresholds, not that exponent. For $z = w - w_{\mathrm{app}}$ the exact equation is

$$(\partial_t + (K * w + b) \cdot \nabla - \kappa\Delta) z = -(K * z) \cdot \nabla w_{\mathrm{app}} - R^J - (b - b_J) \cdot \nabla w_{\mathrm{app}}.$$

The actual foreign field is part of divergence-free transport and cancels from the $L^1, L^p$ norm inequalities. The Biot–Savart $X$ bound gives

$$(\|z\|_X)' \le C_1 \|z\|_X + \left\|R^J\right\|_X + \|(b - b_J) \cdot \nabla w_{\mathrm{app}}\|_X .$$

The initial error is zero. Smooth norm regularizations and compact cutoffs justify the integrated inequality from time zero.

On $\mathcal{U}$ the mismatch is at most $Cm_q(1+a)$. Outside it, $\nabla F = 0$ and the weighted radial heat-gradient norm is $Ce^{-c/(\kappa t)}$, after decreasing $c$ to absorb fixed polynomial heat factors. The actual radial-growth bound and $\|b_J\|_\infty \le aD$ therefore bound the omitted force by $C(1+aD)e^{-c/(\kappa t)}$. Convolve with $e^{C_1(t-v)}$. For the top pure residual, division by $a(t)^{M+1}$ leaves

$$\int_0^t e^{-[(M+1)\sigma - C_1](t-v)}(1+D(v))\,\mathrm{d}v \le [(M+1)\sigma - C_1]^{-1} + B_0.$$

For the viscous residual, division by $a(t)$ leaves at most $C(1+t)^{d_R+1}e^{(C_1-\sigma)_+ t}(1+B_0)$. The constant collar mismatch gives $Cm_q(1+t)e^{C_1 t}$. The Gaussian force uses $\int D \le B_0$ and monotonicity of its exponential, proving (155) after enlarging its harmless prefactor. In the scale family, $a(0) \ge q^{r+\theta}$ and (156) makes both non-Gaussian errors relative to $a(t)$ positive powers of $q$ times fixed logarithms. The Gaussian dominates all these fixed power losses uniformly for smaller viscosities. Finally the finite coefficient sum has norm $Ca$ and the error has the same bound, proving (157). □

### 8.1.4 A normalized finite polynomial loses the forcing waveform

Extend $B$ by zero after $H$, and put $S = H - t_0$. Suppose the decomposition holds on $-t_0 \le s \le S$, with the following future bound for $s \ge 0$:

$$\begin{aligned} B(t_0+s) &= B^{\mathrm{main}}(s) + E(s), \\ \left\|B^{\mathrm{main}}(s)\right\|_{C^R} &\le C_F(1+s)^{d_F}e^{hs}, \quad 0 \le h \le h_* < \sigma, \\ \int_{-t_0}^{S} e^{-\sigma s}\,\|E(s)\|_{C^R}\,\mathrm{d}s &\le \eta_q. \end{aligned} \tag{158}$$

The main and remainder are finite-sector fields of the same class as $B$ and are zero-extended after $S$. Before $s = 0$ only the total weighted bound (149) is used; no past convergence hypothesis is needed for the coefficient reset. Define the *full* first-order growing amplitude

$$C_q = \frac{\alpha}{\varepsilon}e^{\lambda t_0}\mathcal{A}(\Phi) + \int_0^H e^{-\lambda(t-t_0)}\mathcal{A}(-B(t)\cdot\nabla W)\,\mathrm{d}t. \tag{159}$$

It includes every retained input and the initial seed. It equals the infinite-future growing amplitude of the zero-extended first coefficient. It is not replaced by a canonical target coefficient.

Let $F_1(C) = CF_+ + \bar{C}\overline{F_+}$, and, for $j \ge 2$, define

$$F_j(C) = -(j\sigma - \mathcal{L}_{\mathrm{rot}})^{-1}\sum_{a+b=j}(K * F_a(C))\cdot\nabla F_b(C). \tag{160}$$

The inverse is the convergent integral of the finite-sector semigroup $e^{t(\mathcal{L}_{\mathrm{rot}} - j\sigma)}$. It is not a formal inverse. All coefficients are smooth, supported in $\mathcal{C}$, of zero mass, and belong to sectors $4n$ with $|n| \le j$ and $n \equiv j$ modulo two. Induction proves that they are real polynomials of total degree $j$ in $C, \bar{C}$, and that $F_j(C) = |C|^j \mathcal{T}_{\arg C} F_j(1)$ for $C \ne 0$.

**Lemma 8.3** (Finite coefficient memory loss)**.** *Under (149) and (158), for any fixed $0 < d < \min(\sigma - 2\eta, \sigma - h_*)$, the pure coefficients obey*

$$\left\|e^{-j\sigma s}\mathcal{T}_{-\tau s}g_{j0}(t_0+s) - F_j(C_q)\right\|_{H^{r_{j0}}} \le C_j(1+s)^{d_j}e^{-ds} + C_j\eta_q, \qquad 0 \le s \le S. \tag{161}$$

*The constants depend only on the fixed finite orders, input ball, future envelope, and the strict rate gap. They do not depend on the waveform, preparation length, or an incoming canonical-profile error. No lower bound on $|C_q|$ is required for this coefficient statement.*

*Proof.* The coefficient bounds imply that every normalized coefficient at $s = 0$ is uniformly bounded in its reserved order. Write $h_j(s) = e^{-j\sigma s}\mathcal{T}_{-\tau s}g_{j0}(t_0 + s)$. For $j = 1$ the growing component is $F_1(C(s))$, where the partial integral in (159) defines $C(s)$. Its missing main future tail has norm at most $C(1 + s)^{d_F}e^{-(\sigma-h)s}$; its missing remainder has norm at most $C\eta_q$. Its complementary component has propagator decay $e^{-(\sigma-2\eta)s}$ after normalization. Its main forcing has bound $C(1+s)^{d_F}e^{-(\sigma-h)s}$ in this frame. Duhamel convolution of these two decays is bounded by $C(1 + s)^{d_1}e^{-ds}$ for the stated strictly smaller $d$. The remainder convolution is bounded by $C\int_0^s e^{-\sigma v}\,\|E(v)\|_{C^R}\,\mathrm{d}v \le C\eta_q$. This proves the first case.

For $j \ge 2$ the normalized equation is exactly

$$h_j' = (\mathcal{L}_{\mathrm{rot}} - j\sigma)h_j - \sum_{a+b=j}(K * h_a)\cdot\nabla h_b - e^{-\sigma s}(\mathcal{T}_{-\tau s}B(t_0 + s))\cdot\nabla h_{j-1}.$$

Its homogeneous propagator decays as $Ce^{-(j-1)\sigma s}$ by (146). Subtract (160). Each quadratic difference has at least one lower difference, and its other factor is uniformly bounded in the needed reserved order. The reserve supplies one extra derivative in that lower factor, so (148) applies without loss. By induction these terms are bounded by $C(1 + s)^d e^{-ds} + C\eta_q$ in the target order. The direct main external term is $C(1 + s)^{d_F}e^{-(\sigma-h)s}$; its remainder has weighted time norm at most $C\eta_q$. Duhamel and $(j - 1)\sigma > d$ yield the same bound for the normalized difference, including the decaying bounded initial value at zero. Increasing a finite polynomial degree handles a possible convolution polynomial. This proves the finite induction and (161). □

### 8.1.5 Exact protected-coordinate matching

Define the real Banach-valued polynomial

$$\mathcal{P}_M(z) = W + \sum_{j=1}^{M} F_j(z), \qquad z \in \mathbb{C}.$$

Its protected coordinate is $\Psi_M(z) = \mathcal{A}(\mathcal{P}_M(z))$. It is a real polynomial in $z, \bar{z}$, is rotation equivariant, $\Psi_M(e^{i\theta}z) = e^{i\theta}\Psi_M(z)$, and $D\Psi_M(0)$ is the identity on $\mathbb{R}^2$. In particular it has the form $z[1 + O(|z|)]$; the parity actually eliminates its quadratic term, but that stronger fact is not needed.

**Lemma 8.4** (Exact polynomial and ancient cap gauge)**.** *There is a fixed $\rho > 0$ such that $\Psi_M$ has a unique inverse on $|\zeta| < \rho$, with inverse derivative uniformly bounded. Suppose $\|v - \mathcal{P}_M(z)\|_X \le e$, $|z| < \rho$, and $\mathcal{A}(v) = \delta e^{i\theta}$ with $0 < \delta \ll \rho$. Then for $z_* = \Psi_M^{-1}(\delta e^{i\theta})$,*

$$|z - z_*| \le Ce, \qquad \|v - \mathcal{P}_M(z_*)\|_X \le Ce. \tag{162}$$

*For the inherited canonical ancient orbit, Corollary 4.5 supplies every fixed finite jet*

$$\left\|\Omega_{\mathrm{ref}}(u) - \mathcal{P}_M(c_{\mathrm{ref}}e^{\lambda u})\right\|_X \le C_M|c_{\mathrm{ref}}e^{\lambda u}|^{M+1}, \qquad u \ll 0, \quad c_{\mathrm{ref}} \ne 0.$$

*If $u_\delta$ is its small exact cap, the rotation $\phi$ matching its protected coordinate exactly to $\delta e^{i\theta}$ satisfies*

$$\|v - \mathcal{T}_\phi\Omega_{\mathrm{ref}}(u_\delta)\|_X \le C[e + \delta^{M+1}]. \tag{163}$$

*Proof.* Choose $2\rho$ so small that $\sup_{|z|\le 2\rho}\|D\Psi_M(z)-I\|\le 1/4$. The integral mean-value formula then gives the quantitative inverse estimate $|\Psi_M(z)-\Psi_M(z')|\ge (3/4)|z-z'|$ in that ball. The inverse function theorem, or its contraction proof applied to $z=\zeta-[\Psi_M(z)-z]$, supplies the inverse on a smaller ball. Since $\mathcal{A}$ is bounded on $X$, the state error bounds the coordinate error by $Ce$, and the inverse estimate proves the first bound in (162). The derivative of $\mathcal{P}_M$ is bounded on this fixed ball, so the second follows.

At the canonical exact cap, its jet remainder is $O(\delta^{M+1})$ because $|c_{\mathrm{ref}}e^{\lambda u_\delta}|\asymp\delta$. The same inverse estimate changes its linear asymptotic parameter to the unique parameter matching its *full* protected coordinate, with error $O(\delta^{M+1})$. Rotation equivariance makes that parameter, after the exact coordinate-matching rotation, precisely $z_*$. The polynomial and ancient-state errors therefore add to the bound (163). Matching only the linear parameter would instead leave a lower-order error; it is not used here. □

### 8.1.6 State reset and the separate instantaneous cap condition

Combine the lemmas in a scale family with $\eta_q\le q^{\chi_E}L^{d_E}$, $\chi_E>0$, and $0<c_*\le|C_q|\le C^*$. Define

$$S_q=\sigma^{-1}\log\frac{\delta}{\varepsilon|C_q|},\qquad \delta=L^{-4},$$

and assume the prospective interval contains $t_0+[S_q-L_0,S_q+L_0]$, where $L_0$ is fixed. At a time in that window put $z(t)=\varepsilon e^{\lambda(t-t_0)}C_q$. The coefficient memory estimate, the positive-viscosity correctors, the comparison bound, and $\|W_\kappa-W\|_X\le C\kappa H$ give

$$\|w(t)-\mathcal{P}_M(z(t))\|_X\le C\delta^{M+1}+q^{\chi_*}L^{d_*},\tag{164}$$

for some fixed $\chi_*>0$. Indeed $S_q=(r/\sigma)L+O(\log L)$, so the intrinsic memory transient is a positive power of $q$ times fixed logarithms. The other errors have the strict powers already stated. The exponent $\chi_*$ is independent of an incoming canonical state error: that error may change the waveform and $C_q$, but is not used in the memory estimate. Exact coordinate matching by Lemma 8.4 then gives the same order of ancient-state reset at any exact cap in this window.

A weighted time norm alone does not supply a crossing slope. For a protected first-cap conclusion, impose the following *separate pointwise actual-field bounds* on the late window whenever $3\delta/4\le|\mathcal{A}(w)|\le 5\delta/4$, writing $f=w-W_\kappa$:

$$\left|\int T_{\mathrm{adj}}b\cdot\nabla W\right|+\int|f|\,|b|\,|\nabla T_{\mathrm{adj}}|+\left|\int T_{\mathrm{adj}}b\cdot\nabla(W_\kappa-W)\right|\le\rho_q\delta,\qquad\rho_q\to0.\tag{165}$$

These are instantaneous hypotheses, not consequences of (149). In the actual receiver application they follow, respectively, from the absolute packet moments and older collar bounds, the global weighted adjoint lemma, and the weighted smooth heat-profile estimate with the global radial-growth bound. These pointwise criteria are checked separately in the receiver application.

**Corollary 8.5** (Protected actual first cap)**.** *Assume the preceding scale-family bounds, the nonzero coefficient bound,* (165)*, and a sufficiently wide prospective interval. Then for small $q$ there is a unique first crossing $|\mathcal{A}(w(t_q))|=\delta$. It lies in the fixed late window and satisfies*

$$t_q=t_0+S_q+o(1),\qquad \frac{\mathrm{d}}{\mathrm{d}t}|\mathcal{A}(w)|\ge\sigma\delta/2\quad a.e.\quad \text{for } 3\delta/4\le|\mathcal{A}(w)|\le5\delta/4 \text{ in that window.}$$

*The upper slope is $C\delta$ almost everywhere, and its exact-cap state satisfies*

$$\left\|w(t_q)-\mathcal{T}_{\phi_q}\Omega_{\mathrm{ref}}(u_\delta)\right\|_X\le C\delta^{M+1}+q^{\chi_*}L^{d_*},$$

*where the canonical coordinate and actual coordinate are matched exactly. Any continuous family of actual solutions satisfying these uniform bounds has continuous cap times and phases.*

*Proof.* The all-time density bound and boundedness of $\mathcal{A}$ give $|\mathcal{A}(w)| \le Ca(t)$. Choose $L_0$ so large that $Ce^{-\sigma L_0}/c_* < 1/4$. Then no cap occurs for $t - t_0 \le S_q - L_0$. In the fixed late window (164) and the polynomial coordinate yield $\mathcal{A}(w) = z(t) + O(\delta^2) + o(\delta)$ uniformly. The moduli at the window endpoints straddle $\delta$, so continuity gives a crossing. The adjoint identity proved in Section 4.2, with (157) and (165), gives

$$\mathcal{A}' = \lambda\mathcal{A} + E_A, \qquad |E_A| \le C\delta^2 + \rho_q\delta + C\kappa(1+H)\delta.$$

The last term tends to zero relative to $\delta$. The coordinate is absolutely continuous. Its modulus derivative exists almost everywhere and equals $\sigma|\mathcal{A}| + \mathrm{Re}(\overline{\mathcal{A}}E_A)/|\mathcal{A}|$. On the stated level interval it is at least $3\sigma\delta/4 - C\delta^2 - o(\delta) \ge \sigma\delta/2$. The upper bound follows from the same identity. Let $C_{\mathrm{speed}}$ be its fixed upper-speed constant. For all sufficiently small $q$, the crossing lies in the middle half of the late window. Choose

$$d_{\mathrm{cap}} = \min\{L_0/4,\ (8C_{\mathrm{speed}})^{-1}\}.$$

On $[t_q - d_{\mathrm{cap}}, t_q + d_{\mathrm{cap}}]$ the radius stays in the level band: before a first exit its change is at most $C_{\mathrm{speed}}\delta d_{\mathrm{cap}} \le \delta/8$, whereas exit requires a change of $\delta/4$. The almost-everywhere lower slope integrates on this entire interval. At earlier times the envelope margin applies or the radius is below $3\delta/4$; a previous return from a larger radius would require a downward band crossing. Thus for each fixed $d \in (0, d_{\mathrm{cap}}]$ there is the explicit uniform precap margin

$$\sup_{0\le t\le t_q - d} |\mathcal{A}(w(t))| \le \delta - \min\{\delta/4,\ \sigma\delta d/2\}.$$

A second crossing within the protected window would require a downward passage through this level interval, contradicted by the strictly positive slope. The endpoint estimate also gives $t_q - (t_0 + S_q) = o(1)$. Apply the exact-gauge lemma to (164) at this crossing. Finally a uniform early margin, common crossing slope, and uniform convergence of the continuous coordinate under solution parameters prove continuity of the first hit; its nonzero coordinate makes its circle-valued phase continuous. □

The finite density and reset estimates use upper field-history bounds. The protected first-cap corollary additionally uses a nonzero $C_q$ and (165).

# 9 Transmission to an actual receiver

## 9.1 An actual receiver and the loss of source memory

The receiving step starts from the child preparation envelope and its phase-matched exact-cap state. The finite expansion is applied to the actual harmonic coefficients of the coloured Navier–Stokes evolution.

Write $L = \log(1/q)$, $\delta = L^{-4}$, $\alpha = q^K$, where $K > 6$, and fix the smooth reference $W$, its protected coordinate $\mathcal{A}$, its growing eigenvalue $\lambda = \sigma + i\tau$, and its canonical ancient orbit $\Omega_{\mathrm{ref}} := \Omega_{\delta_{\mathrm{ref}}}$. The reference amplitude $\delta_{\mathrm{ref}} > 0$ is fixed, sufficiently small, independently of $q$. After one fixed time translation and rotation, normalize the canonical orbit so $\mathcal{A}(\Omega_{\delta_{\mathrm{ref}}}(0)) = \delta_{\mathrm{ref}} > 0$.

The bilateral transform is multiplied by a nonzero factor by this normalization; its nonvanishing selection is therefore unchanged. Choose a fixed $\beta$ in a sufficiently large interval so that

$$C_* := \gamma_\epsilon \int_{\mathbb{R}} e^{-\mu_\beta u} B_4(\Omega_{\delta_{\mathrm{ref}}}(u))\,\mathrm{d}u \neq 0, \qquad \mu_\beta = [\sigma + i(\tau - 4/5)]/\beta.$$

All choices below are made before $q$.

### 9.1.1 Discounted histories of an arbitrary prepared actual child

The following conditional history estimate applies to a child prepared by an incoming packet. It uses its preparation envelope and exact-cap reset independently of the receiving-stage conclusion.

**Lemma 9.1** (Prepared-child reference history)**.** *Fix the smooth profile, its all-order canonical reference orbit, $\beta > 1$, logarithmic clock constants and finite spatial derivative orders. Write $L = \log(1/q)$, $\delta = L^{-4}$ and $\mu_\beta = [\sigma + i(\tau - 4/5)]/\beta$. Assume the actual colours belong to the geometric stack of Lemma 3.1, with the clock and parameter margins of Corollary 6.9 in child units. Use child units, with all older constant rotations removed. Let the actual child $w_c$ have normalized viscosity $\kappa_c \le q^{3/2}$, an exact cap time $\tau_c$ satisfying $c_0 L \le \tau_c \le C_0 L$, and the common geometric initial colour bounds. Suppose, for fixed finite trial constants $C_{\mathrm{past}}, C_{\mathrm{reset}} > 0$,*

$$\|w_c(t) - e^{\kappa_c t\Delta} W\|_X \le C_{\mathrm{past}} \delta e^{\sigma(t-\tau_c)}, \qquad 0 \le t \le \tau_c.$$

*Let $s_q < 0$ be the canonical exact $\delta$-cap time, and match the constant phase $\phi$ exactly in the protected coordinate. Suppose*

$$\|\mathcal{T}_{-\phi} w_c(\tau_c) - \Omega_{\mathrm{ref}}(s_q)\|_X \le E_q, \qquad E_q = C_{\mathrm{reset}} \delta^{M+1} + q^\chi L^D, \quad \chi > 0,$$

*and*

$$\delta^{-C_W/\sigma - 1/\beta}(1 + |s_q|)\,[E_q + q(1 + |s_q|)] \longrightarrow 0. \tag{166}$$

*Here $C_W$ is the fixed one-sided comparison constant on the small canonical ancient segment. The sufficient fixed choice $M + 1 > C_W/\sigma + 2$ has strict slack in this condition.*

*Put $t_{\mathrm{ref}} = \tau_c - s_q$. On the interval ending at this reference time, assume the actual inactive-prefix estimates and the geometric preparation localization hypotheses of Lemma 7.8, with strict clock slack. On the following prospective interval assume the actual parent/larger constant-core and harmonic conclusions on the provisional parent density stop $\|w_i - W_{\kappa_i}\|_X \le C_s \delta$, where $C_s > 0$ is fixed and finite. In particular the child's older radial drift in a fixed fraction of the parent core has coefficient at most $C(1 + C_s)\delta/\beta$. Use the Gaussian parent-radius handoff of Lemma 7.11 at $t_{\mathrm{ref}}$. Let the following child horizon satisfy*

$$0 \le U_q \le C_U \beta L.$$

*The complete age and parent-unit horizon are assumed to lie in the fixed logarithmic windows of the cited inactive/core estimates.*

*For every such shorter stopped interval the absolute differentiated main-input history is bounded by one fixed ball independent of the trial constants, after a later $q$ threshold depending on those constants. If the complete prospective interval has been reached and $U_q \to \infty$, define*

$$B_q(u) = \begin{cases} B_4(\mathcal{T}_{-\phi} w_c(t_{\mathrm{ref}} + u)), & -t_{\mathrm{ref}} \le u \le U_q, \\ 0, & \text{otherwise.} \end{cases}$$

*Then*

$$\int_{\mathbb{R}} e^{-\sigma u/\beta} |B_q(u) - B_4(\Omega_{\mathrm{ref}}(u))|\, du \longrightarrow 0. \tag{167}$$

*Consequently its complex discounted transform converges to the full canonical transform at $\mu_\beta$.*

*For clarity, $B_q$ is the actual physical history* truncated at the controlled prospective endpoint and zero-extended thereafter. *The canonical history in (167) is defined on the entire real line. No estimate for the genuine actual infinite-time future after parent density/core control has ended is asserted or needed.*

*The quantifiers are as follows. The fixed profile, reference, $\beta$, clocks and finite spatial orders determine the input ball first. For each fixed finite triple $(C_{\mathrm{past}}, C_{\mathrm{reset}}, C_s)$ satisfying the stated preparation/reset assumptions, there is a later $q$ threshold. Within that class every estimate and convergence above is uniform in depth, source phase and allowed nondistinguished seed homotopies. The threshold need not be uniform over unbounded trial constants. Thus the constants may be selected once from the finite-forcing output bounds and then one common $q$ chosen, without multiplying an inherited input constant at every descending stage.*

*Proof.* Write $L_q = -s_q$. The canonical exact-cap asymptotic gives

$$L_q = O(\log L), \qquad e^{-\sigma s_q/\beta} \le C\delta^{-1/\beta}.$$

All phases below have already been removed by $\mathcal{T}_{-\phi}$. Rigid rotations preserve every radial positive-envelope estimate.

*Localization before the reference.* On the preparation interval the child's own nonradial radial velocity has time integral at most $CC_{\mathrm{past}}\delta$. The descendant velocity contributes at most $C\beta q C_0 L$. The actual older nonrotational harmonic field and core leaks on a fixed child ball are smaller, by Corollary 6.9. For any fixed trial $C_{\mathrm{past}}$ these displacements tend to zero. The initial support and a fixed radial margin therefore give, at every preparation time, positive-envelope Gaussian tails outside one fixed ball. Conditional global origin-growth estimates control every fixed post-exit moment; their fixed $q$-power amplification is absorbed by the Gaussian. The exact radial heat base has the corresponding Gaussian moment tails. These conclusions use the actual positive scalar envelope and the common drift, not a gradient of the actual child.

On the cap-to-reference segment, Lemma 7.8 gives the one-sided $X$ comparison with the phase-matched canonical ancient solution

$$\sup_{0\le v\le L_q} \|\mathcal{T}_{-\phi} w_c(\tau_c + v) - \Omega_{\mathrm{ref}}(s_q + v)\|_X \le C\delta^{-C_W/\sigma}[E_q + q(1 + L_q)]. \tag{168}$$

Indeed the canonical segment has fixed derivative bounds and fixed compact support. On that support the descendant field is $O(\beta q)$, the actual larger nonrotational field tends to zero with a strict positive $q$ power, and $\kappa_c \le q^{3/2}$. The comparison forcing is therefore $O(q)$ with fixed parameters. Gronwall over $L_q$ costs at most $C\delta^{-C_W/\sigma}$. No derivative of an actual moment coefficient or actual colour occurs.

The canonical nonradial radial displacement on this segment is $O(\delta_{\mathrm{ref}})$; the comparison error contributes $o(1)$ by (166). Choose the fixed reference amplitude inside the fixed radial margin. Together with the descendant and older-field contributions, this gives the same fixed-radius positive-envelope moment reset at the reference. In particular no factor $\log\log(1/q)$ is introduced into its moment constants. The separate parent-radius handoff gives the Gaussian exterior-tail input required by Proposition 7.3; fixed-radius tightness alone is not used as a substitute for that input.

*The negative history.* Let $u = t - t_{\mathrm{ref}}$. For $u \le s_q$, angular symmetry gives $B_4(e^{\kappa_c t\Delta}W) = 0$. Inside the fixed preparation ball the whole-moment pairing is bounded by a fixed multiple of the perturbation $L^1$ norm. Outside it use the preceding Gaussian positive-envelope and heat tails. Thus

$$|B_q(u)| \le CC_{\mathrm{past}}\delta e^{\sigma(u-s_q)} + \mathcal{G}_q(u), \qquad -t_{\mathrm{ref}} \le u \le s_q,$$

where the Gaussian remainder is superpolynomial on the logarithmic preparation horizon. Its weighted integral is still superpolynomial: the earliest negative endpoint has size $O(L)$, so the discount costs only a fixed power of $q^{-1}$, whereas $\kappa_c \le q^{3/2}$. The main term has the explicit bound

$$\int_{-t_{\mathrm{ref}}}^{s_q} e^{-\sigma u/\beta} CC_{\mathrm{past}}\delta e^{\sigma(u-s_q)}\,du \le \frac{CC_{\mathrm{past}}\delta e^{-\sigma s_q/\beta}}{\sigma(1-1/\beta)} \le CC_{\mathrm{past}}\delta^{1-1/\beta} \longrightarrow 0.$$

The canonical moment on $(-\infty, s_q]$ has the same vanishing discounted bound, since it is $O(e^{\sigma u})$. Therefore the weighted absolute difference on the entire pre-small-cap interval tends to zero. This calculation replaces the selected-source identity involving $\alpha e^{\sigma t_{\mathrm{ref}}}$.

On $[s_q, 0]$, the fixed-radius localization converts (168) into a whole fourth-moment difference with the same bound plus Gaussian errors. Since

$$\int_{s_q}^{0} e^{-\sigma u/\beta}\,du \le C\delta^{-1/\beta}(1 + L_q),$$

its weighted absolute difference tends to zero by (166). The negative canonical history on this segment has one fixed discounted norm. This proves a negative-history input bound consisting of a fixed reference majorant, $CC_{\mathrm{past}}\delta^{1-1/\beta}$, and a vanishing reset contribution.

*The controlled future.* At reference the positive-envelope mass and fourth moment have fixed majorants. The own and descendant velocities have a fixed bounded sum by the individual scalar contractions and the convergent geometric velocity series. The older global radial drift is at most $C|y|/\beta$; inside the parent probe core its coefficient is $C(1 + C_s)\delta/\beta$, with Gaussian leaks. The parent-radius handoff supplies its initial exterior Gaussian tail. Keep the local radial coefficient $a = C(1 + C_s)\delta/\beta$ explicit in the proof of Proposition 7.3. Its unexited fourth moment has a prefactor determined only by the initial envelope moments and fixed source and descendant velocity bound, and a factor $e^{4au}$. All stop-dependent constants in the exited part are absorbed by its Gaussian after a later $q$ threshold. The proposition and Corollary 7.4 thus yield on every shorter stopped interval

$$|B_q(u)| \le C(1 + u)^4 e^{C(1+C_s)\delta u/\beta} + o(q^P), \qquad 0 \le u \le U_q,$$

for every fixed $P$. The prefactor is fixed by the reference-envelope majorants and bounded source and descendant speed. It does not multiply $C_s$. For each fixed trial $C_s$, a later threshold makes

$$C(1 + C_s)C_U\delta L \le \log 2, \qquad C(1 + C_s)\delta < \sigma/2.$$

Consequently the future absolute discounted input is bounded by a fixed multiple of $\int_0^\infty e^{-\sigma u/\beta}(1+u)^4\,du$, and its tail from $J$ is bounded by

$$C_{\beta,\sigma}(1 + J)^4 e^{-\sigma J/(2\beta)} + o(q^P).$$

The errors remain negligible under all logarithmic-window and fixed $q$-power losses. Spatial differentiation of a retained fourth-harmonic stream field differentiates only its fixed smooth

spatial basis, so the same argument supplies every fixed spatial derivative input norm. Together with the negative-history estimates, this gives a fixed input ball independent of the trial constants after their later thresholds. It is an upper bound only and requires neither a nonzero signed transform nor completion of a shorter density stop.

Once the complete prospective interval is available, every fixed window $[0, J]$ lies within it for small $q$. From (168), the actual reference state converges to the canonical reference state in $X$. On each such following window the canonical derivatives are bounded, viscosity tends to zero, and the actual older nonrotational and descendant fields on fixed support tend to zero. The same one-sided comparison therefore gives convergence in $C([0, J]; X)$. Positive-envelope radial localization on this fixed window, followed by conditional moment growth, gives uniformly vanishing fourth-moment tails outside a sufficiently large fixed ball. Whole fourth moments consequently converge on the window. All constants are uniform over the fixed prepared class.

The canonical positive history has its polynomial discounted tail. Truncate at $J$, apply fixed-window whole-moment convergence, and then let $J \to \infty$, using the uniform future tail just established. Combining this with the two negative-history pieces proves (167). The complex transform conclusion follows because $|e^{-\mu_\beta u}| = e^{-\sigma u/\beta}$.

Every estimate used geometric normalized mass, amplitude or velocity sums, fixed spatial orders and the common clocks. Its thresholds depend on the fixed trial constants, not on depth or phase. Selecting those constants once and then taking the minimum of the finitely many thresholds yields the stated invariant-class quantifiers. If the force origin is instead chosen at the actual protected crossing of the fixed reference level, the same comparison and canonical positive slope change its time and constant phase only by $o(1)$; this preserves the history and transform conclusions. □

**Proposition 9.2** (Actual receiving-stage implication)**.** *Consider actual colours $w_l$ at scales $A_l = A_1\beta^{l-1}$, $R_l = R_1 q^{l-1}$, in the common unforced Navier–Stokes velocity. Their initial conditions are the fixed smooth seeded profiles constructed in Section 3.1, with all seed amplitudes at most $\alpha$; every colour is real, fourfold symmetric and has zero circulation. Fix a receiving index $i$ and its immediate child $c = i + 1$. Work in receiver units and remove the common solid rotation of all older cores. Assume $\kappa_c \le q^{3/2}$ and hence $\kappa_i = \beta q^2 \kappa_c \le \beta q^{7/2}$.*

*The inductive child hypothesis is as follows. In child units it has an exact first cap $\tau_c$, with $|\mathcal{A}(w_c(\tau_c))| = \delta$,*

$$c_0 L \le \tau_c \le C_0 L, \qquad \left\| w_c(t) - e^{\kappa_c t \Delta} W \right\|_X \le C_{\mathrm{past}} \delta e^{\sigma(t-\tau_c)} \qquad (0 \le t \le \tau_c), \tag{169}$$

$$\inf_{\phi} \left\| w_c(\tau_c) - \mathcal{T}_\phi \Omega_{\delta_{\mathrm{ref}}}(s_q) \right\|_X \le C_{\mathrm{reset}} \delta^{M+1} + q^\chi L^D, \tag{170}$$

*where $C_{\mathrm{past}}, C_{\mathrm{reset}}$ are fixed finite trial constants, and $s_q < 0$ is the canonical exact-cap time $|\mathcal{A}(\Omega_{\delta_{\mathrm{ref}}}(s_q))| = \delta$; the phase in (170) is chosen in the same exact protected-coordinate gauge. Here $\chi > 0$ is fixed. Choose the fixed finite order $M$ so that $M + 1 > C_W/\sigma + 2$, where $C_W$ is the one-sided cap-to-reference comparison constant. Then (166) holds and controls the discounted history through the advance. The positive scalar envelope transported from $|w_c(0)|$ has at $\tau_c$ uniformly bounded moments of each fixed order and Gaussian tails outside a fixed ball, as supplied by the preparation localization lemma. All these constants are uniform in depth and in the allowed seed homotopies.*

*Choose fixed $C_0, C_2$ with $C_2 > C_0 > \max(K/\sigma, 6/\sigma)$, then $\beta, D$ sufficiently large so that $C_0 > 6\beta/[\sigma(\beta - 1)]$, the unconditional inactive closure holds and*

$$\epsilon_0 := C_2\sigma/\beta < K - 6, \quad C_2\lambda_B < K - 6, \quad C_2(\lambda_P + C_4/\beta) < 1, \quad \epsilon_0 < 1/2. \tag{171}$$

*For small $q$, the actual receiver has a unique protected first cap $|\mathcal{A}(w_i(t_i))| = \delta$, with*

$$t_i = t_0 + \sigma^{-1} \log \frac{\delta}{q^6 |C_*|} + o(1),$$

*where $t_0$ is the child's fixed-reference time in receiver units. The reference phase $\phi_c$ used below is measured in the receiver's frame at $t_0$. If $\theta_c$ is its phase in the child's frame with* all *older rotations removed, then $\phi_c = \theta_c + 4t_0/5$ modulo $2\pi$: the parent core rotation is $-1/5$ in receiver time. Thus the source field has factor $e^{4i(t-t_0)/5} e^{i\phi_c}$, rather than counting the reference rotation a second time. It obeys an exponential preparation envelope*

$$\left\| w_i(t) - e^{\kappa_i t \Delta} W \right\|_X \le C q^6 e^{\sigma(t-t_0)} \qquad (0 \le t \le t_i). \tag{172}$$

*In its exact-cap phase gauge its state satisfies*

$$\|w_i(t_i) - \mathcal{T}_{\phi_i} \Omega_{\delta_{\mathrm{ref}}}(s_q)\|_X \le C\delta^{M+1} + q^{\chi'} L^{D'}, \tag{173}$$

*for some fixed $\chi' > 0, D' < \infty$. The exponent $\chi'$ depends only on the fixed stage parameters, and may be chosen independently of the incoming reset exponent $\chi$. Advancing this cap to the fixed reference preserves an upper clock bound $C_0 L$, for small $q$. At the cap, and whenever the coordinate lies in $[3\delta/4, 5\delta/4]$ within its protected late window,*

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathcal{A}(w_i(t))| \ge \sigma\delta/2.$$

*The corresponding upper derivative bound is $C\delta$. The positive receiver envelope has uniformly bounded fixed moments and Gaussian tails at the cap. The cap time and phase are continuous under all allowed seed homotopies. If the child's reference phase is $\phi_c$, then the receiver cap phase is $\phi_c + \arg C_* + \tau S_q + o(1)$, uniformly, where $S_q = \sigma^{-1} \log[\delta/(q^6|C_*|)]$.*

*Proof.* Let $C_s \ge 1$ be a fixed finite trial density-stop constant and put

$$T_s = \inf\{t \ge 0 : \|w_i(t) - W_{\kappa_i}(t)\|_X = C_s\delta\}, \qquad \inf \varnothing = +\infty.$$

For small $q$ the initial density is $O(q^K) < C_s\delta$. While the child is current, Corollary 6.9 bounds the receiver independently of a signed child transfer. Its logarithmic horizon contains the child cap and its short reference advance by the strict clock slack. Lemma 7.8 therefore supplies the fixed-reference state, phase and positive-envelope moment reset before $T_s$. In receiver units,

$$t_0 \le C_0 L/\beta + O(\log L).$$

A shift to the child's actual protected crossing of the fixed reference level changes this time and phase by $o(1)$ and preserves these inputs. Fix a prospective half-width $L_0 > 0$, to be enlarged below, and define

$$S_q^* = \sigma^{-1} \log \frac{\delta}{q^6 |C_*|}, \qquad H_+ = t_0 + S_q^* + L_0, \qquad \widehat{H} = \min(H_+, T_s).$$

For small $q$, $H_+ \le C_2 L$. Corollary 6.9 controls all older colours on $[0, \widehat{H}]$. The receiver's nonradial mass is at most $2C_s\delta$ in normalized units, so its actual constant core follows from Proposition 5.3: $C_2 L\sqrt{C_s\delta} \to 0$ and $C_2 L\beta q \to 0$ for every fixed trial $C_s$. The associated harmonic and

Gaussian-leak estimates hold on this entire density-stopped interval, including any portion after a hypothetical coordinate crossing.

*Upper child-history bounds on shorter stops.* Apply Lemma 9.1 with $U_s = \beta(\widehat{H} - t_0) \le C_2\beta L$. The parent-radius Gaussian handoff supplies its initial exterior tail. In child units the own and descendant velocities have a fixed bounded sum, the global older radial drift is at most $C|y|/\beta$, and inside a fixed fraction of the parent core it is at most $C(1+C_s)\delta|y|/\beta$ plus Gaussian leaks. These bounds persist to $\widehat{H}$ because they require the density stop rather than a coordinate-cap restriction. The lemma gives

$$|B_c(u)| \le C(1+u)^4 e^{C(1+C_s)\delta u/\beta} + o(q^P), \qquad 0 \le u \le U_s,$$

for every fixed $P$, with the prefactor determined by the fixed reference moments and bounded source and descendant velocities, independently of $C_s$. For each fixed trial $C_s$, a later $q$ threshold makes $C(1+C_s)C_2\delta L \le \log 2$ and $C(1+C_s)\delta < \sigma/2$. Hence the future discounted input is in a fixed ball independently of the trial constant. The pre-small-cap contribution is $O(C_{\mathrm{past}}\delta^{1-1/\beta})$, and the weighted reset contribution tends to zero by (166). These upper bounds hold on every shorter density-stopped interval; no signed-transform lower bound or completion of the full interval has yet been used.

*All other actual harmonic coefficients.* In receiver units the inner packet has total mass $O(\beta q^2)$, $L^p$ norm $O(\beta q^{2/p})$, and global velocity at most $C\beta q$. Inside the protected receiving core the older/current radial remainder has bound $C(1+C_s)\delta r^3$ plus Gaussian leakage. Radial moment comparison and exit at a fixed core radius give, for every fixed $n$,

$$\int |x|^{4n} |\text{inner packet}(t,x)|\,\mathrm{d}x \le C_n\beta q^{4n+2}(1+H)^{4n} + o(q^P) \qquad (0 \le t \le H \le \widehat{H}). \tag{174}$$

Indeed its radial moment root is bounded, before exit, by $Ce^{C_n(1+C_s)\delta t}[q+\beta qt+\sqrt{\kappa_i t}]$; $\kappa_i \le \beta q^{7/2}$ makes the square-root term smaller than $q$ up to logarithms. At exit the probability is $e^{-c/(\kappa_i H)}$, and the global radial growth amplifies each fixed moment by only $e^{C_n H}$, which is absorbed by this Gaussian. This calculation uses the positive aggregate, not its nonexistent summed $L^\infty$ bound.

The descendants below the immediate child have mass $O(\beta q^2)$ in child units. The child's preparation envelope traps their positive aggregate in a fixed child ball up to its reference time; afterwards their fourth moment is bounded by

$$C\beta q^2(1+u)^4 e^{C(1+C_s)\delta u/\beta} + o(q^P).$$

Thus their fourth moment adds only

$$O(q^{2-\epsilon_0}L^D)$$

to the weighted child-reference transform. The factor $q^{-\epsilon_0}$ comes from the finite negative-history endpoint, whose source clock is at most $C_2L + O(\log L)$; its receiving discount is $\sigma/\beta$. This argument controls the actual entire descendant packet and does not assert a separate small-density bound for its colours.

Retain exterior harmonics $4, 8, \dots, 4J$ of the inner packet and interior harmonics $3, 7, \dots, 4J-1$ of the older colours. Use their *actual* time coefficients, and fixed smooth stream cutoffs agreeing on the derivative collars; call the resulting global field $b_J$. The leading child fourth harmonic gives the first-order coordinate forcing

$$\beta q^6 \gamma_\epsilon e^{4i(t-t_0)/5} B_c(\beta(t-t_0)).$$

For each finite $k$, all retained higher inner harmonics have weighted $C^k$ norm, divided by $q^6$, at most $C_J q^{4-\epsilon_0} L^D$ by (174). The unconditional inactive theorem bounds the actual older harmonics on these collars by

$$O(q^{K+2-C_2\lambda_B}) + O(q^{8-C_2(\lambda_P+C_4/\beta)} L^D).$$

After source recentering, their weighted norm divided by $q^6$ is $o(1)$ by (171). The own initial seed contributes only $O(q^{K-6-\epsilon_0} L^D)$ to the first growing coefficient. All these strict powers are uniform in depth.

For the collar density leak, the inner positive aggregate has Gaussian $L^1$ tail outside a fixed protected radius and conserved finite $L^p$ norm. Biot–Savart interpolation therefore gives a Gaussian velocity leak, despite its infinite sum of initial $L^\infty$ amplitudes. Older leaks have the individual core estimates. The omitted harmonic remainder is $O(q^{4J+6} L^D)$ for the inner packet, and gains arbitrarily many $q$-powers for the older Taylor series. Choose $J$ so large that, writing $P = 4J+6$,

$$P > 6 + \epsilon_0 + C_2(C_1 - \sigma)_+ + \chi_1$$

for a fixed $\chi_1 > 0$; older omission orders are chosen to meet the same inequality. This is a relative envelope requirement, not only an absolute endpoint error bound: the first envelope is $a(0) = q^6 e^{-\sigma t_0} \gtrsim q^{6+\epsilon_0} L^{-D}$, and a $q^P$ forcing error divided by $a(t)$ amplifies by at most $e^{(C_1-\sigma)_+ C_2 L}$. The fixed-profile Kato exponent $C_1$ is selected before $J$.

*The actual nonlinear density bootstrap.* Set $\varepsilon = q^6$, $s = t - t_0$ and $a(t) = \varepsilon e^{\sigma s}$. Choose the finite orders first. The main child history and fixed stream basis determine a fixed derivative input ball. For every fixed triple $(C_{\mathrm{past}}, C_{\mathrm{reset}}, C_s)$ the preceding estimates place the actual input in this same ball after a later $q$ threshold. Thus for every spatial derivative order needed in the finite recursion,

$$\int_0^H \varepsilon^{-1} e^{-\sigma(t-t_0)} \, \|b_J(t)\|_{C^k} \, \mathrm{d}t \le C_k. \tag{175}$$

This bound holds for every $H \le \widehat{H}$ and follows from Lemma 9.1 for the main harmonic and the strict power estimates for all other harmonics. Spatial differentiation acts on the fixed stream basis and leaves the actual time coefficients undifferentiated.

Apply Lemmas 8.1 and 8.2 to this actual run. In their notation take $\varepsilon = q^6$, $r = 6$, $\theta = \epsilon_0$, $C_H = C_2$ and $\gamma = 3$: $\kappa_i \le \beta q^{7/2} \le q^3$ for small $q$. For each retained coefficient use the explicit reserve $r_{j\ell} = k_* + 2(M-j) + 2(N-\ell)$ and $R = k_* + 2M + 2N + 3$, with $k_* = 4$ for the $X$ conclusion. The finite stream field has polar sectors at most $4J$; its order-$j$ coefficients therefore have sectors at most $4Jj$. The seed bound in (149) follows from $q^{K-6} e^{\sigma t_0} \, \|\Phi\|_{H^{r_{10}}} = o(1)$. The other input bound is exactly (175) for this finite derivative order $R$. Actual harmonics agree on the fixed broad annulus $\mathcal{U}$, with omission $m_q \le q^P L^D$; Gaussian density leaks can be absorbed in that bound. The actual global radial bound is $|b \cdot e_r| \le C(|x| + \beta q)$, uniformly in depth. Choose $(M+1)\sigma > C_1$ and a finite $N$ so that $3(N+1) > C_2(C_1 - \sigma)_+ + \chi_2$. The harmonic order was chosen to satisfy $P > 6 + \epsilon_0 + C_2(C_1-\sigma)_+ + \chi_2$. These are precisely the relative-error inequalities (156). The coefficient recursion including every viscosity term is (150); its exact cancelled residual is (153), not a pointwise assumed forcing remainder. The actual foreign field stays in the divergence-free error transport. Consequently (157) gives

$$\|w_i - w_{\mathrm{app}}\|_X \le C a(t)^{M+1} + q^{\chi_2} L^D a(t), \qquad \|w_i - W_{\kappa_i}\|_X \le C_a a(t). \tag{176}$$

No signed growing-coefficient lower bound is used in this application.

The constant $C_a$ is computed in the fixed input ball, before choosing $C_s$. Enlarge the fixed $L_0$ so that its half-width meets the early-margin requirement of Corollary 8.5, using the coefficient target lower bound $|C_*|/2$. Then choose

$$C_s > \max\{1, 2C_a e^{\sigma L_0}/|C_*|\}.$$

Only afterwards decrease $q$ to validate the input estimates and core margins for this fixed choice, as well as the fixed child preparation and reset constants. Since $a(H_+) = \delta e^{\sigma L_0}/|C_*|$, (176) gives $\|w_i - W_{\kappa_i}\|_X < C_s\delta/2$ throughout $[0, \widehat{H}]$. Continuity excludes $T_s \le H_+$. Thus the full deterministic prospective interval has been reached using upper input bounds alone.

*The complete actual child history and growing amplitude.* Now $\beta t_0 \to \infty$ and $U_+ = \beta(H_+ - t_0) \to \infty$. Extend the actual phase-measured child moment $B_c(u)$ by zero outside $[-\beta t_0, U_+]$. Lemma 9.1 proves weighted absolute convergence on this complete interval and therefore

$$\int_{\mathbb{R}} e^{-\mu_\beta u} B_c(u)\,\mathrm{d}u = e^{i\phi_c}\mathcal{F}_{\mathrm{ref}}(\mu_\beta) + o(1). \tag{177}$$

The zero extension is only of the controlled finite history; no estimate of the actual infinite future after $H_+$ is used. Extend the retained field $b_J$ by zero after $H_+$, and define its *full actual* first growing amplitude by

$$C_q = \frac{\alpha}{\varepsilon} e^{\lambda t_0}\mathcal{A}(\Phi) - \frac{1}{\varepsilon}\int_0^{H_+} e^{-\lambda(t-t_0)}\left[\int T_{\mathrm{adj}} b_J(t)\cdot\nabla W\,\mathrm{d}x\right]\mathrm{d}t. \tag{178}$$

This includes the own seed and every retained actual older, descendant and higher-sector input. Equivalently it is $\lim_{s\to\infty} e^{-\lambda s}\mathcal{A}(g_{10}(s + t_0))$ for this zero-extended first coefficient. Retained sectors other than $\pm 4$ have zero direct growing projection, but remain in its full spatial coefficient and in the nonlinear recursion. Its difference from the actual main-child discounted transform is $O(q^{\chi_4} L^D)$ for some fixed $\chi_4 > 0$, by the preceding power estimates. The main-child transform converges to the canonical value by (177); that convergence is only $o(1)$ here, since the finite ancient-truncation error may be logarithmic. Hence

$$C_q = e^{i\phi_c} C_* + o(1), \qquad |C_q| \ge |C_*|/2.$$

*Universal coefficient reset.* Use Lemma 8.3 with main field equal to the actual immediate-child fourth harmonic. Its post-reference $C^R$ norm is $C(1 + s)^4 e^{C(1+C_s)\delta s}$, so $h_* = \sigma/2$ is admissible for the final small $q$. Every other retained field has weighted remainder norm $\eta_q \le q^{\chi_3} L^D$ by the preceding strict power estimates. The pattern rotation is the vector pushforward/scalar rotation of Section 8.1; all angular sectors are rotated separately. The full coefficient in (178) equals (159), with the field zero-extended at $H_+$. Choose a fixed $0 < d < \min(\sigma - 2\eta, \sigma/2)$. Lemma 8.3 proves, with the explicit derivative reserve, convergence to $F_j(C_q)$ for each pure coefficient. At $s = S_q + O(1) \asymp L$ its transient is a positive $q$-power. The actual main waveform need not be close to a canonical waveform for this statement: only its finite input ball and future envelope are used. Its canonical closeness was used for the lower bound on $C_q$. Formula (164), with the positive-viscosity correctors and $\|W_{\kappa_i} - W\|_X \le C\kappa_i H_+$, gives a state error $C\delta^{M+1} + q^{\chi'} L^{D'}$ against the universal polynomial. Lemma 8.4 matches its *full* protected coordinate by the two-real-variable inverse, and identifies the matching all-finite-jet ancient state. This proves (173) once the exact first cap is found. Neither the incoming waveform error nor an incoming reset exponent enters this intrinsic transient exponent.

*Protected first crossing.* The global density envelope excludes a cap before $H_+ - 2L_0$ for the fixed $L_0$ chosen above. In the ensuing fixed window the settled first growing coordinate is $\varepsilon e^{\lambda s} C_q + O(a^2) + o(a)$. The polynomial future tail and the small weighted remainders make the error uniform there. Its moduli at the two ends straddle $\delta$. The actual adjoint-coordinate identity gives, when the coordinate is in $[3\delta/4, 5\delta/4]$ within this protected late window,

$$\mathcal{A}' = \lambda \mathcal{A} + O(\delta^2) + o(\delta).$$

For the instantaneous radial-base forcing, use (174), not merely the weighted history: the collar field is $O(q^6 L^D)$ plus the small older field. For the exact foreign cross-advection pairing use Lemma 7.1:

$$\int |f|\,|b|\,|\nabla T_{\mathrm{adj}}| \le C\chi_q \,\|f\|_1 = o(\delta), \qquad \chi_q = C\left(\beta q + \frac{\delta q^2 + q^4 + \varepsilon_{\mathrm{core}}}{\beta}\right).$$

This uses the global test weight and the actual older cores; it requires no claim that the receiver's own heat-base defect is Gaussian in its core. The heat-profile foreign correction is $O(\kappa_i H)$, using the global radial bound on $b$ and the weighted smooth heat Taylor bound. It is $o(\delta)$; the remaining heat errors and the quadratic self error have the stated orders too. Thus the three separate pointwise criteria of (165) hold with $\rho_q \to 0$. Corollary 8.5, applied with half-width $L_0/2$, proves the protected first crossing and its lower and upper slopes. Its center uses $|C_q|$; replacing that by $|C_*|$ changes the center by $o(1)$. Consequently this narrower window is contained in the controlled prospective interval for small $q$; no forcing beyond $H_+$ is used. This gives the asserted time and phase asymptotics. Uniqueness of recrossing is asserted in the protected window; no later Euler or Navier–Stokes return to the level is excluded.

For the universal polynomial, let $a = |z|$ denote the radius of its parameter in the growing phase gauge. Its coordinate modulus is $R(a) = a[1+O(a)]$. $\mathrm{d}R/\mathrm{d}a = 1+O(a) > 0$. Since $a' = \sigma a$, its time derivative is $\mathrm{d}R/\mathrm{d}t = \sigma a[1+O(a)]$. Matching the actual first coordinate therefore selects its time/amplitude and phase gauges uniquely; the matching error has the same orders as (173). This step uses the exact cap, rather than identifying it with the linear preparation time.

Finally, (172) gives integrated receiver nonradial drift $O(\delta)$; the younger global velocity gives $O(\beta q H)$, and the older field is smaller. The positive receiver envelope has Gaussian tails outside a fixed ball and bounded fixed moments at the cap by the radial exit lemma and global moment growth. The same uniform preparation and crossing estimates hold under the stated homotopies. Positive-viscosity continuous dependence, followed by the strictly positive crossing slope, proves cap-time and cap-phase continuity. Its own cap clock is $[6/\sigma + C_0/\beta]L + O(\log L) < C_0 L$; the additional reference advance is only $O(\log L)$, so the same strict clock margin remains. The lower clock bound persists since $t_i \ge (6/\sigma)L - O(\log L)$. This completes the actual receiving-stage implication. □

# 10 Uniform descent and fixed-datum phase selection

## 10.1 Uniform constants, the descending induction, and phase selection

We record the constant selection explicitly. An assertion that every finite stage has some finite constant would not suffice for the construction below. In this section the selected seed has amplitude exactly $\alpha$; other seed amplitudes may lie in $[0, \alpha]$.

**Proposition 10.1** (A common invariant class of actual stages)**.** *Fix $p > 2$, the smooth profile and a sufficiently small fixed reference orbit, normalized once by a time shift and rotation so that $\mathcal{A}(\Omega_{\mathrm{ref}}(0)) = \delta_{\mathrm{ref}} > 0$. There are choices of $K \in (6, 643/100)$, fixed $\beta > 1$, $q > 0$, and constants $C_{\mathrm{past}}, C_{\mathrm{reset}}, R_0, c_0, C_0, C_2 > 0$, all independent of the selected index $N$, such that the following holds. Put $\alpha = q^K$, $\delta = (\log(1/q))^{-4}$, $A_j = \beta^{j-1}$, $R_j = q^{j-1}$, and $\nu_N = q^{3/2} A_N R_N^2$. For arbitrary seed phases and amplitudes in $[0, \alpha]$, with the $N$th amplitude exactly $\alpha$, every actual colour $j = N, N-1, \dots, 1$ has a protected first cap. For the selected viscosity, $\kappa_j = q^{3/2}(\beta q^2)^{N-j} \le q^{3/2}$ for every $j \le N$. In its normalized older-core rotating frame, its cap time $\tau_j$ from the physical initial time obeys*

$$c_0 \log(1/q) \le \tau_j \le C_0 \log(1/q) < C_2 \log(1/q),$$

*and its preparation satisfies*

$$\left\| w_j(t) - e^{\kappa_j t \Delta} W \right\|_X \le C_{\mathrm{past}} \delta e^{\sigma(t-\tau_j)}, \qquad 0 \le t \le \tau_j.$$

*Its state in its exact-cap gauge lies within $C_{\mathrm{reset}} \delta^{M+1} + q^{\chi} (\log(1/q))^D$ of the corresponding state on $\Omega_{\mathrm{ref}}$, for fixed finite $M, \chi > 0, D < \infty$. The positive envelope is localized uniformly throughout preparation and at its fixed-reference time. Its fixed moments and fixed-radius Gaussian tails have common bounds; for $j \ge 2$ the separate parent-radius Gaussian tails also have common bounds. The first caps have common protection constants at each fixed physical scale and depend continuously on the seed parameters.*

*Proof.* Choose $K$, for example $K = 31/5$. The generic profile constants, including a one-sided $X$ comparison exponent $C_W$ uniform on a fixed small ancient segment and the finite residual exponent $C_1$, are fixed first. Fix a small reference on that segment; the finite expansion orders are selected below after the clock and amplitude ratios, and before the invariant class constants. The later advance from the shrinking actual cap to this reference uses one-sided comparison; it does not require the selected source's terminal expansion amplitude to exceed the reference amplitude. Choose fixed clocks $C_0 > K/\sigma$ and $C_2 > \max\{C_0, K/\sigma + 1\}$ with strict slack. Fix

$$0 < c_0 < \min\{K/\sigma, 6/\sigma\}.$$

Choose a sufficiently large interval for $\beta$ so that throughout it

$$\frac{6\beta}{\sigma(\beta - 1)} < C_0, \qquad K > 6 + \frac{K}{\beta - 1}, \qquad \epsilon_0 := C_2 \sigma/\beta < \min\{K - 6, 1/2\}.$$

The unweighted inactive rate is made small by this large amplitude ratio. After selecting $\beta$ as below, choose the fixed finite weight $D$ so that the weighted inactive rate and its remaining geometric rows satisfy

$$C_2 \lambda_B < K - 6, \qquad C_2(\lambda_P + C_4/\beta) < 1.$$

Their residual off-diagonal smallness is supplied by the later choice of $q$, as in Corollary 6.9. The bilateral canonical transform is analytic and not identically zero, so a $\beta$ in this interval can be chosen with $C_* \neq 0$. The preceding finite weight choice gives the required weighted-row smallness; the large $\beta$ controls the unweighted rate. All these parameters are fixed before $q$.

Fix a broad localization ball $B_{R_0}$ containing the initial profile and small ancient segment with a fixed radial margin. We verify that a common input class, rather than an inherited arbitrary constant, feeds the finite receiving recursion. Suppose provisionally that a child is in a trial class with past-envelope constant $C_{\mathrm{past}}$ and cap-reset constant $C_{\mathrm{reset}}$. Its own nonradial velocity

integrates to at most $CC_{\mathrm{past}}\delta$ before its cap. The younger velocity contributes $C\beta qC_2\log(1/q)$, and the older harmonic remainder is smaller. For any fixed trial constant these vanish as $q \downarrow 0$. The radial exit argument therefore gives the same localization ball and moment constants, independent of this trial constant, once $q$ is small. This holds at every preparation time, since the integrated radial drift up to that time has the same bound. For the stronger parent-radius tail, run a separate exit argument at radius $c/q$ in source units. The older stopped-core radial coefficient there is $C\delta/\beta$, and the source and descendants have bounded total speed. Their age is $O(\log(1/q))$, so the deterministic radial displacement is much smaller than $q^{-1}$. The exit probability is $C\exp[-cq^{-2}/(\kappa_j C_2\log(1/q))]$. Conditional global moment growth adds only fixed powers of $q^{-1}$; the same Gaussian, with reduced exponent constant, controls the parent-radius weighted tails.

Let $s_q$ be the reference orbit's exact $\delta$-cap time. Advancing to the fixed reference state takes $-s_q = O(\log\log(1/q))$. On this segment canonical nonradial velocity integrates to $O(\delta_{\mathrm{ref}})$, and the actual comparison error has integral tending to zero if $M$ is large enough. Choose $\delta_{\mathrm{ref}}$ small within the fixed radial margin. A second positive-envelope localization at the reference state hence gives the same fixed radius and universal moment constants, without a factor $\log\log(1/q)$.

In the reference source clock, the pre-small-cap contribution to the weighted forcing norm is at most

$$CC_{\mathrm{past}}\delta^{1-1/\beta}. \tag{179}$$

Indeed the past envelope is $C_{\mathrm{past}}\delta e^{\sigma v}$ in small-cap coordinates $v \le 0$, while the reference discount contributes $e^{-\sigma s_q/\beta} = O(\delta^{-1/\beta})$. The cap-to-reference segment is bounded by the fixed ancient envelope and the vanishing forward-comparison error. The positive reference history is bounded by Lemma 9.1 on the entire prospective interval through $\min(H_+, T_s)$, including the part after an attained coordinate cap. Here $T_s$ is the trial density stop of Proposition 9.2. Its moment prefactor is universal: the trial-stop factor $e^{CC_s\delta u/\beta}$ is at most two on $u \le C_2\beta L$ after a later decrease of $q$ for each fixed $C_s$. The controlled history is zero extended at its prospective endpoint; no uncontrolled future is used. Consequently every needed spatially differentiated forcing norm is bounded by

$$M_{\mathrm{input}} \le M_{\mathrm{ref}} + CC_{\mathrm{past}}\delta^{1-1/\beta} + o(1).$$

The constant $M_{\mathrm{ref}}$ depends only on the fixed reference/profile and fixed radius, not on either trial constant or depth. The small error depends on the trial reset constant, but its coefficient tends to zero. Once that full prospective interval has been reached, the main signed transform is uniformly within $o(1)$ of $e^{i\phi_c}C_*$; fixed-window comparison followed by the two discounted tails proves this uniformly on the trial class. Thus $|C_q| \ge |C_*|/2$ once a sufficiently long prospective receiving interval has been reached. The upper input bound, which is valid on any shorter stop, is used first to reach that interval.

Select all finite expansion orders before computing the class constants. Choose $M+1 > C_W/\sigma + 2$ and $(M+1)\sigma > C_1$; choose enough actual harmonic terms and viscosity orders that their generic residual losses are smaller than the entire growing envelope on $0 \le t \le C_2\log(1/q)$. In particular the omitted harmonic power $P$ satisfies $P > 6 + C_2\sigma/\beta + C_2(C_1-\sigma)_+ + 1$ for the receiving stage, and the analogous selected-source inequality with $a(0) = q^K$. There are only finitely many required derivatives and sectors.

Compute finite-expansion bounds using the fixed input ball $M_{\mathrm{input}} \le M_{\mathrm{ref}} + 1$. Choose $C_{\mathrm{past}}$ larger than twice the resulting cap-relative envelope constant, divided by the fixed lower growing coefficient, and larger than twice the selected-source past constant. Now choose $q$ so that the term (179) and all other input errors are at most one. The output past envelope improves this same

$C_{\mathrm{past}}$ ball. The fixed comparison and coefficient bounds give a finite output cap-reset constant independent of the incoming reset error: the source waveform changes $C_q$, which exact-cap time and phase normalization remove; the universal coefficient reset has only intrinsic positive-$q$-power transients. Choose $C_{\mathrm{reset}}$ larger than twice this output constant and the selected-source constant. Decrease $q$ again so that its inherited reset error advances to the fixed reference by $o(1)$. This constructs a genuinely invariant class of constants.

Choose a common positive exponent $\chi$ smaller than the intrinsic base and receiving exponents and take a common finite logarithmic power. No input reset exponent is successively lost.

Theorem 7.7 gives the actual base $j = N$ in this invariant class. Its normalized cap clock is $(K/\sigma)\log(1/q) + O(\log\log(1/q))$. Assume the class holds for $j+1$. Proposition 9.2, with the common input ball just established, supplies it for $j$. The receiving normalized clock from the physical initial time obeys

$$\tau_j \le \frac{\tau_{j+1}}{\beta} + \frac{6}{\sigma}\log(1/q) + O(\log\log(1/q)).$$

The geometric sum and strict clock slack keep it below $C_0\log(1/q)$ after a final decrease of $q$. The receiving growth interval gives a common positive lower clock $c_0\log(1/q)$. All core, tail, moment and inactive estimates therefore stay inside their $C_2$ windows, including the short reference advance. Descending induction proves the assertions for every finite $N$ with the same constants. Only finite older prefixes are stopped; all infinite younger packets were controlled by their convergent mass, $L^p$ and velocity sums.

Finally decrease $q$ so that $\beta q^{2/p} \le 1/2$ and $\beta q \le 1/2$. Then the datum series and every required geometric sum converge, and $\beta q^2 < 1$ gives the selected viscosities decreasing to zero.

The estimates hold throughout a fixed late interval on both sides of each prospective cap and give a strict early margin and an integrated positive radial slope on that late interval. The quantitative first-cap lemma yields common protection constants. At the outer physical scale they give fixed common numbers $e, h, c, L_{\mathrm{obs}} > 0$ in Lemma 2.8: choose a sufficiently small two-sided late half-width $e$, use the slope $c = \sigma\delta/2$ (after a harmless reduction), and take the early margin $h > 0$ furnished by the settled-coordinate and prewindow estimates. The upper scalar derivative gives a common Lipschitz bound $L_{\mathrm{obs}}$. At any other fixed scale, the normalized constants are converted by its fixed clock factor $A_j$. Positive-viscosity continuous dependence and the protected crossing imply continuous cap times and phases under seed homotopies. All estimates were uniform in those phases and in all nondistinguished amplitude reductions. □

**Proposition 10.2** (Surjective outer phase for each selected viscosity)**.** *With the preceding fixed parameters, fix $N \ge 2$, all phases except the $N$th, and put all seed amplitudes equal to $\alpha$. The map taking $\vartheta_N \in \mathbb{R}/(2\pi\mathbb{Z})$ to the unit phase of the outer colour at its first cap for viscosity $\nu_N$ is continuous and surjective.*

*Proof.* Multiply every seed except the distinguished $N$th seed by $r \in [0,1]$. The invariant actual-stage proposition applies throughout this homotopy. At $r = 0$ the initial datum is the radial stack plus just that seed. Rotational covariance of the unforced equation and its colour equations shows that changing its phase rotates the entire solution, leaves the cap time unchanged and rotates the outer cap coordinate once. Thus the outer phase map has degree one (with an immaterial sign if the opposite rotation convention is used). Common protected caps give a continuous circle-valued homotopy as $r$ goes to one. Its degree stays nonzero, and a continuous circle map of nonzero degree is surjective. For completeness, a map omitting a point factors through the circle minus that point, an interval, and is homotopic to a constant; it has degree zero. This proves the last assertion without a differentiability assumption. □

### 10.1.1 Proof of the non-unique-selection theorem

*Proof of Theorem 1.1.* Use the fixed parameters in Proposition 10.1. All actual seeds will have the same fixed amplitude $\alpha$. Set the outer seed phase to zero and provisionally every other phase to zero. The compact admissible stack lemma gives all required norm, support and circulation properties for every phase choice. Choose $T_f > C_2 \log(1/q)$ plus the common outer protection window. Let

$$a^\nu(t) = \int T_{\mathrm{adj}}(x) w_1^\nu(t,x)\,\mathrm{d}x$$

be the bounded smooth adjoint observable of the outer passive colour. There is no older-core rotation at this outer scale. This is one fixed bounded $C^2$ test with angular-four covariance, allowed in (4). Pairing convergence follows from $|\int T_{\mathrm{adj}}(f_n - f)| \le \|T_{\mathrm{adj}}\|_\infty \|f_n - f\|_1$. The scalar identity for this noncompact test is obtained using expanding cutoffs; its bounded derivatives and the bounded physical velocity make all cutoff errors tend to zero. The invariant stage theorem gives cap times in one compact subinterval of $(0, T_f)$, a common precap margin and common positive radial slope near the first crossing of radius $\delta$. The observable is uniformly Lipschitz, since its scalar equation gives

$$|(a^\nu)'| \le \|w_1(0)\|_1 \left[\|u^\nu\|_\infty \|\nabla T_{\mathrm{adj}}\|_\infty + \nu \|\Delta T_{\mathrm{adj}}\|_\infty\right].$$

The velocity bound is uniform by the fixed-data $L^1, L^p$ contractions and the kernel split, and $\nu_N \le 1$ after choosing $q$ small.

We choose sparse indices $N_k \to \infty$ and their phases successively. At the $k$th step keep all earlier assigned phases and vary only $\vartheta_{N_k}$. Proposition 10.2 permits its outer cap phase to be exactly $+1$ for even $k$, and $-1$ for odd $k$. Changing a phase at depth $N$ changes the initial $L^1 \cap L^p$ norm by at most $C\alpha A_N(R_N^2 + R_N^{2/p}) \to 0$. Choose each next index so far out that all subsequent tail phase changes alter every earlier selected-viscosity outer observable by less than the desired summable tolerance. This is possible by continuous dependence at each fixed positive viscosity and absolute convergence of the remaining initial tail. In detail, after step $k$ there are finitely many positive viscosities and continuity radii; choose $N_{k+1}$ so that the norm of the entire freely variable tail is below all those radii. The fixed smooth outer colour is the same in every datum. Its scalar continuity follows from the fixed-viscosity mild equation. For two velocities, put $h$ for their outer-colour difference and $v$ for their velocity difference. The heat-kernel gradient estimate gives, for $r = 1, 2$,

$$\|h(t)\|_r \le C\nu^{-1/2} \int_0^t (t-s)^{-1/2} \left[\|u(s)\|_\infty \|h(s)\|_r + \|v(s)\|_\infty \|w_1(s)\|_r\right] \mathrm{d}s.$$

Volterra comparison gives uniform colour continuity at fixed $\nu > 0$ from uniform velocity continuity; no actual colour gradient is assumed. At step $k$ choose a continuity radius $r_k > 0$ at that step's assigned datum, for its selected viscosity and target accuracy $2^{-k}$. Require the worst-case norm of every variable phase tail starting at $N_{k+1}$ to be below $\min_{i \le k} r_i$. Protected first-cap continuity then gives the stated phase tolerances. Tolerances may be chosen so the $k$th final cap phase differs from its target by at most $2^{-k}$.

The resulting infinite phase assignment defines one fixed $\omega_0$. No viscosity-dependent initial datum is used. The selected viscosities $\nu_{N_k} = q^{3/2}(\beta q^2)^{N_k - 1}$ tend to zero. The fixed-data compactness theorem yields subsequences of the even and odd selected solutions converging strongly in $C([0,T_f]; L^1 \cap L^p)$ to Euler solutions $\omega^+$ and $\omega^-$. Their velocities converge strongly, and the passive-colour theorem identifies the limits of $w_1^\nu$ as the unique bounded transport colour with the

fixed initial outer density, determined by the total Euler path. Thus their observables converge uniformly on $[0, T_f]$.

Extract also convergent cap times. Common precap protection and radial slope ensure each limiting time is the first crossing of radius $\delta$ for its limiting observable: before the common cap window there is a strict margin, and inside it the integrated radial lower slope excludes an earlier crossing. The limiting cap coordinates are respectively $+\delta$ and $-\delta$. If $\omega^+ = \omega^-$, uniqueness of the bounded passive colour in that total velocity gives the same observable. It would then have the same first cap time and coordinate, contradicting these two values. Therefore the total Euler paths differ. The compactness theorem already gives their strong initial trace and renormalization, completing the proof. □

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