

GLOBAL WEAK EULER FLOWS WITH FINITE SIGNED INITIAL VORTICITY

ABSTRACT. We construct infinitely many global weak solutions of the two-dimensional Euler equations on the plane for every finite signed initial vorticity μ_0 in the inhomogeneous space H^{-1} . The velocity approaches its Biot–Savart initial value $u_0 = K * \mu_0$ in the sense that $\|u(t) - u_0\|_{L^2} \rightarrow 0$ as $t \downarrow 0$, although u_0 itself need not belong to L^2 . The vorticity is integrable at every positive time, continuous into H^{-1} , and weakly-* continuous as a finite measure. For an L^1 initial density, it is strongly continuous in L^1 up to the initial time. We first construct an approximate Euler flow whose stress tends to zero at the initial time. We then remove this stress using moving dipoles, with corrections that are summable in velocity L^2 and vorticity L^1 . The corrections leave the velocity unchanged at the endpoints of each time interval. Nonzero circulation is allowed. No energy inequality, sign preservation, or Lagrangian transport property is imposed.

1. INTRODUCTION

The two-dimensional Euler equations describe the motion of an ideal incompressible fluid:

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0, \quad \operatorname{div} u = 0. \quad (1.1)$$

Here u is the velocity and p is the pressure. For a smooth solution, the scalar vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ satisfies

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

The absence of vortex stretching is the basic reason that the planar problem has a global classical theory. It does not, however, settle the existence problem for singular vorticity. A vortex sheet carries vorticity on a curve, rather than on a set of positive area. More generally, the initial vorticity may be a finite signed measure. The question is then whether such data generate a velocity solving the nonlinear Euler equation, and in what class that velocity should be sought.

For compactly supported measures, membership in $H^{-1}(\mathbb{R}^2)$ is the natural local kinetic-energy condition. In the presence of vorticity of both signs, this condition alone does not supply the compactness used in the classical vortex-sheet existence theory. The difficulty is not merely to define the initial velocity: it is to pass to the quadratic term without an uncontrolled concentration defect. Delort explicitly raised the question of removing the sign assumption in his contemporaneous account of the vortex-sheet theorem [13, §4, pp. II–10–II–11].

In this paper we take a different route. We construct weak velocity solutions directly, without requiring them to arise as limits of a specified regularization and without imposing an energy inequality. The initial vorticity may be any finite signed measure in the inhomogeneous space $H^{-1}(\mathbb{R}^2)$. At every positive time, the constructed vorticity is an L^1 function. The velocity difference $u(t) - u_0$ tends to zero in global L^2 as $t \downarrow 0$, and the

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initial vorticity is attained strongly in H^{-1} and weakly-* as a measure. The distinction between this existence statement and an admissible or transport-based vortex-sheet theory is essential.

1.1. The main result. Let $\mathcal{M} = \mathcal{M}(\mathbb{R}^2)$ denote the finite signed Radon measures, equipped with the total variation norm. Throughout, $H^{-1} = H^{-1}(\mathbb{R}^2)$ is inhomogeneous. We use the Biot–Savart kernel

$$K(x) = \frac{(-x_2, x_1)}{2\pi|x|^2}, \quad x \neq 0.$$

For $\mu_0 \in \mathcal{M} \cap H^{-1}$, the velocity given by the Biot–Savart law, $u_0 = K * \mu_0$, belongs to L^2_{loc} ; see Lemma 2.1. Nonzero circulation is allowed, so u_0 need not have finite global kinetic energy. We therefore estimate velocity differences in $L^2(\mathbb{R}^2)$, rather than the full velocity.

Weak velocity solutions. Below, a weak velocity solution with initial value u_0 is a divergence-free field $u \in C([0, \infty); L^2_{\text{loc}})$ with $u(0) = u_0$ such that, for every $T > 0$ and every divergence-free $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^2; \mathbb{R}^2)$,

$$\int_0^T \int_{\mathbb{R}^2} (u \cdot \partial_t \varphi + (u \otimes u) : \nabla \varphi) dx dt + \int_{\mathbb{R}^2} u_0 \cdot \varphi(0) dx = 0. \quad (1.2)$$

Theorem 1.1. *Let $\mu_0 \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ be arbitrary, with no restriction on its sign, and set $u_0 = K * \mu_0$. There are infinitely many distinct global weak velocity solutions u with initial value u_0 such that*

$$u - u_0 \in C([0, \infty); L^2), \quad (u - u_0)(0) = 0. \quad (1.3)$$

Their vorticity $\mu(t) = \text{curl } u(t)$ satisfies

$$\begin{aligned} \mu &\in C_{w^*}([0, \infty); \mathcal{M}) \cap C([0, \infty); H^{-1}), & \mu(0) &= \mu_0, \\ \mu &\in C((0, \infty); L^1). \end{aligned} \quad (1.4)$$

If μ_0 has an L^1 density, then $\mu \in C([0, \infty); L^1)$ as well.

At every $t \geq 0$, the velocity is given by the Biot–Savart law $u(t) = K * \mu(t)$, and total circulation is conserved: $\mu(t)(\mathbb{R}^2) = \mu_0(\mathbb{R}^2)$.

Here $C_{w^*}([0, \infty); \mathcal{M})$ means continuity against every test function in $C_0(\mathbb{R}^2)$. When a measure has a density, we use the same symbol for that density. The continuity assertions imply locally uniform bounds on $\|u(t) - u_0\|_2$, $\|\mu(t)\|_{H^{-1}}$, and $\|\mu(t)\|_{\mathcal{M}}$; for the last bound, use uniform boundedness on $C_0(\mathbb{R}^2)$. No total-variation continuity at the initial time is asserted for singular data: it would be incompatible with positive-time absolute continuity, as explained in Remark 6.3.

Scope of the theorem. The equation in the theorem is the weak *velocity* equation (1.2). We do not claim an energy inequality, preservation of either sign of vorticity, propagation of higher L^p norms, or renormalized transport. At the stated integrability, $u\mu$ need not belong to L^1_{loc} , and the ordinary distributional vorticity transport equation cannot simply replace (1.2). In particular, nonuniqueness here is not nonuniqueness within the bounded-vorticity class of Yudovich. Nor does the theorem identify a vanishing-viscosity limit or a physically selected continuation of a sheet.

1.2. Overview of the proof. There are two main steps: constructing an approximate solution whose stress vanishes at the initial time, and removing stress on a closed time interval without changing either endpoint velocity.

The first step starts from exact bounded-vorticity Euler flows with smoothed initial data. A usual nonnegative smoothing kernel is not sufficient for the compact tensor used to join adjacent scales. We therefore use a signed radial kernel of mass one and vanishing second moment. Two radial Poisson equations then provide an integrable, compactly supported symmetric tensor for the difference between adjacent smoothed velocities. The joining times are chosen between two competing scales: they must be long enough to make the gluing stress small, but short enough that each exact flow remains close to its initial velocity. An auxiliary sequence that decreases slowly enough makes this possible without any prescribed rate of approximation of μ_0 . This is carried out in Section 3.

The second step is local in time. We first hold the stress constant on each of finitely many time intervals, then approximate it in space by finitely many constant rank-one matrices supported on rectangles, up to a scalar matrix that can be absorbed into the pressure. A single dipole visits the rows of these rectangles in succession. Its radius stays constant during each pass, and its amplitude vanishes before the next pass begins. The mean force over a complete cycle approximates the required stress divergence. Integrating the force minus its cycle average gives a time correction that vanishes at both ends of the cycle. A compact radial potential spreads the force from the dipole radius to a larger scale with an L^1 stress bound that has no logarithmic loss. The outcome is the estimate

$$\|w\|_{C_t L_x^2} + \|\operatorname{curl} w\|_{C_t L_x^1} \leq C \|R\|_{L_t^\infty L_x^1}^{1/2},$$

with w equal to zero at both endpoints. Sections 4 and 5 contain this construction.

Finally, we iterate independently on the intervals $[2^j, 2^{j+1}]$. The corrections are summable simultaneously in velocity L^2 and vorticity L^1 . Their endpoint values are zero, so the intervals join without a temporal defect and the velocity remains unchanged at every dyadic time. The vanishing initial stress makes the total correction tend to zero at $t = 0$. These observations give the initial limits, the Biot–Savart identity, and circulation conservation in Section 6. To obtain nonuniqueness, we change the starting approximate flow near the fixed time $t = 1$ before repairing it. Since that time is an endpoint of a dyadic interval, the resulting solutions remain separated there; see Section 7.

1.3. Historical background and relation to earlier results.

Classical solutions and geometric vortex sheets. The classical planar theory, from Euler’s formulation [18] through the global existence results of Wolibner, Hölder, and Kato [40, 20, 21], leads to Yudovich’s bounded-vorticity theory [41]. On the plane, $L^1 \cap L^\infty$ vorticity gives a unique flow and controls the velocity at infinity. We use the whole-plane formulation of Crippa and Stefani [9, Definition 3.2 and Theorem 3.3], which does not require a globally L^2 background velocity.

For a vortex sheet, vorticity is concentrated on a curve and its strength is the tangential velocity jump. The Birkhoff–Rott formulation [28, 1] describes the curve’s motion by a singular integral. Analytic evolution and its limitations have been studied through local existence [33], global solutions in a special regime [17], and singularity formation and Sobolev ill-posedness [6]. These results concern the persistence or breakdown of a geometric sheet, rather than existence in the broader weak velocity class considered here.

Compactness and the sign of singular vorticity. DiPerna and Majda [14, 15] showed why an L^2_{loc} velocity bound alone does not suffice to pass to the nonlinear equation: weak convergence can leave concentration in the quadratic products. Vorticity bounds in L^p , $p > 1$, provide a compactness mechanism that is lost for arbitrary measures.

Delort's theorem [12] obtains a global weak solution for compactly supported, one-signed vorticity in H^{-1} . The sign condition excludes the concentration defect that would otherwise obstruct passage to the limit. The theorem also allows a regular component of either sign: Delort's seminar account records, following a remark of P.-L. Lions, an $L^1_{\text{comp}} \cap H^{-1}$ perturbation of the one-signed part [13, §4, p. II–9 and its footnote]. Vecchi and Wu further developed the L^1 theory [35]. The relevant distinction is therefore not simply signed versus unsigned functions, but the possible signs of the *singular* part and the compactness of its approximations.

Related existence arguments, Hardy-space estimates, and concentration-cancellation formulations appear in [25, 19, 30]. Special signed configurations can also be treated, including sheets with reflection symmetry and opposite distinguished signs in the two half-planes [24]. More general criteria exclude concentration [23]; quantitative refinements use sparseness to control compactness in H^{-1} [16]. We do not assert that arbitrary signed approximations satisfy these criteria. Instead, we correct their equation errors by explicit perturbations.

Weak nonuniqueness and convex integration. Scheffer's space-time compactly supported flow [29] and Shnirelman's time-compact flow on the torus [31] showed how far weak solutions can depart from classical ones. De Lellis and Székelyhidi introduced the differential-inclusion and convex-integration framework [10, 11]: first solve Euler with a Reynolds stress, then remove that stress by small velocity perturbations.

Wiedemann obtained infinitely many global weak solutions for every solenoidal L^2 datum on the torus [39, Theorem 1]. That construction imposes no vorticity bound, attains the initial datum only weakly in L^2 , and has an initial energy jump. The strong initial limit of the L^2 velocity difference and the positive-time L^1 vorticity in Theorem 1.1 are additional requirements.

For sheet data, Székelyhidi constructed admissible nonunique solutions from the flat sheet [34], but their vorticity is not a bounded measure. Mengual and Székelyhidi treated regular closed sheets without distinguished sign [27]. Mengual also obtained admissible nonuniqueness for suitable initial vorticity in $L^1 \cap L^p$ for every finite $p > 2$ [26]. In the last result, the L^p condition concerns the *initial datum*, not a propagated bound for the nonunique solutions.

Nonuniqueness with controlled vorticity. Vishik proved nonuniqueness in finite L^p vorticity classes for a *forced* equation [36, 37]. For unforced Euler, Bruè and Colombo obtained weak- L^1 vorticity [2], and Buck and Modena obtained vorticity in Hardy spaces below the L^1 threshold [4]. Neither class is the L^1 class of Theorem 1.1.

Bruè, Colombo, and Kumar crossed this threshold. They constructed unforced planar Euler flows with vorticity in $L^\infty_t L^p_x$ for some $p > 1$ [3, Theorems 1.1–1.3]. Their nonuniqueness result concerns a dense class of initial data on the torus and uses rapidly travelling, concentrated dipoles. We use their moving-dipole and time-averaging mechanism for a different purpose: to treat every finite signed initial measure in H^{-1} on the plane, with a strong initial velocity limit and possibly nonzero circulation. We require L^1 , rather than L^p with $p > 1$, control of the evolved vorticity.

The dipole itself is classical [8, 22]; its use to cancel stress after averaging in time is due to Bruè, Colombo, and Kumar [3, §§1.2, 2.2, 4.7–4.8]. The time correction has precedents in Buckmaster and Vicol [5, §4.3, (4.20)]; the integral of the force minus its cycle average used here appears in [3, (4.76)–(4.78)]. We provide all profile identities and estimates needed for our L^1 vorticity construction.

1.4. Notation and the Euler–Reynolds equation. Unless stated otherwise, spatial function spaces are over $\mathbb{R}^2$ and $\|f\|_p = \|f\|_{L^p(\mathbb{R}^2)}$. Vector and matrix norms are Euclidean. For vectors v, w and a matrix field S , our conventions are

$$\begin{aligned}\nabla^\perp &= (-\partial_2, \partial_1), & \operatorname{curl} v &= \partial_1 v_2 - \partial_2 v_1, & (\nabla v)_{ij} &= \partial_j v_i, \\ (v \otimes w)_{ij} &= v_i w_j, & (\operatorname{div} S)_i &= \sum_j \partial_j S_{ij}, & S : T &= \sum_{i,j} S_{ij} T_{ij}.\end{aligned}$$

We write I_2 for the identity matrix. A divergence-free velocity u and a symmetric stress R form an *Euler–Reynolds pair* if there is a scalar distribution p such that

$$\partial_t u + \operatorname{div}(u \otimes u) + \nabla p = \operatorname{div} R \quad (1.5)$$

in distributions. All pressures below are spatial pressures; their addition by functions of time is immaterial. Replacing R by $R - qI_2$ replaces p by $p - q$. We will write the pressure explicitly when deriving the principal identities, but no pressure estimate is needed for passage to the final weak velocity equation.

A letter C denotes a constant that may change from line to line. Constants for a fixed finite rectangle approximation may depend on that approximation. In contrast, the coefficient in the norm estimate of Lemma 5.1 depends only on the fixed profiles, not on the old velocity or the chosen rectangles. This distinction will be maintained when selecting the parameters.

2. WHOLE-PLANE ESTIMATES AND BOUNDED-VORTICITY FLOWS

The whole-plane normalization matters at two points. First, finite vorticity need not generate a globally square-integrable velocity. Second, a square-integrable velocity correction cannot change the total circulation. We record both facts before turning to the exact flows used in the initial approximation.

2.1. Biot–Savart velocities and L^2 differences. We take

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx, \quad m(\xi) = -\frac{i\xi^\perp}{|\xi|^2}.$$

Then $\widehat{K} = m$. We normalize the inhomogeneous norm by

$$\|f\|_{H^{-1}}^2 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{|\widehat{f}(\xi)|^2}{1 + |\xi|^2} d\xi.$$

Lemma 2.1 (Biot–Savart velocities and an L^2 div–curl principle). *For $\mu \in \mathcal{M}$, the convolution $K * \mu$ is defined almost everywhere, belongs to L^1_{loc} , and has divergence zero, curl μ , and Fourier transform $m\widehat{\mu}$. If also $\mu \in H^{-1}$, then $K * \mu \in L^2_{\text{loc}}$.*

If $z \in L^2(\mathbb{R}^2; \mathbb{R}^2)$, $\operatorname{div} z = 0$, and $\nu = \operatorname{curl} z \in \mathcal{M}$, then

$$z = K * \nu, \quad \nu(\mathbb{R}^2) = 0. \quad (2.1)$$

Proof. For every bounded ball B , Tonelli's theorem gives

$$\int_B \int_{\mathbb{R}^2} |K(x-y)| d|\mu|(y) dx \leq \|\mu\|_{\mathcal{M}} \sup_y \int_B |K(x-y)| dx < \infty.$$

The kernel identities yield the distributional claims. They extend from compactly supported measures to arbitrary finite measures by truncation in total variation: the Fourier transforms converge uniformly, and m is locally integrable and integrable against rapidly decreasing test functions.

Choose a smooth frequency cutoff χ equal to one near zero. Since $|\hat{\mu}| \leq \|\mu\|_{\mathcal{M}}$, $\chi m \hat{\mu} \in L^1$. The H^{-1} assumption gives $(1-\chi)m \hat{\mu} \in L^2$. Fourier inversion therefore writes $K * \mu$ as a C_0 field plus an L^2 field, proving local L^2 . This decomposition also shows why the inhomogeneous H^{-1} assumption is sufficient: the low frequencies are controlled by the finite mass of the measure, while the Sobolev assumption controls the high frequencies.

For the second assertion, the Fourier div-curl equations imply $\hat{z} = m \hat{\nu}$ almost everywhere away from the origin. Consequently

$$\|z\|_2^2 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{|\hat{\nu}(\xi)|^2}{|\xi|^2} d\xi.$$

Continuity of $\hat{\nu}$ at zero and the logarithmic divergence of $\int_{B_1} |\xi|^{-2} d\xi$ force $\hat{\nu}(0) = \nu(\mathbb{R}^2) = 0$. The locally integrable function $m \hat{\nu}$ is the Fourier transform of $K * \nu$ and agrees almost everywhere with $\hat{z}$. Thus $z = K * \nu$. $\square$

In particular, nonzero circulation rules out a globally L^2 Biot-Savart velocity. We will never infer such a bound from an inhomogeneous H^{-1} bound. Instead, all successive corrections will belong to the Banach space

$$\mathcal{X} = \{z \in L^2(\mathbb{R}^2; \mathbb{R}^2) : \operatorname{div} z = 0, \operatorname{curl} z \in L^1\}, \quad \|z\|_{\mathcal{X}} = \|z\|_2 + \|\operatorname{curl} z\|_1. \quad (2.2)$$

This space is complete because divergence and curl pass to distributional limits. We also use

$$\|\operatorname{curl} z\|_{H^{-1}} \leq \|z\|_2 \quad (z \in L^2). \quad (2.3)$$

2.2. Velocity bounds and time increments. Let

$$\mathbb{P} = I - \nabla \Delta^{-1} \operatorname{div}$$

be the Leray projection. For $1 < p < \infty$, the operators $\mathbb{P}$ and $-K * \operatorname{div}$ are bounded on L^p , since their entries are combinations of Riesz transforms. These bounds follow from the classical Calderón-Zygmund theory [7]; see also [32, Chapters II–III] for the Riesz-transform formulation. On L^2 their operator norms are at most one. No boundedness of either operator on L^1 is used.

For $\nu \in L^1 \cap L^\infty$ and $2 < p \leq \infty$, splitting K at radius r and applying Young's inequality gives

$$\|K * \nu\|_p \leq C_p \left(r \|\nu\|_p + r^{2/p-1} \|\nu\|_1 \right).$$

Interpolation of $\|\nu\|_p$ and the choice $r = (\|\nu\|_1 / \|\nu\|_\infty)^{1/2}$ yield

$$\|K * \nu\|_p \leq C_p \|\nu\|_1^{1/2+1/p} \|\nu\|_\infty^{1/2-1/p}, \quad 2 < p \leq \infty. \quad (2.4)$$

The zero datum is treated separately.

Lemma 2.2 (Exact bounded-vorticity backgrounds). *Every signed $\nu_0 \in L^1 \cap L^\infty$ has a global Euler evolution ν , with velocity $V = K * \nu$, such that every vorticity L^p norm, $1 \leq p \leq \infty$, is conserved. On every finite horizon,*

$$\nu \in C([0, T]; L^1 \cap L^2), \quad V \in C([0, T]; L^\infty \cap L^3), \quad V - V(0) \in C([0, T]; L^2).$$

For $p = 2, 3/2$ and $s, t \in [0, T]$,

$$\|V(t) - V(s)\|_p \leq C_p |t - s| \|V\|_{L_{t,x}^\infty} \|\nu_0\|_p, \quad C_2 = 1. \quad (2.5)$$

Only the difference, not necessarily $V(t)$, belongs to these global L^p spaces.

Proof. We use the bounded-vorticity theory originating with Yudovich [41]. For the precise whole-plane statement, apply Crippa and Stefani [9, Theorem 3.3 and Definition 3.2]. Its kernel assumptions hold for K : it is divergence free and

$$|K(z)| = (2\pi|z|)^{-1}, \quad |K(z) - K(w)| = \frac{|z - w|}{2\pi|z||w|}.$$

The initial vorticity belongs to $L^1 \cap L^3$, and the kernel estimates in that theorem give a bounded Hölder velocity. Its L^∞ vorticity bound improves the modulus to the log-Lipschitz one. The resulting velocity has an Osgood log-Lipschitz modulus and a unique continuous flow X_t . Since $\operatorname{div} V = 0$, this flow preserves Lebesgue measure. One can verify this by spatially mollifying V : the smooth flows preserve volume, and Osgood stability gives convergence of flows and inverse flows on compact sets. Bounded displacement permits passage to the limit against compactly supported continuous functions.

Choose the representative defined at every time by $\nu(t) = \nu_0 \circ X_t^{-1}$. Volume preservation proves norm conservation. Approximation of ν_0 in L^1 by compactly supported continuous functions, together with continuity of the inverse flow and bounded displacement, proves L^1 time continuity. Then

$$\|\nu(t) - \nu(s)\|_2^2 \leq 2 \|\nu_0\|_\infty \|\nu(t) - \nu(s)\|_1$$

proves L^2 continuity. Applying (2.4) to vorticity and to its time differences proves the L^∞ and L^3 velocity claims.

Integrate the vorticity equation and convolve with K :

$$V(t) - V(s) = -K * \operatorname{div} \int_s^t V(\tau) \nu(\tau) d\tau.$$

The preceding L^p operator bounds prove (2.5). Finally $\partial_t V, \nabla V \in L^\infty(0, T; L^2)$ and V is bounded. Transport implies

$$\operatorname{curl}(\partial_t V + (V \cdot \nabla)V) = 0,$$

and the acceleration is therefore a spatial gradient. Thus there is a scalar distribution p_V with

$$\partial_t V + \operatorname{div}(V \otimes V) + \nabla p_V = 0.$$

This recovers the velocity equation without assuming $V \in L^2$. For example, since $V \in L^3$, a pressure representative can be chosen by applying the double Riesz transforms to $V \otimes V$; it belongs to $L^{3/2}$ on each finite time interval. $\square$

3. AN APPROXIMATE FLOW WITH VANISHING INITIAL STRESS

We now construct the initial Euler–Reynolds pair. Merely smoothing u_0 would give an approximate velocity, but not the small stress required for a strong initial trace after iteration. The useful approximation consists of exact Euler flows at successively smaller vorticity scales, joined over suitably chosen time intervals. The next lemma supplies the compact tensor needed for the joining term.

3.1. A compact smoothing kernel with the required cancellation. Choose a nonnegative radial $a \in C_c^\infty(\mathbb{R}^2)$ with integral one. For any spatial kernel f , write $f_\varepsilon(x) = \varepsilon^{-2}f(x/\varepsilon)$. Set

$$b = \frac{4a - a_2}{3}, \quad C_b = \|b\|_1. \quad (3.1)$$

Then b is real, radial, compactly supported, and smooth, with

$$\int b \, dx = 1, \quad \int |x|^2 b(x) \, dx = 0. \quad (3.2)$$

Indeed, dilation by a factor of two multiplies the second moment by four. The coefficients in (3.1) therefore preserve the mass while cancelling that moment. A nonnegative nontrivial smooth kernel cannot have this cancellation. No positivity of the approximating vorticity will be used.

Lemma 3.1 (Smoothing and an adjacent-scale tensor). *Let $\mu_0 \in \mathcal{M} \cap H^{-1}$ and $u_0 = K * \mu_0$. Then*

$$e(\varepsilon) := \varepsilon^2 \|b_\varepsilon * \mu_0\|_\infty \longrightarrow 0, \quad \|K * (b_\varepsilon * \mu_0) - u_0\|_2 \longrightarrow 0. \quad (3.3)$$

There are symmetric tensors $B_\varepsilon \in C_c^\infty$ satisfying

$$\begin{aligned} \operatorname{div} B_\varepsilon &= K * (b_\varepsilon - b_{\varepsilon/2}), \\ \|B_\varepsilon\|_1 &\leq C\varepsilon^2, \quad \|\operatorname{div} B_\varepsilon\|_{3/2} \leq C\varepsilon^{1/3}. \end{aligned} \quad (3.4)$$

After convolution with μ_0 , the last two estimates acquire only the factor $\|\mu_0\|_{\mathcal{M}}$.

Proof. For $0 < \varepsilon \leq 1$, $\|b_\varepsilon\|_{H^1} \leq C\varepsilon^{-2}$ and $\|b_\varepsilon\|_2 = C\varepsilon^{-1}$. Given $\gamma > 0$, choose $f \in L^2$ with $\|\mu_0 - f\|_{H^{-1}} < \gamma$. Uniformly in spatial translations, Sobolev duality and Cauchy–Schwarz give

$$e(\varepsilon) \leq C\gamma + C\varepsilon \|f\|_2.$$

First let $\varepsilon \downarrow 0$, then $\gamma \downarrow 0$.

Radiality and (3.2) imply that all moments of orders one, two, and three vanish. Thus $|1 - \widehat{b}(\zeta)| \leq C \min\{|\zeta|^4, 1\}$. For $\varepsilon \leq 1$,

$$(1 + |\xi|^2)|m(\xi)|^2 |\widehat{b}(\varepsilon\xi) - 1|^2 \leq C.$$

The multiplier on the left tends to zero at each nonzero frequency. Plancherel’s theorem and dominated convergence against the integrable function $(1 + |\xi|^2)^{-1}|\widehat{\mu}_0(\xi)|^2$ prove the second limit. This argument controls the global L^2 difference, including its low-frequency part.

We construct the tensor by two compact radial Poisson solves. Put $f = b - b_{1/2}$. It is radial and has zero mass and zero second moment. The radial equation $\Delta g = f$ has a

smooth solution chosen to vanish outside a fixed ball, because

$$g'(\rho) = \frac{1}{2\pi\rho} \int_{B_\rho} f \, dx$$

vanishes for all radii larger than the support radius of f . We choose the remaining additive constant so that g is zero there. Near the origin the radial formula gives a smooth function, because f is smooth and radial. Integration by parts gives

$$\int g \, dx = \frac{1}{4} \int |x|^2 f(x) \, dx = 0.$$

A second radial solve therefore gives $h \in C_c^\infty$ with $\Delta h = g$, hence $\Delta^2 h = f$. Set

$$h_\varepsilon(x) = \varepsilon^2 h(x/\varepsilon), \quad B_\varepsilon = \nabla(\nabla^\perp h_\varepsilon) + [\nabla(\nabla^\perp h_\varepsilon)]^T.$$

Then

$$\operatorname{div} B_\varepsilon = \nabla^\perp \Delta h_\varepsilon = K * (b_\varepsilon - b_{\varepsilon/2}).$$

For the last equality, the compactly supported field $\nabla^\perp \Delta h_\varepsilon$ is divergence-free and has curl $b_\varepsilon - b_{\varepsilon/2}$, so Lemma 2.1 applies. Moreover,

$$B_\varepsilon(x) = B_1(x/\varepsilon), \quad \operatorname{div} B_\varepsilon(x) = \varepsilon^{-1}(\operatorname{div} B_1)(x/\varepsilon).$$

Consequently their L^1 and $L^{3/2}$ norms scale as ε^2 and $\varepsilon^{1/3}$, respectively. This proves (3.4). Young's inequality proves the claims after convolution with a finite measure. $\square$

3.2. Joining exact Euler flows.

Proposition 3.2 (Approximate flow with vanishing initial stress). *For each datum in Theorem 1.1 there is a positive-time Euler–Reynolds pair (u^s, R^s) such that*

$$\begin{aligned} u^s - u_0 &\in C([0, \infty); L^2), & (u^s - u_0)(0) &= 0, \\ R^s &\in C([0, \infty); L^1), & R^s(0) &= 0, \\ u^s &\in C((0, \infty); L^3), & \operatorname{curl} u^s &\in C((0, \infty); L^1), \\ \sup_{t>0} \|\operatorname{curl} u^s(t)\|_1 &\leq C_b \|\mu_0\|_{\mathcal{M}}. \end{aligned} \tag{3.5}$$

For $t > 0$, the approximate velocity satisfies $u^s(t) = K * \operatorname{curl} u^s(t)$. If $\mu_0 \in L^1$, then also $\operatorname{curl} u^s \in C([0, \infty); L^1)$ with value μ_0 at zero.

Proof. Put $\varepsilon_n = 2^{-n}$ and $e_n = e(\varepsilon_n)$, $n \geq 0$. Let (U_n, ν_n) be the exact global background with $\nu_n(0) = b_{\varepsilon_n} * \mu_0$. Constants in this proof may depend on b and $\|\mu_0\|_{\mathcal{M}}$, but not on n or time. Norm conservation and (2.4) give

$$\begin{aligned} \|\nu_n(t)\|_1 &\leq C_b \|\mu_0\|_{\mathcal{M}}, & \|\nu_n(t)\|_\infty &= \varepsilon_n^{-2} e_n, \\ \|U_n(t)\|_\infty &\leq C \varepsilon_n^{-1} e_n^{1/2}, & \|U_n(t)\|_3 &\leq C \varepsilon_n^{-1/3} e_n^{1/6}. \end{aligned} \tag{3.6}$$

Interpolation of vorticity norms in (2.5) then yields

$$\begin{aligned} \|U_n(t) - U_n(0)\|_2 &\leq C t \varepsilon_n^{-2} e_n, \\ \|U_n(t) - U_n(0)\|_{3/2} &\leq C t \varepsilon_n^{-5/3} e_n^{5/6}. \end{aligned} \tag{3.7}$$

Choice of the joining times. The stress associated with changing the smoothing scale has size ε_n^2/t_n . On the other hand, the exact flow changes by an amount of order $t_n\varepsilon_n^{-2}e_n$. We need both quantities to tend to zero. Since $e_n \rightarrow 0$ can be arbitrarily slow, choose a positive sequence α_n such that

$$\alpha_n \rightarrow 0, \quad \frac{1}{2}\alpha_n \leq \alpha_{n+1} \leq \alpha_n, \quad \alpha_n^2 \geq \max\{e_n, e_{n+1}\}.$$

Such a sequence always exists; its construction is given in Appendix A.1. Set

$$E_n = \max\{e_n, e_{n+1}\}, \quad t_n = \frac{\varepsilon_n^2}{\alpha_n}.$$

Then $t_n \downarrow 0$, $t_{n+1}/t_n \in [1/4, 1/2]$, and

$$\frac{\varepsilon_n^2}{t_n} = \alpha_n \rightarrow 0, \quad t_n \varepsilon_n^{-2} E_n = \frac{E_n}{\alpha_n} \leq \alpha_n \rightarrow 0. \quad (3.8)$$

Thus both the joining stress and the change in the exact flows tend to zero, without assuming a decay rate for e_n . Choose smooth increasing cutoffs χ_n on $[t_{n+1}, t_n]$, equal to zero near the left endpoint and one near the right endpoint, with $|\chi'_n| \leq C/t_n$. Define

$$u^s(t) = \begin{cases} \chi_n(t)U_n(t) + (1 - \chi_n(t))U_{n+1}(t), & t \in [t_{n+1}, t_n], \\ U_0(t), & t \geq t_0. \end{cases} \quad (3.9)$$

The formulas agree near every joining time.

Write $\rho_n = \|U_n(0) - u_0\|_2 \rightarrow 0$. For $0 \leq t \leq t_n$,

$$\|U_n(t) - u_0\|_2 + \|U_{n+1}(t) - u_0\|_2 \leq \rho_n + \rho_{n+1} + C\alpha_n \rightarrow 0. \quad (3.10)$$

This proves that the velocity difference tends to zero in L^2 at the initial time. Positive-time continuity follows from the background lemma. Linearity of curl and of $K*$ gives the Biot–Savart identity and the asserted curl bound.

The gluing identity and its three bounds. Set

$$D_n = U_n - U_{n+1}, \quad T_n = U_n \otimes U_n - U_{n+1} \otimes U_{n+1}, \quad \mathcal{B}_n = B_{\varepsilon_n} * \mu_0.$$

Let p_n be a pressure for U_n , and define

$$P_n(t) = \int_0^t (p_n(s) - p_{n+1}(s)) ds.$$

The adjacent-scale tensor gives $D_n(0) = \operatorname{div} \mathcal{B}_n$. Integrating the difference of the two Euler equations therefore yields

$$D_n(t) = \operatorname{div} \left(\mathcal{B}_n - \int_0^t T_n(s) ds \right) - \nabla P_n(t).$$

Expanding the tensor product in (3.9) now gives an Euler–Reynolds pair with pressure

$$p^s = \chi_n p_n + (1 - \chi_n) p_{n+1} + \chi'_n P_n$$

and stress

$$R^s = \chi'_n \mathcal{B}_n - \chi'_n \int_0^t T_n(s) ds - \chi_n(1 - \chi_n) D_n \otimes D_n \quad \text{on } [t_{n+1}, t_n]. \quad (3.11)$$

For $t \geq t_0$, take $R^s = 0$. All terms are symmetric. To justify their global integrability, first use (3.4) and (3.7) to obtain

$$\|D_n(t)\|_{3/2} \leq C\varepsilon_n^{1/3} + Ct\varepsilon_n^{-5/3}E_n^{5/6}.$$

Since $T_n = D_n \otimes U_n + U_{n+1} \otimes D_n$, Hölder gives

$$\begin{aligned} \sup_{0 \leq t \leq t_n} \|T_n(t)\|_1 &\leq C \left(E_n^{1/6} + t_n \varepsilon_n^{-2} E_n \right) \\ &\leq C \left((\alpha_n)^{1/3} + \alpha_n \right) \longrightarrow 0. \end{aligned} \quad (3.12)$$

The first term in (3.11) is bounded by $C\varepsilon_n^2/t_n = C\alpha_n$. The second is bounded by $C \sup_{0 \leq s \leq t_n} \|T_n(s)\|_1$, since $|\chi'_n|t_n \leq C$. The third is bounded by $C \sup_{0 \leq s \leq t_n} \|D_n(s)\|_2^2$. Combining these estimates gives

$$\sup_{t_{n+1} \leq t \leq t_n} \|R^s(t)\|_1 \leq C [\alpha_n + \alpha_n^{1/3} + (\rho_n + \rho_{n+1} + C\alpha_n)^2] \longrightarrow 0.$$

The use of $L^{3/2}$ for D_n in the second term is important: U_n and U_{n+1} need not be in global L^2 , so a global $L^2 \times L^2$ estimate of that term would not be available. For each fixed n , D_n is continuous into $L^2 \cap L^{3/2}$, U_n, U_{n+1} are continuous into L^3 , and hence T_n and R^s are continuous into L^1 . The stress vanishes near the joining times. Extension by zero at $t = 0$ proves its asserted continuity.

The additional L^1 trace. Suppose $\mu_0 \in L^1$, and let X_j be the volume-preserving flow of U_j . For $j = n, n+1$ and $0 \leq t \leq t_n$,

$$\sup_x |X_j(t, x) - x| \leq C t_n \varepsilon_n^{-1} \sqrt{E_n} = C \varepsilon_n \frac{\sqrt{E_n}}{\alpha_n} \leq C \varepsilon_n \longrightarrow 0.$$

The inverses have the same displacement bound. Approximation by compactly supported continuous functions and volume preservation show that $\mu_0 \circ X_j(t)^{-1} \rightarrow \mu_0$ in L^1 , uniformly over these times. Also $b_{\varepsilon_j} * \mu_0 \rightarrow \mu_0$ in L^1 , since the signed kernels form an approximate identity with uniformly bounded L^1 norms. Therefore

$$\|\nu_j(t) - \mu_0\|_1 \leq \|b_{\varepsilon_j} * \mu_0 - \mu_0\|_1 + \|\mu_0 \circ X_j(t)^{-1} - \mu_0\|_1 \longrightarrow 0.$$

Taking the same convex combinations as in (3.9) proves the additional trace. $\square$

4. THE TRANSLATING DIPOLE AND ITS COMPACT FORCE CORRECTION

The perturbation is based on the classical Lamb–Chaplygin dipole [8, 22]. We use the normalization employed in the moving-dipole construction of Bruè, Colombo, and Kumar [3, §2.2], with signs adjusted to our convention for $\nabla^\perp$. The Bessel formula and its boundary matching are verified in Appendix A.2. Here we record the properties needed for the repair and prove the compact radial force estimate.

4.1. The fixed translating profile. We use a fixed divergence-free profile D , translating in the direction $e_1 = (1, 0)$. It is C^1 , bounded in its core, and has an integrable curl supported in $\overline{B}_1$. Outside that core,

$$D(x) = \nabla \left(-\frac{x_1}{|x|^2} \right), \quad D(x) = O(|x|^{-2}).$$

The explicit construction in Appendix A.2 proves these properties and the translating Euler equation below.

For a unit vector ξ , let O_ξ be the rotation carrying e_1 to ξ , and put

$$D_{r,\xi}(x) = r^{-1} O_\xi D(O_\xi^T x/r), \quad r > 0.$$

Rotation and scaling give

$$\begin{aligned} -r^{-1}\partial_\xi D_{r,\xi} + \operatorname{div}(D_{r,\xi} \otimes D_{r,\xi}) + \nabla p_{r,\xi}^D &= 0, \\ \operatorname{div} D_{r,\xi} &= 0. \end{aligned} \quad (4.1)$$

The decay and compactly supported curl give, uniformly in ξ ,

$$\begin{aligned} \|D_{r,\xi}\|_2 + \|\operatorname{curl} D_{r,\xi}\|_1 &\leq C_D, \\ \|D_{r,\xi}\|_{3/2} &= r^{1/3} \|D\|_{3/2}, \quad \|D_{r,\xi}\|_3 = r^{-1/3} \|D\|_3. \end{aligned} \quad (4.2)$$

The first line is the critical scaling used in the iteration: concentrating the dipole does not increase either its L^2 velocity norm or its L^1 vorticity norm. Its $L^{3/2}$ velocity norm, however, tends to zero with the radius. This is what makes products with the old L^3 velocity small.

4.2. The radial representation and compact tensor. The dipole also has a radial representation. There is a radial scalar function $W \in C^1(\mathbb{R}^2)$, supported in $\overline{B}_1$, with $\int W dx = 1$, such that

$$D_{r,\xi} = 2\pi r \mathbb{P}(W_r \xi). \quad (4.3)$$

Here $W_r(x) = r^{-2}W(x/r)$. The identity follows by comparing the curls of the two divergence-free L^2 fields. The formula for W and this verification are in Appendix A.2.

Fix a nonnegative radial $\kappa \in C_c^\infty(B_1)$ of integral one.

Lemma 4.1 (Compact radial force correction). *For $0 < r < H$ and $|\xi| = 1$ there is a symmetric tensor $\mathcal{B}_{r,H,\xi}$ supported in $\overline{B}_H$ such that*

$$\begin{aligned} \frac{D_{r,\xi}}{r} + \nabla \pi_{r,\xi} &= 2\pi \kappa_H \xi + \operatorname{div} \mathcal{B}_{r,H,\xi}, \\ \|\mathcal{B}_{r,H,\xi}\|_1 &\leq CH, \end{aligned} \quad (4.4)$$

where $\pi_{r,\xi} = 2\pi \xi \cdot \nabla \Delta^{-1} W_r$. The constant is independent of r , H , and ξ .

Proof. The radial function $W_r - \kappa_H$ has zero mass and is supported in $\overline{B}_H$. Let $g = g_{r,H}$ solve $\Delta g = W_r - \kappa_H$, with the radial solution chosen to vanish outside B_H . As in the compact Poisson construction above,

$$\begin{aligned} g'(\rho) &= \frac{1}{2\pi\rho} \int_{B_\rho} (W_r - \kappa_H) dx, \\ \|\nabla g\|_1 &= \int_0^H \left| \int_{B_\rho} (W_r - \kappa_H) dx \right| d\rho \leq H(\|W\|_1 + \|\kappa\|_1). \end{aligned}$$

The elementary symmetric divergence inverse in two dimensions has the form $\xi \otimes \nabla g + \nabla g \otimes \xi - (\xi \cdot \nabla g)I_2$; compare the general formula in [3, (A.7)]. Here the radial Poisson solution makes that tensor compactly supported and gives its L^1 estimate directly. Set

$$\mathcal{B}_{r,H,\xi} = 2\pi (\xi \otimes \nabla g + \nabla g \otimes \xi - (\xi \cdot \nabla g)I_2).$$

Direct differentiation gives $\operatorname{div} \mathcal{B}_{r,H,\xi} = 2\pi(W_r - \kappa_H)\xi$. Combine this with (4.3) and the identity

$$2\pi \mathbb{P}(W_r \xi) = 2\pi W_r \xi - \nabla \pi_{r,\xi}.$$

The L^1 bound follows from the estimate for ∇g above. In particular, no integral of $1/\rho$ remains in this bound, even when r/H is very small. $\square$

4.3. One dipole pass at fixed radius. During one pass fix r , ξ , and H , and let the amplitude $\lambda(t) \geq 0$ and centre $X(t)$ satisfy

$$X'(t) = \frac{\lambda(t)}{r} \xi.$$

Set $v(t, x) = \lambda(t) D_{r, \xi}(x - X(t))$. Differentiating gives

$$\partial_t v(t, x) = \lambda'(t) D_{r, \xi}(x - X(t)) - \frac{\lambda(t)^2}{r} \partial_\xi D_{r, \xi}(x - X(t)).$$

The second term combines with $\operatorname{div}(v \otimes v)$ through (4.1). Applying Lemma 4.1 to the first term gives

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p_v &= F + \operatorname{div} E, \\ F(t, x) &= 2\pi r \lambda'(t) \kappa_H(x - X(t)) \xi, \quad E(t, x) = r \lambda'(t) \mathcal{B}_{r, H, \xi}(x - X(t)). \end{aligned} \quad (4.5)$$

One may take

$$p_v(t, x) = \lambda(t)^2 p_{r, \xi}^D(x - X(t)) + r \lambda'(t) \pi_{r, \xi}(x - X(t)).$$

In particular,

$$\|E\|_1 \leq CH |r \lambda'|. \quad (4.6)$$

The radius stays fixed throughout the pass. At its end, the amplitude will vanish, so the next pass may use a different centre, direction, and radius without a jump in velocity or vorticity. No spatial cutoff of the velocity is needed, and (4.6) has no logarithmic loss.

5. STRESS REPAIR WITHOUT CHANGING ENDPOINT VELOCITIES

We approximate the stress uniformly in time by finitely many fixed tensors. Continuous stresses, and stresses with finitely many continuous one-sided pieces, admit such an approximation. A general L^∞ stress need not; the required approximation property is stated explicitly below. Values at isolated times are immaterial.

Lemma 5.1 (Repair preserving the endpoint velocities). *Let $I = [a, b] \Subset (0, \infty)$, and let (u, R) be an Euler–Reynolds pair on its interior with*

$$u \in C(I; L^3), \quad R \in L^\infty(I; L^1(\mathbb{R}^2; \mathbb{R}_{\text{sym}}^{2 \times 2})).$$

Assume that R can be approximated in $L^\infty(I; L^1)$ by functions that are constant on each interval of a finite time partition. For every $\varepsilon > 0$ there are a divergence-free total correction $w \in C(I; \mathcal{X}) \cap C(I; L^3)$ and a symmetric $\tilde{R} \in L^\infty(I; L^1)$ with the same time-approximation property such that $(u + w, \tilde{R})$ is an Euler–Reynolds pair and

$$\begin{aligned} w(a) = w(b) &= 0, \quad \|\tilde{R}\|_{L^\infty(I; L^1)} \leq \varepsilon, \\ \|w\|_{C(I; L^2)} + \|\operatorname{curl} w\|_{C(I; L^1)} &\leq C \sqrt{\|R\|_{L^\infty(I; L^1)}}. \end{aligned} \quad (5.1)$$

The constant depends only on the fixed profiles, not on the interval, the spatial support, the old velocity, or its derivatives. The small scale may depend on the interval, the chosen rectangles and buffer lengths, and $\|u\|_{C(I; L^3)}$.

The idea is to cancel the stress through the time average of a travelling force, rather than pointwise through the quadratic velocity perturbation. This is the mechanism of [3, §1.2.3 and §§4.7–4.8]. We approximate the stress by constant matrices on rectangles and use one dipole to traverse their rows. A time correction removes the difference between the force and its cycle average. Repeating complete cycles increasingly rapidly makes the remaining errors small while preserving both endpoint velocities.

The choices are made in this order: the approximation tolerance, the time partition and rectangle families, the entry and exit buffer lengths, and finally the small scale (equivalently, the number of complete cycles). We describe the ingredients in that order, then check the estimates in the proof of the lemma.

5.1. Approximating the stress by rank-one blocks. Adding a scalar matrix to the stress only changes the pressure. After this adjustment, the negative of each constant symmetric stress can be written as a nonnegative matrix of rank at most one. The next lemma makes this observation compatible with a rectangle approximation.

Lemma 5.2 (Block approximation). *For every symmetric $S \in L^1$ and $\sigma > 0$, there exist a compactly supported $q \in L^1$, finitely many rectangles Q_j , constants $c_j > 0$, and unit vectors ξ_j parallel to one side of Q_j , such that*

$$A = \sum_{j=1}^J c_j \mathbf{1}_{Q_j} \xi_j \otimes \xi_j, \quad \|S + A - qI_2\|_1 < \sigma, \quad \sum_{j=1}^J c_j |Q_j| + \|q\|_1 \leq C \|S\|_1. \quad (5.2)$$

The family may be empty.

Proof. Approximate S in L^1 by its cell averages on a sufficiently fine finite square grid in a sufficiently large box. This gives a step tensor $S^* = \sum_{\ell} S_{\ell} \mathbf{1}_{P_{\ell}}$ with $\|S - S^*\|_1 < \sigma/2$ and $\|S^*\|_1 \leq \|S\|_1$. For each constant symmetric matrix, the spectral theorem gives

$$-S_{\ell} + \lambda_{\max}(S_{\ell})I_2 = c_{\ell} \xi_{\ell} \otimes \xi_{\ell}, \quad c_{\ell} = \lambda_{\max}(S_{\ell}) - \lambda_{\min}(S_{\ell}) \geq 0.$$

Here ξ_{ℓ} is a unit eigenvector for the smaller eigenvalue; blocks with $c_{\ell} = 0$ require no vector choice. Put $q = \sum_{\ell} \lambda_{\max}(S_{\ell}) \mathbf{1}_{P_{\ell}}$. For each positive c_{ℓ} , fill all but an arbitrarily small area of P_{ℓ} by finitely many disjoint rectangles with sides parallel to ξ_{ℓ} and $\xi_{\ell}^{\perp}$. A sufficiently fine grid in these rotated coordinates supplies such an interior approximation. Choose the uncovered weighted areas to total less than $\sigma/2$. This gives the first estimate in (5.2). The mass bound follows from $|\lambda_{\max}(S_{\ell})| + c_{\ell} \leq C|S_{\ell}|$ and the fact that only interior rectangles were retained. $\square$

5.2. How one dipole traverses all the rows. We approximate each rectangle by parallel rows. A single dipole traverses them one at a time, increasing its amplitude before entering a row and decreasing it after leaving. Its amplitude is zero between passes, so its position and direction can be reset without a velocity jump. One complete cycle visits every row of every rectangle. The parameters are chosen so that the cycle has a prescribed duration and its average force approximates the required stress divergence.

The construction uses four related scales:

Scale	Role in the construction
H	Radius over which the force is smoothed
$d = H^2$	Distance between neighbouring rows
$\tau = H^2$	Duration of one complete cycle
r_j (of order H^4)	Dipole radius during a pass through block j

The dipole radius r_j is of order H^4 once the rectangle data and buffer lengths have been fixed. Figure 1 shows a single pass and its amplitude schedule.

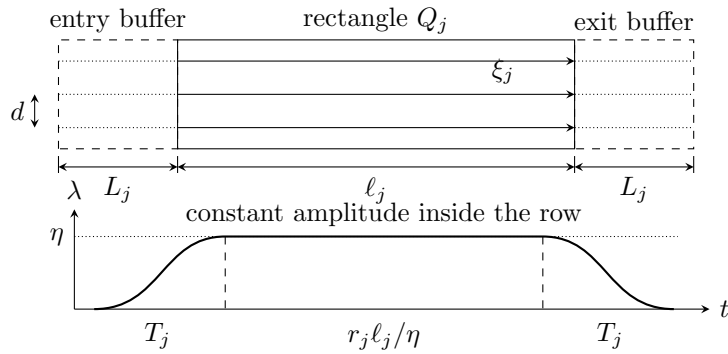


FIGURE 1. One dipole pass. The amplitude rises from zero in the entry buffer, remains constant along the row, and returns to zero in the exit buffer. The next pass can then reset the centre, direction, and radius. A complete cycle concatenates every row pass; the diagram is not to scale.

Fix a nonempty family in Lemma 5.2. Write each rectangle as

$$Q_j = \{x_j + s\xi_j + y\xi_j^\perp : 0 < s < \ell_j, 0 < y < h_j\}.$$

Choose a smooth increasing ramp $\vartheta : [0, 1] \rightarrow [0, 1]$, flat at both endpoints, with $\vartheta(0) = 0$, $\vartheta(1) = 1$, and $\vartheta(s) + \vartheta(1 - s) = 1$. Thus $\int_0^1 \vartheta(s) ds = 1/2$. For each block choose an entry and exit buffer length $L_j > 0$. These lengths are fixed before the fine scale and can be chosen so that

$$M_{\text{buf}} := 4 \sum_j c_j h_j L_j < \sigma. \quad (5.3)$$

The buffers extend each row longitudinally outside its rectangle; they may intersect other rectangles, since only one dipole is active.

Let $H > 0$ be small. Set

$$d = H^2, \quad \tau = H^2, \quad N_j = \lfloor h_j/d \rfloor, \quad y_{jk} = (k + \tfrac{1}{2})d \quad (0 \leq k < N_j). \quad (5.4)$$

Take H small enough that every $N_j \geq 1$. For each row use a constant radius r_j and a *common* peak amplitude η across all blocks. Define them by

$$\eta^2 = \frac{d}{2\pi} \sum_j c_j N_j (\ell_j + 4L_j), \quad r_j = \frac{\tau d c_j}{2\pi \eta}. \quad (5.5)$$

On entry, use $\lambda(t) = \eta\vartheta(t/T_j)$ for a duration

$$T_j = \frac{2r_j L_j}{\eta}.$$

Because $X' = \lambda\xi_j/r_j$, the entry displacement is exactly L_j . Next traverse the rectangle at amplitude η , for duration $r_j\ell_j/\eta$, then use the reversed ramp for the exit buffer. The amplitude is zero at the two ends of every row pass. Reset the centre, direction, and radius there, then continue through every row of every block. The total time is exactly

$$\sum_j N_j \frac{r_j}{\eta} (\ell_j + 4L_j) = \tau. \quad (5.6)$$

Thus all blocks, not individual directions, form one complete cycle.

Let $M = \sum_j c_j |Q_j|$. Since $dN_j \leq h_j$,

$$\eta^2 \leq \frac{M + M_{\text{buf}}}{2\pi}. \quad (5.7)$$

As $H \downarrow 0$, $dN_j \rightarrow h_j$, so η tends to a positive finite limit. Once the rectangles and buffers are fixed, there are constants $c_0, C_0, H_0 > 0$ such that

$$c_0 H^4 \leq r_j \leq C_0 H^4 < H \quad \text{for every } j \text{ and } 0 < H < H_0. \quad (5.8)$$

In the interior of a row $\lambda' = 0$, whereas on a ramp

$$|r_j \lambda'| \leq \frac{\eta^2 \|\vartheta'\|_\infty}{2L_j}.$$

Consequently

$$B_* := \sup_{\text{cycle}} |r \lambda'|$$

is bounded independently of H once the rectangle data and buffers are fixed. The constants may depend on the positive coefficients c_j of this finite family; no uniform lower bound across different families is required.

5.3. The cycle average approximates the required stress. On each pass use F and E from (4.5). Since λ vanishes at the endpoints,

$$\frac{d}{dt} (r \lambda \kappa_H(x - X(t))) = r \lambda' \kappa_H(x - X(t)) - \lambda^2 \partial_\xi \kappa_H(x - X(t))$$

implies

$$\overline{F} := \frac{1}{\tau} \int_{\text{cycle}} F dt = \text{div } A_{\text{av}}, \quad A_{\text{av}} = \frac{2\pi}{\tau} \int_{\text{cycle}} \lambda(t)^2 \kappa_H(\cdot - X(t)) \xi(t) \otimes \xi(t) dt. \quad (5.9)$$

For an interior row, $dt = (r_j/\eta) ds$ and $2\pi\eta r_j/\tau = dc_j$. Its contribution to A_{av} is therefore the desired weighted row integral.

For completeness, for one rectangle Q with side lengths ℓ, h , coefficient c , and direction ξ , put

$$F_H = cd \sum_{k=0}^{N-1} \int_0^\ell \kappa_H(\cdot - x_0 - s\xi - (k + \frac{1}{2})d\xi^\perp) ds, \quad N = \lfloor h/d \rfloor.$$

Then

$$\|F_H - c\mathbf{1}_Q\|_1 \leq Cc \left(\frac{d}{H} |Q| + d\ell + H \text{Per}(Q) \right). \quad (5.10)$$

Here $\text{Per}(Q)$ is the perimeter of Q . Replacing each row by the uniform strip of width d around it costs at most $CcdH^{-1}|Q|$, using $\|\kappa_H(\cdot - z) - \kappa_H\|_1 \leq C|z|/H$. The unused transverse strip has area at most $d\ell$. Finally

$$\|\mathbf{1}_Q * \kappa_H - \mathbf{1}_Q\|_1 \leq CH \text{Per}(Q),$$

which follows directly from the area of the symmetric difference of a rectangle and its translate. These three comparisons prove (5.10).

On a ramp, $\lambda^2 \leq \eta^2$. Thus the combined L^1 norm of all ramp contributions to A_{av} is at most

$$4 \sum_j dc_j N_j L_j \leq M_{\text{buf}}.$$

Summing (5.10) and using $d = H^2$ gives

$$\|A_{\text{av}} - A\|_1 \leq M_{\text{buf}} + C_{\text{blocks}} H \quad (0 < H \leq 1). \quad (5.11)$$

In particular the same small buffer mass controls both the average error and the leading amplitude in (5.7).

5.4. The time correction and the remaining stress. The time correction is the time integral of the force minus its cycle average, followed by the Leray projection [3, (4.76)–(4.78)]. It vanishes at both ends of the cycle [3, Remark 4.1]. If t_* is the beginning of a complete cycle, define

$$z(t) = -\mathbb{P} \int_{t_*}^t (F(s) - \bar{F}) ds. \quad (5.12)$$

Then $\text{div } z = 0$, $z(t_*) = z(t_* + \tau) = 0$, and

$$\partial_t z + F + \nabla p_z = \bar{F} = \text{div } A_{\text{av}}, \quad (5.13)$$

where $p_z = -\Delta^{-1} \text{div}(F - \bar{F})$ is one possible pressure. The average in this formula is over the *entire* block schedule. A separate average over each row would not yield the tensor required for the repair.

The L^p bounds for $\mathbb{P}$ and scaling of κ_H show, for $p > 1$,

$$\|z\|_p \leq C_p \tau B_* H^{2/p-2}.$$

For its curl, the identity $\text{curl } \mathbb{P}F = \text{curl } F$ removes the projection, so the estimate uses only $\|\nabla \kappa_H\|_1 \leq CH^{-1}$. In the norms needed below,

$$\|z\|_2 + \|\text{curl } z\|_1 \leq C\tau B_* H^{-1} = CB_* H, \quad \|z\|_{3/2} \leq CB_* H^{4/3}. \quad (5.14)$$

Let v be the dipole perturbation and set $w = v + z$, the total correction. Adding the old Euler–Reynolds equation, (4.5), and (5.13) gives the exact symmetric stress

$$\begin{aligned} \tilde{R} &= R + A_{\text{av}} - qI_2 + E \\ &\quad + u \otimes w + w \otimes u + v \otimes z + z \otimes v + z \otimes z. \end{aligned} \quad (5.15)$$

The corresponding pressure is $p + p_v + p_z - q$, where p is the old pressure. Expanding $(u + w) \otimes (u + w)$ separates the old product, the dipole product $v \otimes v$, and the five cross and correction products displayed above. Consequently, for a frozen tensor S ,

$$\begin{aligned} \|\tilde{R}\|_1 &\leq \|R - S\|_1 + \|S + A - qI_2\|_1 + \|A_{\text{av}} - A\|_1 + \|E\|_1 \\ &\quad + 2\|u\|_3 \|w\|_{3/2} + 2\|v\|_2 \|z\|_2 + \|z\|_2^2. \end{aligned} \quad (5.16)$$

The first three errors come from approximating the stress in time and space. The term E comes from changing the dipole amplitude, and the remaining terms are products involving the corrections. At fixed coarse data, (4.2), (4.6), (5.8), and (5.14) give

$$\begin{aligned} \|E\|_1 + \|z\|_{\mathcal{X}} &\leq C_{\text{coarse}} H, \\ \|v\|_{3/2} + \|z\|_{3/2} &\leq C_{\text{coarse}} H^{4/3}, \\ \|v\|_{\mathcal{X}} &\leq C_D \eta. \end{aligned} \tag{5.17}$$

Here C_{coarse} may depend on the fixed rectangles, coefficients, and buffer lengths, but not on H . The last estimate has the universal constant C_D . Once these coarse data are fixed, every fine-scale stress error in (5.16) tends to zero.

5.5. Proof of the repair lemma.

Proof of Lemma 5.1. Let $\delta = \|R\|_{L^\infty(I; L^1)}$. If $\delta = 0$, take $w = 0$ and $\tilde{R} = 0$. Assume $\delta > 0$ and fix

$$0 < \sigma < \min\{\delta, \varepsilon/6\}.$$

By the time-approximation assumption, choose finitely many time cells I_k and frozen symmetric tensors $S_k \in L^1$ with

$$\operatorname{ess\,sup}_{t \in I_k} \|R(t) - S_k\|_1 < \sigma, \quad \|S_k\|_1 \leq \delta + \sigma.$$

Apply Lemma 5.2 on each cell with error σ and choose buffers satisfying (5.3). All block families and all positive buffer lengths are now fixed. Their total core mass is at most $C(\delta + \sigma)$ on each cell.

If a block family is empty, take $w = 0$ there and $\tilde{R} = R - q_k I_2$; its norm is below 2σ . For every nonempty family, divide the time cell into N equal complete cycles and choose

$$\tau = |I_k|/N, \quad H = \sqrt{\tau}, \quad d = \tau.$$

The exact schedule (5.5) fills each cycle. Let $N \rightarrow \infty$, separately on each of the finitely many cells. The first three terms in (5.16) have sum less than $3\sigma + C_{\text{blocks}} H$: one σ comes from freezing time, one from the rectangle approximation, and one from the buffer contributions. The remaining errors tend to zero by (5.17) and the finite old norm $\|u\|_{C(I; L^3)}$. Thus sufficiently large cycle counts give $\|\tilde{R}\|_{L^\infty(I; L^1)} \leq \varepsilon$.

From (5.7) and $\sigma \leq \delta$,

$$\|v\|_{\mathcal{X}} \leq C\sqrt{\delta}.$$

Increasing the same cycle counts so that $\|z\|_{\mathcal{X}} \leq \sqrt{\delta}$ gives $\|w\|_{C(I; \mathcal{X})} \leq C\sqrt{\delta}$, and hence (5.1) after adjusting the universal constant. The bound on $\|u\|_3$ only affects how small H must be, not this constant. In particular, the construction needs no additive amplitude allowance and no derivative estimate for u .

This verifies the order of choices stated above: all approximation data and buffer lengths were fixed before increasing the cycle counts. Constants that grow when a buffer is shortened are harmless, because the small scale is chosen after those constants are known.

It remains to justify the endpoint matching and the stated time spaces. Each fixed construction has finitely many passes with positive radii. Translation continuity of $D_{r, \xi}$ in $L^2 \cap L^3 \cap L^{3/2}$ and of its curl in L^1 , together with the zero amplitude at pass endpoints,

gives the corresponding continuity of v . The time correction (5.12) is continuous in these velocity spaces, and

$$\operatorname{curl} z(t) = - \int_{t_*}^t \operatorname{curl}(F(s) - \bar{F}) ds$$

is continuous with values in L^1 . Both v and z , including their curls, vanish at complete-cycle endpoints. Therefore w has zero traces at every cell endpoint, and in particular at a, b .

On each fixed cell, the compact stress E is continuous in L^1 : its coefficient $r\lambda'$ vanishes at resets and its spatial profile is a translated fixed L^1 tensor. Products between perturbations are continuous in L^1 by their L^2 continuity. Products involving the old velocity are continuous in L^1 by $C_t L^3 \times C_t L^{3/2}$. The remaining change $A_{av} - q_k I_2$ is constant on each cell. Thus $\tilde{R} - R$ has finitely many continuous one-sided pieces and admits uniform time-step approximation in $L^\infty(I; L^1)$. Since R has the same approximation property, so does $\tilde{R}$.

Finally, apply the equation on each time cell and add the weak formulations. The boundary terms from $\partial_t(u + w)$ cancel because the velocity traces agree. A jump of the stress does not produce a temporal Dirac mass, since the equation contains only its spatial divergence. The same argument applies at pass and cycle boundaries. This proves the lemma on all of I . $\square$

6. GLOBAL ITERATION AND THE INITIAL TRACE

The endpoint condition in Lemma 5.1 allows us to work on disjoint intervals. There is no need to overlap local corrections or to differentiate a time partition of unity. We first give the argument for a general approximate flow, so that the same result can later be applied to the modified approximate flows used for nonuniqueness.

Proposition 6.1 (Global repair preserving values at dyadic times). *Suppose (u^s, R^s) is a positive-time Euler–Reynolds pair satisfying the approximate-flow properties (3.5), with the curl bound there replaced, if necessary, by a bound on each finite time interval. The starting velocity need not be given by the Biot–Savart law for this proposition. Then there is a correction*

$$w \in C([0, \infty); \mathcal{X}), \quad w(0) = 0, \quad (6.1)$$

such that $u = u^s + w$ solves (1.2) and

$$u(2^j) = u^s(2^j) \quad (j \in \mathbb{Z}). \quad (6.2)$$

The solution has all the regularity properties of Theorem 1.1, satisfies the Biot–Savart law, and conserves circulation. Its L^1 initial vorticity trace holds whenever the starting approximate flow has that trace.

Proof. For $j \in \mathbb{Z}$, put

$$I_j = [2^j, 2^{j+1}], \quad \delta_j = 2^j + \sup_{t \in I_j} \|R^s(t)\|_1.$$

These numbers are positive, tend to zero as $j \rightarrow -\infty$, and are bounded over the intervals meeting any fixed finite time horizon.

Start with $(u^{(0)}, R^{(0)}) = (u^s, R^s)$. On each I_j , apply Lemma 5.1 successively, at stage q with requested stress tolerance $16^{-(q+1)}\delta_j$. It gives

$$\begin{aligned} u^{(q+1)} &= u^{(q)} + w_q, & \|R^{(q)}\|_{L^\infty(I_j; L^1)} &\leq 16^{-q}\delta_j, \\ \sup_{t \in I_j} \|w_q(t)\|_{\mathcal{X}} &\leq C4^{-q}\sqrt{\delta_j}, & w_q(2^j) &= w_q(2^{j+1}) = 0. \end{aligned} \quad (6.3)$$

The induction starts because $\|R^s\|_{L^\infty(I_j; L^1)} \leq \delta_j$. On each compact positive-time interval only finitely many I_j occur. Thus the separate repairs paste to global positive-time Euler–Reynolds pairs. At every finite stage the velocity is continuous into L^3 on compact positive-time intervals, so the next local repair is legitimate. No uniform-in-stage L^3 bound is asserted or used.

Extend every w_q by zero at $t = 0$. Its continuity there follows from $\delta_j \rightarrow 0$. For $T > 0$, set

$$\delta_T^* = \sup_{j: 2^j \leq T} \delta_j < \infty.$$

One summation controls both the velocity in L^2 and its curl in L^1 :

$$\sum_{q=0}^{\infty} \|w_q\|_{C([0, T]; \mathcal{X})} \leq C\sqrt{\delta_T^*} \sum_{q=0}^{\infty} 4^{-q} < \infty \quad (6.4)$$

produces $w = \sum_q w_q$ with (6.1). It also preserves the velocity at every dyadic time, as in (6.2). In particular

$$u - u_0 = (u^s - u_0) + w \in C([0, \infty); L^2), \quad (u - u_0)(0) = 0.$$

The equation and its initial term. On compact positive-time intervals, the stresses tend to zero in $L_t^\infty L_x^1$ and the velocities converge uniformly in local L^2 . For each compact spatial set B ,

$$\|u^{(q)} \otimes u^{(q)} - u \otimes u\|_{L^1(B)} \leq \left(\|u^{(q)}\|_{L^2(B)} + \|u\|_{L^2(B)} \right) \|u^{(q)} - u\|_{L^2(B)}.$$

Hence the limit solves Euler on $(0, \infty)$. A continuous local L^2 trace extends this equation to the initial time. Indeed, test the positive-time equation with $\rho_\epsilon(t)\varphi(t, x)$, where ρ_ϵ increases from zero to one on $(\epsilon, 2\epsilon)$. The term with ρ'_ϵ tends to $\int u_0 \cdot \varphi(0) dx$ by strong local L^2 continuity and $\int \rho'_\epsilon = 1$. The omitted linear and quadratic integrals tend to zero by local L^2 boundedness. This proves (1.2).

Vorticity regularity. For $t > 0$,

$$\mu(t) = \operatorname{curl} u^s(t) + \operatorname{curl} w(t).$$

The approximate-flow properties and (6.1) imply $\mu \in C((0, \infty); L^1)$ and a measure bound on every finite horizon. At $t = 0$ the curl is μ_0 . The identity

$$\mu(t) = \mu_0 + \operatorname{curl}(u(t) - u_0), \quad \|\mu(t)\|_{H^{-1}} \leq \|\mu_0\|_{H^{-1}} + \|u(t) - u_0\|_2 \quad (6.5)$$

and (2.3) give strong H^{-1} continuity and its local bounds. Distributional time continuity extends to weak-* measure continuity by uniform approximation of C_0 test functions by C_c^∞ functions, using the locally uniform total variation bound. When the approximate vorticity has an L^1 initial trace,

$$\|\mu(t) - \mu_0\|_1 \leq \|\operatorname{curl} u^s(t) - \mu_0\|_1 + \|\operatorname{curl} w(t)\|_1 \longrightarrow 0.$$

The Biot–Savart law and circulation. For each fixed $t \geq 0$, apply Lemma 2.1 to

$$z = u(t) - u_0 \in L^2, \quad \nu = \mu(t) - \mu_0 \in \mathcal{M}.$$

It gives $z = K * \nu$ and $\nu(\mathbb{R}^2) = 0$, hence

$$u(t) = K * \mu(t), \quad \mu(t)(\mathbb{R}^2) = \mu_0(\mathbb{R}^2).$$

The approximate-flow bounds, (6.4), and (6.5) also give the locally uniform bounds stated after Theorem 1.1. $\square$

Remark 6.2 (Why the initial stress must tend to zero). The initial correction on I_j is bounded by $C\sqrt{\delta_j}$. This tends to zero because the starting stress tends to zero. A merely bounded initial stress would not give the strong initial velocity or H^{-1} vorticity trace by this argument. The endpoint condition is equally important: it lets us sum the estimates independently on the dyadic intervals, with no additional error from gluing the perturbations in time.

Remark 6.3 (The singular part of the initial measure). If $\mu_0 = f dx + \mu_0^{\text{sing}}$ is the Lebesgue decomposition, then the positive-time vorticity has an L^1 density, and mutual singularity gives

$$\|\mu(t) - \mu_0\|_{\mathcal{M}} = \|\mu(t) - f\|_1 + \|\mu_0^{\text{sing}}\|_{\mathcal{M}} \geq \|\mu_0^{\text{sing}}\|_{\mathcal{M}}, \quad t > 0.$$

Thus a nonzero singular part precludes total-variation continuity at zero for these solutions. This does not conflict with the strong H^{-1} and weak-* measure traces proved above.

7. NONUNIQUENESS BY PRESERVING THE VELOCITY AT $t = 1$

Proof of Theorem 1.1. Propositions 3.2 and 6.1 already give one solution with all the stated properties. To obtain infinitely many, choose nonzero $\phi \in C_c^\infty(\mathbb{R}^2)$ and set

$$a = \nabla^\perp \phi, \quad z_0 = \Delta a, \quad B_0 = \nabla a + (\nabla a)^T.$$

Then B_0 is symmetric, $\text{div } B_0 = z_0$, and $\text{div } z_0 = 0$. Moreover $z_0 \neq 0$, since a compactly supported harmonic a would vanish, forcing the compactly supported ϕ to be constant and hence zero.

Let (u^s, R^s) be the approximate flow from Proposition 3.2. Choose $\chi \in C_c^\infty((0, \infty))$ with $\chi(1) = 1$. For each integer $n \geq 1$ define

$$\begin{aligned} h_n &= n\chi z_0, & u_n^s &= u^s + h_n, \\ R_n^s &= R^s + n\chi' B_0 + u^s \otimes h_n \\ &\quad + h_n \otimes u^s + h_n \otimes h_n. \end{aligned} \tag{7.1}$$

Because $\partial_t h_n = \text{div}(n\chi' B_0)$, these are exact Euler–Reynolds pairs. The mixed stress is continuous into L^1 by $C_t L^3 \times C_t L^{3/2}$, and all added fields are supported away from the initial time. The modified pairs therefore retain the hypotheses of Proposition 6.1, with bounds allowed to depend on n .

Repair each modified approximate flow. Since 1 is a dyadic endpoint, the repair leaves its velocity there unchanged:

$$u_n(1) = u_n^s(1) = u^s(1) + nz_0.$$

For $n \neq m$,

$$\|u_n(1) - u_m(1)\|_2 = |n - m| \|z_0\|_2 > 0. \tag{7.2}$$

The difference $u_n - u_m$ is continuous into global L^2 , so its norm remains positive on an interval about 1. Thus these are distinct space-time weak solutions, not just different representatives at an isolated time. All have the same initial datum and every property of the theorem. $\square$

APPENDIX A. AUXILIARY CONSTRUCTIONS

A.1. A sequence for the joining times. The joining argument in Section 3 needs a positive sequence that tends to zero, controls two adjacent approximation errors, and decreases by at most a factor of two at each step. Here is an explicit construction, valid for every nonnegative sequence $e_n \rightarrow 0$.

Set

$$\gamma_k = \max\{2^{-k}, \sqrt{e_k}, \sqrt{e_{k+1}}\}, \quad \alpha_n = \sup_{k \geq 0} 2^{-(n-k)+} \gamma_k, \quad (s)_+ = \max\{s, 0\}.$$

Then $\alpha_n \geq \gamma_n > 0$, so $\alpha_n^2 \geq \max\{e_n, e_{n+1}\}$. Each term in the supremum decreases by a factor between $1/2$ and 1 as n increases by one. Hence

$$\frac{1}{2}\alpha_n \leq \alpha_{n+1} \leq \alpha_n.$$

Finally, $\gamma_k \rightarrow 0$. Given $\varepsilon > 0$, choose $N \geq 1$ so that $\gamma_k < \varepsilon$ for $k \geq N$. For $n \geq N$,

$$\alpha_n \leq \max \left\{ \max_{0 \leq k < N} 2^{-(n-k)} \gamma_k, \varepsilon \right\}.$$

The finite maximum tends to zero, proving $\alpha_n \rightarrow 0$.

A.2. The dipole profile. Let J_0 and J_1 be the Bessel functions of the first kind, and let β be the first positive zero of J_1 . Existence of this zero and the recurrence identities used below are classical; see Watson [38]. The zero is simple: otherwise uniqueness for the Bessel differential equation at $\beta > 0$ would imply $J_1 \equiv 0$. The recurrence $J'_1 = J_0 - J_1/z$ then gives $J_0(\beta) \neq 0$. For $x = (\rho \cos \theta, \rho \sin \theta)$, define

$$\psi(\rho, \theta) = \begin{cases} -\rho \sin \theta + \frac{2J_1(\beta\rho)}{\beta J_0(\beta)} \sin \theta, & \rho \leq 1, \\ -\frac{\sin \theta}{\rho}, & \rho \geq 1, \end{cases} \quad D = \nabla^\perp \psi. \quad (\text{A.1})$$

This is the Lamb–Chaplygin profile, with the signs fixed by our conventions; compare [3, §2.2]. We record the verification needed here rather than an exact energy normalization.

The Bessel identities give, from either side of $\rho = 1$,

$$\psi = -\sin \theta, \quad \partial_\rho \psi = \sin \theta, \quad \partial_{\rho\rho} \psi = -2 \sin \theta.$$

Thus D is C^1 across the boundary and smooth on either side. Set $q = \psi + x_2$. Then

$$D - e_1 = \nabla^\perp q, \quad q|_{\partial B_1} = 0, \quad \text{curl } D = -\beta^2 q \mathbf{1}_{B_1}.$$

Writing $w = D - e_1$ and using $(w \cdot \nabla)w = (\text{curl } w)w^\perp + \nabla(|w|^2/2)$, we obtain

$$-\partial_1 D + \text{div}(D \otimes D) = \nabla \left(\frac{|D - e_1|^2}{2} + \frac{\beta^2 q^2}{2} \mathbf{1}_{B_1} \right).$$

The vanishing trace of q excludes a boundary measure. Hence this is an exact translating Euler profile. Outside its core,

$$D(x) = \nabla \left(-\frac{x_1}{|x|^2} \right), \quad D(x) = O(|x|^{-2}).$$

Rotation and scaling of the translating equation give (4.1). The bounded core, exterior decay, and compactly supported curl give the norm scalings in (4.2).

Define the compact radial scalar profile

$$W(x) = \frac{1}{\pi} \left(1 - \frac{J_0(\beta|x|)}{J_0(\beta)} \right) \mathbf{1}_{B_1}(x). \quad (\text{A.2})$$

Its value and first radial derivative vanish at $|x| = 1$. Since $\int_0^1 \rho J_0(\beta\rho) d\rho = J_1(\beta)/\beta = 0$,

$$\int_{\mathbb{R}^2} W dx = 1.$$

We now verify the radial identity (4.3). For $r = 1$ and $\xi = e_1$, both sides are divergence-free L^2 fields, and

$$\operatorname{curl}(2\pi\mathbb{P}(We_1)) = -2\pi\partial_2 W = -\frac{2\beta J_1(\beta\rho)}{J_0(\beta)} \sin\theta \mathbf{1}_{B_1} = \operatorname{curl} D.$$

Their difference is a divergence-free, curl-free L^2 field and hence vanishes. Scaling and rotation prove (4.3).

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